

Hertzsprung patterns in Cayley permutations

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Abstract

We adapt the cluster method of Goulden and Jackson to Hertzsprung patterns in Cayley permutations. For a finite antichain T , we compute the cluster series by encoding pattern overlaps as walks in a finite weighted digraph, and use the polynomials $B_n(t)$, which enumerate Cayley permutations by singleton fibres, to derive generating functions for the joint distribution of the patterns in T on Cayley permutations and their primitive counterparts. As applications, we enumerate the Cayley permutations avoiding any single pattern of length three or any set of patterns of length two. We also derive cluster formulas for restricted growth functions and their primitive counterparts. Finally, we show that the proportions of semisimple and simple Cayley permutations tend to $1/(2\sqrt{2})$ and $1/4$, respectively, that almost every restricted growth function is semisimple, and that almost every primitive restricted growth function is simple.

Keywords: Cayley permutation, Hertzsprung pattern, cluster method, overlap digraph, semisimple Cayley permutation, restricted growth function, asymptotics

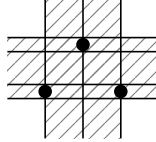
1 Introduction

A permutation τ is a *Hertzsprung factor* [10] of a permutation π when some factor of π is a translate $\tau + c$ of τ . Enumerating permutations that avoid a given set of patterns as Hertzsprung factors generalizes Hertzsprung's problem [10, 15]: *how many ways are there to arrange the numbers $1, 2, \dots, n$ so that the absolute difference between two adjacent numbers is different from 1?* The classic formulation asks for the number of permutations that avoid 12 and 21 as Hertzsprung factors; or, equivalently, those avoiding the mesh patterns [3]



Claesson [10] used a variant of the cluster method of Goulden and Jackson [12] and the transfer-matrix method to determine the joint distribution of occurrences of any set of (incomparable) Hertzsprung patterns.

The main goal of this paper is to study Hertzsprung avoidance on Cayley permutations, defined as words of positive integers where if a letter appears in the word, then all positive integers smaller than that letter also appear. The systematic study of classical pattern avoidance in this context was developed by Cerbai, Claesson, Ernst, and Golab [9]. The literal translation of the preceding definition of Hertzsprung factor is too weak on Cayley permutations, because a value of the factor may recur outside it. The equivalent definition in terms of mesh patterns [5, 6], however, carries over: given an underlying Cayley permutation τ , every box of its plot is shaded except the four corner boxes. For example, the Hertzsprung pattern 121 is



The two vertical strips require the three selected positions to be consecutive, while the horizontal strip forbids any letter outside this factor from lying in its value band. Thus an occurrence occupies a factor whose value band is saturated in this sense. Starting from this saturation condition, we defined simplicity, semisimplicity and the substitution decomposition of Cayley permutations [7], and studied hereditary classes with finitely many simple members [8]. Here, we use the tools from the former papers to study Hertzsprung-pattern avoidance. We adapt the cluster method of Goulden and Jackson, in the form used by Claesson [10], to Hertzsprung patterns in Cayley permutations. For a finite antichain T , we encode pattern overlaps as walks in a finite weighted digraph. We then combine the resulting cluster series with the polynomials $B_n(t)$ tracking the number of singleton fibres to derive the generating function for T -avoiding Cayley permutations. The same technique applies to primitive Cayley permutations and restricted growth functions. The paper is outlined below.

In Section 2 we recall the relevant results and definitions from our previous papers.

In Section 3, we develop the cluster decomposition and its counterpart on primitive Cayley permutations. We also obtain a cluster formula for non-self-overlapping patterns.

In Section 4, we define the overlap digraph and use it to compute the cluster generating function (Theorem 4.2). We treat patterns of length two explicitly (Proposition 4.5) and provide examples of longer patterns and sets of patterns.

In Section 5, we apply the cluster formulas to the asymptotic enumeration of semisimple and simple Cayley permutations. Namely, we show that the proportions of Cayley permutations that are semisimple and that are simple tend to $1/(2\sqrt{2})$ and $1/4$, respectively (Theorem 5.5). Only the obstructions involving two adjacent positions affect these limits: proper H-intervals of length more than two occur with probability $O(1/n)$ and are thus negligible (Lemma 5.4). The proof adapts the argument of Albert, Atkinson, and Klazar [2] for simple permutations.

In Section 6, we provide the corresponding formulas and asymptotics for restricted growth functions (RGFs). We prove that almost every RGF is semisimple, and almost every primitive RGF is simple (Theorem 6.6).

In Section 7, we indicate some directions for further work.

2 Hertzsprung intervals and inflation

We begin by recalling the relevant terminology on simple Cayley permutations [7].

A *Cayley permutation* is an endofunction ω on $[n] = \{1, 2, \dots, n\}$ where $\text{im}(\omega) = [m]$ for some $m \leq n$. We write Cay_n for the set of Cayley permutations of length n and $\text{Cay} = \bigcup_{n \geq 0} \text{Cay}_n$. We also write $\text{Cay}_+ = \bigcup_{n \geq 1} \text{Cay}_n$ for the set of nonempty Cayley permutations.

Given a word ω , we say that β is a *factor* of ω if there are words α and γ such that $\omega = \alpha\beta\gamma$. If α is empty we also say that β is a *prefix* of ω , and if, in addition, γ is nonempty then β is a *proper prefix* of ω . Similarly, if γ is empty we say that β is a *suffix* of ω , and that β is a *proper suffix* if, in addition, α is nonempty. The *standardization* of ω is the word $\text{st}(\omega)$ obtained by replacing the smallest letter of ω by 1, the next smallest by 2, and so on, keeping repetitions equal. Note that $\text{st}(\omega)$ is a Cayley permutation.

Let $\omega = w_1 \cdots w_n \in \text{Cay}_n$. If $I = [a, b] \subseteq [n]$ is a nonempty positional interval, write

$$\omega|_I = w_a w_{a+1} \cdots w_b \quad \text{and} \quad \omega(I) = \{w_a, w_{a+1}, \dots, w_b\}$$

for the factor of ω at positions I and for its set of values. More generally, if $S = \{i_1 < \cdots < i_r\} \subseteq [n]$, write $\omega|_S = w_{i_1} \cdots w_{i_r}$ for the corresponding subword and $\omega(S)$ for its values. We say that I is a *Hertzsprung interval*, or *H-interval*, of ω if there is a value interval $J = [c, d]$ such that

$$\omega(I) = J \quad \text{and} \quad \omega^{-1}(J) = I.$$

Thus, an H-interval occupies consecutive positions I and uses every value in the consecutive range J , which we call its *value band*. The equality $\omega^{-1}(J) = I$ is the *saturation condition*: I contains every position of ω whose value lies in J . In particular, $[n]$ is always an H-interval, while a singleton $[i, i] = \{i\}$ is an H-interval precisely when the value $\omega(i)$ occurs nowhere else in ω . We also regard the empty set as an H-interval with empty value band. When ω is a permutation, its H-intervals are precisely the intervals [4], or blocks [1], of the permutation patterns literature. Following our previous convention [7], here we reserve “interval” for positional and value intervals and “block” for the blocks of a ballot (see the paragraph leading to Proposition 2.3). This choice is justified by the relation between the H-intervals of a Cayley permutation and the modules of the associated relational structure [7, Lemma 5.1].

An H-interval I of ω is an *occurrence* of $\tau \in \text{Cay}$ if $\text{st}(\omega|_I) = \tau$. Equivalently, for nonempty τ , a positional interval I of ω is an occurrence of τ if and only if

$$\omega|_I = \tau + c$$

for an integer $c \geq 0$ and no letter of ω outside I lies in $[c + 1, c + \max(\tau)]$. For example, in $\omega = 12341$ the interval $[2, 4]$ is an occurrence of 123, whereas $[1, 3]$ is not. When such an occurrence exists, we call τ a *Hertzsprung factor* of ω . If I is a prefix or suffix interval, τ is a *Hertzsprung prefix* or *Hertzsprung suffix*, with the qualifier *proper* when I is proper. We identify an occurrence with its positional interval, and a pattern may occur several times: in $\omega = 12341$ the intervals $[2, 3]$ and $[3, 4]$ are

distinct occurrences of 12. Every H-interval is an occurrence of exactly one pattern, namely $\text{st}(\omega|_I)$, and a singleton H-interval is an occurrence of 1. The empty H-interval is an occurrence of the empty pattern.

We shall use the following lemma repeatedly.

Lemma 2.1. *Let $I = [a, b]$ be a nonempty H-interval of a Cayley permutation ω , and put $\sigma = \text{st}(\omega|_I)$. For $a \leq p \leq q \leq b$, set $J = [p, q]$ and $J' = [p - a + 1, q - a + 1]$. Then,*

$$J \text{ is an H-interval of } \omega \iff J' \text{ is an H-interval of } \sigma, \text{ and } \text{st}(\sigma|_{J'}) = \text{st}(\omega|_J).$$

Consequently, J is an occurrence of τ in $\omega \iff J'$ is an occurrence of τ in σ .

Proof. Let $[c, d]$ be the value band of I . Then $\omega|_I = \sigma + (c - 1)$, and saturation of I says that no letter outside I has value in $[c, d]$. The factor $\sigma|_{J'}$ is therefore obtained from $\omega|_J$ by subtracting $c - 1$, so its values are consecutive if and only if those of $\omega|_J$ are. Within I , the same translation gives a bijection between the positions carrying the two value sets; outside I none of the values of $\omega|_J$ occurs. Thus the saturation condition holds for one factor if and only if it holds for the other. Translation of the values does not change standardization, giving the displayed equality and then the claim about occurrences. $\square$

Definition 2.2. A Cayley permutation $\omega \in \text{Cay}_n$ is *semisimple* if every H-interval of ω is empty, a singleton, or $[n]$; these H-intervals are called *trivial*. The word is *primitive* if adjacent letters are distinct. It is *simple* if it is semisimple and primitive, or if $\omega = 11$.

With this definition, simplicity is equivalent to indecomposability of the associated relational structure [7, Theorem 5.2].

There is a bijection between Cayley permutations and ballots (ordered set partitions): a ballot $\varpi = F_1 F_2 \cdots F_m$ of $[n]$ corresponds to the Cayley permutation $\omega = w_1 \cdots w_n$ whose i th letter is the unique index j such that $i \in F_j$. Conversely, the ballot associated with ω is

$$\varpi = F_1 F_2 \cdots F_m, \quad F_j = \omega^{-1}(j),$$

where the F_j are the nonempty fibres of ω . We call these fibres *blocks* and regard a ballot as the word of its blocks, so that a run of consecutive blocks is a factor. This representation gives a convenient description of H-intervals and occurrences.

Proposition 2.3 ([7]). *Let $\omega \in \text{Cay}_n$ have ballot $F_1 F_2 \cdots F_m$, and let $\emptyset \neq I \subseteq [n]$. Then I is an H-interval of ω if and only if I is an interval of $[n]$ and*

$$I = F_c \cup F_{c+1} \cup \cdots \cup F_d$$

for some $1 \leq c \leq d \leq m$. Such an I is an occurrence of $\tau \in \text{Cay}_+$ if and only if subtracting $\min(I) - 1$ from every element of the factor $F_c \cdots F_d$ turns it into the ballot of τ .

Thus, the ballot formulation of H-intervals incorporates saturation by requiring every block in the run to be included in full. An occurrence of τ corresponds to a factor of consecutive blocks whose translated ballot is the ballot of τ . For the patterns of

length two, an occurrence of 11 is a block $\{i, i + 1\}$, an occurrence of 12 is a pair of consecutive blocks $\{i\}\{i + 1\}$, and an occurrence of 21 is a pair of consecutive blocks $\{i + 1\}\{i\}$. We enumerate the ballots avoiding each set of them in Proposition 4.5.

Two H-intervals that overlap in position share an entire block, namely the block containing any common position. We shall repeatedly use the following two closure properties.

Lemma 2.4 ([7]). *Let I and I' be two H-intervals of a Cayley permutation ω such that $I \cap I' \neq \emptyset$.*

1. *The union $I \cup I'$ is an H-interval.*
2. *The intersection $I \cap I'$ is an H-interval with $\omega(I \cap I') = \omega(I) \cap \omega(I')$.*

Let us now define inflation of Cayley permutations [7]. Given $\sigma \in \text{Cay}_k$, associate a nonempty Cayley permutation α_i with each position i , subject to the restriction¹

$$|\sigma^{-1}(\sigma(i))| > 1 \implies \alpha_i = 1. \quad (1)$$

The *inflation* $\sigma[\alpha_1, \dots, \alpha_k]$ is the Cayley permutation obtained by replacing each entry $\sigma(i)$ with a factor order isomorphic to α_i while preserving the relative order of the entries of σ . We call the positional interval occupied by the entries corresponding to α_i the *i th part* of the inflation. Under (1), a position lying in a non-singleton fibre of σ is necessarily inflated by the one-letter Cayley permutation. Further, every non-singleton part is an H-interval.

We showed [7, Theorem 3.4] that every Cayley permutation of length at least two is an inflation $\sigma[\alpha_1, \dots, \alpha_k]$ of a unique semisimple Cayley permutation σ of length $k \geq 2$, with the components obeying (1). For $\sigma \notin \{12, 21\}$, the components are uniquely determined. For $\sigma = 12$, they are unique if α_1 is plus indecomposable; for $\sigma = 21$, they are unique if α_1 is minus indecomposable. That is, α_1 is not a nontrivial direct sum $\alpha \oplus \beta = 12[\alpha, \beta]$ in the first case or a nontrivial skew sum $\alpha \ominus \beta = 21[\alpha, \beta]$ in the second.

An *H-interval decomposition* of ω is a partition of its positions into consecutive intervals, each of which is an H-interval or a single position. Replacing each part by its smallest letter and standardizing the resulting word gives a Cayley permutation, called the *quotient*. Conversely, inflating the quotient by the standardized parts recovers ω . For example, consider the H-interval decomposition

$$\omega = 4 \mid 13321 \mid 56 \mid 4.$$

The quotient is 2132: the value bands $\{1, 2, 3\}$ and $\{5, 6\}$ become 1 and 3, while the two singleton parts carrying 4 both become 2. The standardized parts are 1, 13321, 12, and 1, and inflation recovers the original Cayley permutation:

$$2132[1, 13321, 12, 1] = 4 \mid 13321 \mid 56 \mid 4 = 413321564.$$

¹In our previous paper [7], we also employ a different type of inflation obtained by imposing a weaker restriction where non-singleton fibres can be inflated by constant words.

Only positions in singleton fibres can be inflated nontrivially. The polynomials $B_n(t)$ track their distribution and play a key role in the cluster method and the asymptotic estimates below. Given $\omega \in \text{Cay}$, let $\mu(\omega) = \#\{j : |\omega^{-1}(j)| = 1\}$ and define

$$B(x, t) = \sum_{\omega \in \text{Cay}_+} x^{|\omega|} t^{\mu(\omega)} = \sum_{n \geq 1} B_n(t) x^n, \quad \text{where } B_n(t) = \sum_{\omega \in \text{Cay}_n} t^{\mu(\omega)}.$$

For convenience, set $B_0(t) = 1$. The ballot viewpoint leads to the following results.

Proposition 2.5 ([7]). *The polynomials $B_n(t)$ are determined by*

$$1 + \sum_{n \geq 1} B_n(t) \frac{x^n}{n!} = \frac{1}{2 - e^x - (t-1)x}.$$

For the primitive analogue, let

$$P(x, t) = \sum_{\substack{\omega \in \text{Cay}_+ \\ \omega \text{ primitive}}} x^{|\omega|} t^{\mu(\omega)} = \sum_{n \geq 1} P_n(t) x^n,$$

and set $P_0(t) = 1$.

n	$B_n(t)$	$P_n(t)$
1	t	t
2	$1 + 2t^2$	$2t^2$
3	$1 + 6t + 6t^3$	$2t + 6t^3$
4	$7 + 8t + 36t^2 + 24t^4$	$2 + 18t^2 + 24t^4$
5	$21 + 100t + 60t^2 + 240t^3 + 120t^5$	$2 + 36t + 6t^2 + 144t^3 + 120t^5$

Table 1: The polynomials $B_n(t)$ and $P_n(t)$ for $1 \leq n \leq 5$.

Proposition 2.6 ([7]). *The two singleton-fibre series are related by*

$$P(x, t) = B\left(\frac{x}{1+x}, (1+x)t - x\right).$$

Table 1 lists $B_n(t)$ and $P_n(t)$ for $1 \leq n \leq 5$.

3 Hertzsprung avoidance and the cluster method

Let T be a finite set of Hertzsprung patterns. Denote by $\text{Cay}_n(T)$ the set Cayley permutations of length n that avoid every pattern in T . For a single pattern $T = \{\tau\}$, we simply write $\text{Cay}_n(\tau)$. Our aim is to determine the generating function of the numbers $|\text{Cay}_n(T)|$. We assume throughout that every pattern in T has length at least two. Indeed, occurrences of 1 are precisely singleton fibres and are therefore already tracked by $B(x, t)$. In particular, $B(x, 0)$ counts Cayley permutations avoiding 1. Under the ballot bijection, these are the ordered set partitions in which every block has size at least two. The coefficients of $1 + B(x, 0)$ are recorded as sequence

A032032 in the OEIS [16]; their exponential generating function is $1/(2 + x - e^x)$ by Proposition 2.5. To enumerate Cayley permutations avoiding T , we shall adapt the cluster method of Goulden and Jackson [12]. The method was originally formulated for words over a finite alphabet. Here, we follow Claesson's [10] treatment of Hertzsprung patterns on (standard) permutations.

A T -marked Cayley permutation is a pair (ω, M) in which $\omega \in \text{Cay}$ and M is a set of occurrences in ω of patterns from T . The members of M are called *marked occurrences*, and ω is the *underlying Cayley permutation*. Since occurrences are positional intervals, M distinguishes copies of the same pattern at different positions. The *marked-occurrence graph* of (ω, M) has vertex set M and an edge between distinct occurrences $I, I' \in M$ whenever $I \cap I' \neq \emptyset$. The T -marked Cayley permutation (ω, M) is a T -cluster if ω is nonempty, the marks cover every position of ω , and the marked-occurrence graph is connected. When T is known from context, we simply call (ω, M) a marked Cayley permutations or a cluster.

As an example, let $T = \{123\}$ and $\omega = 123456$. Each of the intervals $[1, 3]$, $[2, 4]$, $[3, 5]$, and $[4, 6]$ is an occurrence of 123. In the following diagrams, an ellipse indicates a marked occurrence:

$$\begin{array}{ll}
1 \text{ } \underbrace{(2 \ 3 \ 4)} \text{ } 5 \text{ } 6 & M = \{[2, 4], [3, 5]\}: \text{ not a cluster (1 and 6 not covered)} \\
\underbrace{(1 \ 2 \ 3)} \text{ } \underbrace{(4 \ 5 \ 6)} & M = \{[1, 3], [4, 6]\}: \text{ not a cluster (disconnected)} \\
\underbrace{(1 \ 2 \ 3)} \text{ } \underbrace{(4 \ 5 \ 6)} & M = \{[1, 3], [3, 5], [4, 6]\}: \text{ a cluster} \\
\underbrace{(1 \ 2 \ 3 \ 4)} \text{ } \underbrace{(5 \ 6)} & M = \{[1, 3], [2, 4], [3, 5], [4, 6]\}: \text{ a cluster}
\end{array}$$

Inflation by marked Cayley permutations is defined in the natural way by carrying over the marked occurrences from each component. For example, using ellipses again to indicate marked occurrences:

$$12134 \left[1, \underbrace{(1 \ 2 \ 3 \ 4)}, 1, \underbrace{(1 \ 2 \ 3)}, 1 \right] = 1 \underbrace{(2 \ 3 \ 4 \ 5)} \ 1 \underbrace{(6 \ 7 \ 8)} \ 9$$

The second component 1234 contains two marked occurrences of 123 in positions $[1, 3]$ and $[2, 4]$; after inflation, they occupy positions $[2, 4]$ and $[3, 5]$. Similarly, the marked occurrence of the fourth component occupies positions $[7, 9]$ after inflation. As in (1), the two positions carrying 1 in the quotient lie in a non-singleton fibre and are therefore inflated by $(1, \emptyset)$. The final position, which carries the singleton value 4, is also inflated by $(1, \emptyset)$ in this example.

Theorem 3.1. *Any marked Cayley permutation, with respect to a finite set T of patterns of length at least two, can be uniquely written*

$$\kappa[(\sigma_1, M_1), (\sigma_2, M_2), \dots, (\sigma_k, M_k)],$$

where $\kappa \in \text{Cay}_k$ and each (σ_i, M_i) is either $(1, \emptyset)$ or a T -cluster. Moreover, $(\sigma_i, M_i) = (1, \emptyset)$ whenever position i lies in a non-singleton fibre of κ . The empty marked Cayley permutation is the case $k = 0$, in which κ is the empty Cayley permutation and the list of components is empty.

Proof. Let (ω, M) be a marked Cayley permutation. For each connected component K of its marked-occurrence graph, let

$$U_K = \bigcup_{I \in K} I.$$

Since K is connected, repeated application of the first item of Lemma 2.4 shows that U_K is an H-interval. Further, the sets U_K belonging to distinct components are disjoint. Together with each uncovered position as a singleton part, they therefore form an H-interval decomposition of ω .

Order the parts of this H-interval decomposition from left to right as $I_1, \dots, I_k$. Each I_i is either U_K for a connected component K or an uncovered singleton. Put $\sigma_i = \text{st}(\omega|_{I_i})$. Let M_i be the marks contained in I_i , translated by $1 - \min I_i$. Lemma 2.1 shows that the translate of each mark $J \subseteq I_i$ is an occurrence of $\text{st}(\omega|_J)$ in σ_i . If $I_i = U_K$, the marks in K cover I_i , and translation preserves their connected intersection graph. Thus (σ_i, M_i) is a cluster. Every other part is an uncovered singleton and gives $(\sigma_i, M_i) = (1, \emptyset)$. Let κ be the quotient of this decomposition. Inflating it by the marked components recovers (ω, M) .

Every nonsingleton part is an H-interval, so its value band is disjoint from the values outside it. Its letter in κ therefore occurs only once. Hence a position in a non-singleton fibre of κ has component $(1, \emptyset)$.

Conversely, consider any representation of the stated form. The marks in each cluster are connected and cover their part, while marks in distinct parts cannot intersect. The nonempty sets of component marks are therefore precisely the connected components of the marked-occurrence graph, and their parts are the corresponding unions U_K . Every remaining part is an uncovered singleton. Thus the parts, the σ_i , the M_i , and the quotient κ are uniquely determined. $\square$

Associate an indeterminate u_τ with each $\tau \in T$, and write $\mathbf{u} = (u_\tau)_{\tau \in T}$ for the vector they form. For a marked Cayley permutation (ω, M) , define the monomial

$$\mathbf{u}_M = \prod_{I \in M} u_{\text{st}(\omega|_I)},$$

where $\text{st}(\omega|_I)$ is the unique pattern of which I is an occurrence. The *cluster generating function* associated with T is

$$C(\mathbf{u}; x) = \sum_{(\omega, M)} \mathbf{u}_M x^{|\omega|},$$

where the sum is over all T -clusters. Similarly, let $G(\mathbf{u}; x) = \sum_{(\omega, M)} \mathbf{u}_M x^{|\omega|}$, where the sum is over all marked Cayley permutations. In the example above, the cluster $(123456, \{[1, 3], [3, 5], [4, 6]\})$ contributes the monomial $u_{123}^3 x^6$, and the cluster with all four occurrences marked contributes $u_{123}^4 x^6$. For a pattern τ , let $\tau(\omega)$ denote the number of occurrences of τ in ω , and set

$$\mathbf{u}^{T(\omega)} = \prod_{\tau \in T} u_\tau^{\tau(\omega)}.$$

Define the *modified cluster generating function*

$$\widehat{C}_T(\mathbf{u}; x) = 1 + C(\mathbf{u} - \mathbf{1}; x)/x,$$

where $\widehat{C}_T(\mathbf{0}; x) = 1 + C(-\mathbf{1}; x)/x$ and $C(-\mathbf{1}; x)$ is the cluster generating function with every u_τ set to -1 .

Theorem 3.2. *Let T be a finite set of patterns, each of length at least two. Then*

$$\sum_{\omega \in \text{Cay}} \mathbf{u}^{T(\omega)} x^{|\omega|} = \sum_{n \geq 0} B_n(\widehat{C}_T(\mathbf{u}; x)) x^n.$$

In particular,

$$\sum_{n \geq 0} |\text{Cay}_n(T)| x^n = \sum_{n \geq 0} B_n(\widehat{C}_T(\mathbf{0}; x)) x^n.$$

Proof. A cluster has length at least two, so $C(\mathbf{u}; x)$ is divisible by x^2 . In particular, $\widehat{C}_T(\mathbf{u}; x)$ is a formal power series with constant term 1.

Fix a quotient $\kappa \in \text{Cay}_n$ in Theorem 3.1. Each position in a non-singleton fibre contributes x . Each singleton fibre may be inflated by a single letter or a cluster, and contributes $x + C(\mathbf{u}; x)$. Thus the marked inflations of κ have total weight

$$x^{n-\mu(\kappa)} (x + C(\mathbf{u}; x))^{\mu(\kappa)} = x^n \left(1 + \frac{C(\mathbf{u}; x)}{x} \right)^{\mu(\kappa)}.$$

Summing over the quotients gives

$$G(\mathbf{u}; x) = \sum_{n \geq 0} B_n \left(1 + \frac{C(\mathbf{u}; x)}{x} \right) x^n.$$

On the other hand, each occurrence of τ may be marked, with weight u_τ , or left unmarked. Hence

$$G(\mathbf{u}; x) = \sum_{\omega \in \text{Cay}} x^{|\omega|} \prod_{\tau \in T} (1 + u_\tau)^{\tau(\omega)}.$$

Replacing $\mathbf{u}$ by $\mathbf{u} - \mathbf{1}$ proves the first identity. Setting every $u_\tau = 0$ in this identity retains precisely the T -avoiding Cayley permutations on the left and replaces $C(\mathbf{u} - \mathbf{1}; x)$ by $C(-\mathbf{1}; x)$ on the right, proving the second identity. $\square$

Theorem 3.2 also applies to sets of Cayley permutations that are closed under inflation in the following sense.

Corollary 3.3. *Let $\mathcal{R} \subseteq \text{Cay}$ contain the empty word and 1, and suppose that every inflation satisfies*

$$\kappa[\sigma_1, \dots, \sigma_k] \in \mathcal{R} \iff \kappa, \sigma_1, \dots, \sigma_k \in \mathcal{R}.$$

Let T be a finite set of patterns, each of length at least two, and suppose that the underlying word of every T -cluster belongs to $\mathcal{R}$. Define

$$B_n^{\mathcal{R}}(t) = \sum_{\omega \in \mathcal{R} \cap \text{Cay}_n} t^{\mu(\omega)}$$

for $n \geq 1$, and set $B_0^{\mathcal{R}}(t) = 1$. Then

$$\sum_{\omega \in \mathcal{R}} \mathbf{u}^{T(\omega)} x^{|\omega|} = \sum_{n \geq 0} B_n^{\mathcal{R}}(\widehat{C}_T(\mathbf{u}; x)) x^n.$$

In particular,

$$\sum_{n \geq 0} |\mathcal{R} \cap \text{Cay}_n(T)| x^n = \sum_{n \geq 0} B_n^{\mathcal{R}}(\widehat{C}_T(\mathbf{0}; x)) x^n.$$

Proof. The unique marked decomposition of Theorem 3.1 restricts to $\mathcal{R}$ under our assumptions. The proof of Theorem 3.2 therefore applies with the sum over quotients restricted to $\mathcal{R}$, replacing $B_n(t)$ by $B_n^{\mathcal{R}}(t)$. $\square$

The cluster formula restricts to primitive Cayley permutations in a similar manner. A factor of a primitive Cayley permutation is primitive, so only primitive patterns can occur in one. Conversely, a cluster of primitive patterns is primitive. To see this, consider any cut between two adjacent positions of the cluster. Some mark must cross the cut, since otherwise the marks on the two sides would form different connected components. The two letters at the cut are therefore adjacent letters of a primitive pattern and hence are distinct. Finally, primitivity is compatible with inflation [7]; that is,

$$\kappa[\sigma_1, \dots, \sigma_k] \text{ is primitive} \iff \kappa, \sigma_1, \dots, \sigma_k \text{ are primitive.}$$

Corollary 3.3 therefore applies with $\mathcal{R}$ equal to the set of primitive Cayley permutations, for which $B_n^{\mathcal{R}}(t) = P_n(t)$, and the next result follows.

Corollary 3.4. *Let T be a finite set of primitive patterns, each of length at least two. Then*

$$\sum_{\substack{\omega \in \text{Cay} \\ \omega \text{ primitive}}} \mathbf{u}^{T(\omega)} x^{|\omega|} = \sum_{n \geq 0} P_n(\hat{C}_T(\mathbf{u}; x)) x^n.$$

In particular,

$$\sum_{n \geq 0} \#\{\omega \in \text{Cay}_n(T) : \omega \text{ is primitive}\} x^n = \sum_{n \geq 0} P_n(\hat{C}_T(\mathbf{0}; x)) x^n.$$

The cluster formula for permutations by Claesson [10] can be obtained as a corollary of the machinery developed here for Cayley permutations. Consider marked Cayley permutations whose underlying words are permutations, and suppose that every pattern in T is also a permutation. Then every T -cluster is a permutation: its marks cover its underlying word, and every value fibre intersecting a mark is a singleton. Moreover, an inflation is a permutation exactly when its quotient and all its components are permutations. Writing $\mathcal{S}$ for the set of all permutations, Corollary 3.3 therefore applies with $\mathcal{R} = \mathcal{S}$. There are $n!$ permutations of length n , each with n singleton fibres, so $B_n^{\mathcal{R}}(t) = n!t^n$, leaving us with Claesson's formula:

$$\sum_{\pi \in \mathcal{S}} \mathbf{u}^{T(\pi)} x^{|\pi|} = \sum_{n \geq 0} n! (x + C(\mathbf{u} - \mathbf{1}; x))^n.$$

Let us now turn our attention to overlaps between patterns. Some patterns τ can overlap with themselves in the sense that there is a Cayley permutation ω of length less than $2|\tau|$ such that τ is both a proper Hertzprung prefix and a proper Hertzprung suffix of ω . We call such patterns *self-overlapping*. This is equivalent to asking that some Cayley permutation contain two distinct overlapping occurrences of τ : the union of two such occurrences is an H-interval by Lemma 2.4, and standardizing its factor gives the required ω by Lemma 2.1. For instance, 12 is self-overlapping, since the intervals $[1, 2]$ and $[2, 3]$ are overlapping occurrences of 12 in 123; but 11 is not, because an occurrence of 11 uses its value exactly twice and a second occurrence would use it a third time, violating saturation. For non-self-overlapping patterns, the number of avoiders depends solely on the length of the pattern and Theorem 3.2 simplifies to an explicit formula analogous to Claesson's [10] formula for permutations.

Corollary 3.5. *Let $\tau \in \text{Cay}_k$ with $k \geq 2$ be non-self-overlapping. Then*

$$\sum_{\omega \in \text{Cay}} u^{\tau(\omega)} x^{|\omega|} = \sum_{n \geq 0} B_n (1 + (u - 1)x^{k-1}) x^n.$$

In particular, setting $u = 0$ gives

$$\sum_{n \geq 0} |\text{Cay}_n(\tau)| x^n = \sum_{n \geq 0} B_n (1 - x^{k-1}) x^n,$$

so the number of Cayley permutations avoiding a single non-self-overlapping pattern depends only on the length of the pattern.

Proof. The marked-occurrence graph of a cluster is connected, so a cluster with two or more marks contains an overlapping pair. For non-self-overlapping τ the only cluster is therefore a single covering mark, and then the underlying Cayley permutation is τ itself. Hence $C(u; x) = ux^k$, so that $\hat{C}_\tau(u; x) = 1 + C(u - 1; x)/x = 1 + (u - 1)x^{k-1}$, and Theorem 3.2 gives both formulas. $\square$

For the non-self-overlapping pattern $\tau = 11$, the previous corollary recovers the case $T = \{11\}$ of Proposition 4.5 below.

4 The overlap digraph

For the rest of this section, let T be a finite antichain under Hertzprung-factor containment, with each pattern of length at least two. We adapt Claesson's [10] overlap digraph for permutations to Cayley permutations.

For $\sigma, \tau \in T$ let $\mathcal{O}(\sigma, \tau)$ be the set of Cayley permutations π such that σ is a proper Hertzprung prefix of π , τ is a proper Hertzprung suffix of π , and $|\pi| < |\sigma| + |\tau|$. Let

$$\Omega(\sigma, \tau) = \sum_{\pi \in \mathcal{O}(\sigma, \tau)} x^{|\pi| - |\sigma|}.$$

Put $k = |\sigma|$ and $\ell = |\tau|$. If the prefix and suffix of $\pi \in \mathcal{O}(\sigma, \tau)$ share i positions, then $1 \leq i < \min(k, \ell)$ and $|\pi| = k + \ell - i$. Write these two factors as $\sigma + c$ and $\tau + d$, with $c, d \geq 0$. At the first shared position, $\sigma(k - i + 1) + c = \tau(1) + d$, which determines the difference $d - c$. For an overlap to exist, the remaining shared positions must give the same difference. The prefix and suffix cover π and have minima $1 + c$ and $1 + d$, so $\min(\pi) = 1 + \min(c, d)$. As π is a Cayley permutation, $\min(\pi) = 1$, and therefore $\min(c, d) = 0$. Together, the known difference $d - c$ and this minimum condition leave only one possible pair (c, d) , and hence only one possible π . Thus $|\mathcal{O}(\sigma, \tau)| \leq \min(k, \ell) - 1$. Different values of i give different exponents $\ell - i$, so $\Omega(\sigma, \tau)$ is a polynomial with coefficients in $\{0, 1\}$. For $\sigma = \tau = 123$, overlaps of one and two positions give

$$\mathcal{O}(123, 123) = \{1234, 12345\}, \quad \Omega(123, 123) = x + x^2.$$

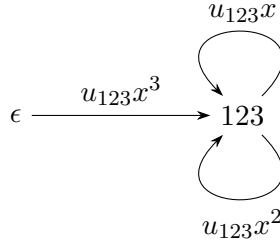
Construct an edge-weighted directed multigraph D_T with vertex set $V = \{\epsilon\} \cup T$. For each $\tau \in T$, it has one edge $\epsilon \rightarrow \tau$ of weight $u_\tau x^{|\tau|}$. For $\sigma, \tau \in T$, the edges $\sigma \rightarrow \tau$ are indexed by $\pi \in \mathcal{O}(\sigma, \tau)$, so different overlap words give parallel edges. The

edge indexed by π has weight $u_\tau x^{|\pi|-|\sigma|}$. There are no other edges. The weighted adjacency matrix therefore has entries

$$A_{\sigma,\tau} = \begin{cases} 0 & \text{if } \tau = \epsilon, \\ u_\tau x^{|\tau|} & \text{if } \sigma = \epsilon, \\ u_\tau \Omega(\sigma, \tau) & \text{otherwise.} \end{cases}$$

Give a T -cluster (ω, M) weight $\mathbf{u}_M x^{|\omega|}$, and a walk in D_T the product of its edge weights.

For $T = \{123\}$, the overlaps 1234 and 12345 give the two loops at 123:



The upper and lower loops are indexed by 1234 and 12345, respectively.

Lemma 4.1. *The T -clusters are in weight-preserving bijection with the walks in D_T starting at ϵ and ending in T .*

Proof. Given a cluster (ω, M) , order its marks $Q_1, \dots, Q_r$ by left endpoint and put $\tau_k = \text{st}(\omega|_{Q_k})$. No mark properly contains another. Indeed, by Lemma 2.1, such a containment would make one member of T a Hertzsprung factor of a different member, contrary to the antichain hypothesis. Thus, ordering the marks by left endpoint also orders them by right endpoint. Since the marked-occurrence graph is connected, consecutive marks overlap. For $k < r$, the word $\pi_k = \text{st}(\omega|_{Q_k \cup Q_{k+1}})$ lies in $\mathcal{O}(\tau_k, \tau_{k+1})$. Indeed, the marks Q_k and Q_{k+1} overlap properly, and their union is an H-interval by Lemma 2.4, so Lemma 2.1 applies. These data give the walk $\epsilon \rightarrow \tau_1 \rightarrow \dots \rightarrow \tau_r$, taking the edge $\tau_k \rightarrow \tau_{k+1}$ indexed by π_k .

Conversely, take such a walk, with π_k indexing its edge $\tau_k \rightarrow \tau_{k+1}$. Place a copy of τ_1 at $Q_1 = [1, |\tau_1|]$. At each step identify the copy of τ_k already placed with the Hertzsprung prefix of π_k ; its Hertzsprung suffix then determines the positions and values of a copy of τ_{k+1} at Q_{k+1} . Both endpoints of Q_k increase strictly with k . If nonconsecutive copies overlap, their common positions belong to every intervening copy, so the agreement of consecutive copies makes all their letters agree. We obtain a word ω , which we translate so that its smallest value is 1, and mark $M = \{Q_1, \dots, Q_r\}$.

Put $J_k = \omega(Q_k)$. The two marks in $Q_k \cup Q_{k+1}$ are H-intervals of a translate of π_k . Their saturation gives

$$\begin{aligned} \omega(Q_k \cap Q_{k+1}) &= J_k \cap J_{k+1}, \\ \omega(Q_k \setminus Q_{k+1}) &= J_k \setminus J_{k+1}, \quad \omega(Q_{k+1} \setminus Q_k) = J_{k+1} \setminus J_k. \end{aligned}$$

Both differences are nonempty, so consecutive value intervals intersect and neither contains the other.

For two such intervals, the lower and upper endpoints are ordered in the same direction. The direction cannot change along the chain. Suppose, for instance, that $\min J_{k-1} < \min J_k > \min J_{k+1}$. Then both $J_{k-1} \cap J_k$ and $J_k \cap J_{k+1}$ contain $\min J_k$. The corresponding positional intervals $Q_{k-1} \cap Q_k$ and $Q_k \cap Q_{k+1}$ are a saturated prefix and suffix of Q_k . They both contain every position carrying $\min J_k$, so they intersect and cover Q_k . One also contains a position carrying $\max J_k$, so its value interval is all of J_k . Saturation then makes it all of Q_k , contrary to the overlap being proper. Complementing values excludes the reverse inequality. Thus the lower and upper endpoints of the J_k are strictly monotone.

We can now check that every mark is saturated in ω . Fix i and a position p to the right of Q_i , and let ℓ be the first mark containing p . Then $\ell > i$ and $p \notin Q_{\ell-1}$, so the displayed equalities give

$$\omega(p) \in J_\ell \setminus J_{\ell-1}.$$

If the chain ascends, this difference lies above $J_{\ell-1}$ and hence above J_i ; if it descends, it lies below both. Thus $\omega(p) \notin J_i$. For a position to the left of Q_i , take instead the last mark containing it and apply the same argument with the order reversed. No value of J_i occurs outside Q_i , so Q_i is saturated. Hence every Q_k is an occurrence of $\tau_k = \text{st}(\omega|_{Q_k})$.

The union of the J_k is a value interval because consecutive ones intersect. The normalization of its minimum to 1 therefore makes ω a Cayley permutation. Its marks cover every position and have a connected intersection graph, so (ω, M) is a T -cluster. The two constructions are inverse to each other. Indeed, a cluster determines its marks and adjacent overlap words, while the chosen edges determine all positions and values of the glued word. The first edge accounts for $|\tau_1|$ positions; the edge indexed by π_k adds $|\pi_k| - |\tau_k|$ positions. Thus the walk has weight

$$u_{\tau_1} x^{|\tau_1|} \prod_{k=1}^{r-1} u_{\tau_{k+1}} x^{|\pi_k| - |\tau_k|} = \mathbf{u}_M x^{|\omega|}. \quad \square$$

Theorem 4.2. *Let T be a finite antichain of patterns, each of length at least two, and let A be the weighted adjacency matrix of D_T , with ϵ first and the elements of T in any order. Write I for the identity matrix of the same size. Then*

$$C(\mathbf{u}; x) = \sum_{i=1}^{|T|} ((I - A)^{-1})_{1, i+1} = \frac{1}{\det(I - A)} \sum_{i=1}^{|T|} (-1)^i \det(I - A : i + 1, 1),$$

where $(N : i, j)$ denotes the matrix obtained by removing the i th row and j th column of N . In particular, $C(\mathbf{u}; x)$ belongs to the field of rational functions $\mathbb{Q}(x)(u_\tau : \tau \in T)$.

Proof. By Lemma 4.1, $C(\mathbf{u}; x)$ counts walks starting at ϵ and ending in T . Every edge weight has positive degree in x , so the matrix identity

$$(I - A)^{-1} = I + A + A^2 + A^3 + \dots$$

is coefficientwise well defined over $\mathbb{Q}[u_\tau : \tau \in T][[x]]$. Therefore $C(\mathbf{u}; x)$ is the sum of the entries $(1, i + 1)$ of $(I - A)^{-1}$ for $1 \leq i \leq |T|$. The determinant $\det(I - A)$ has constant term 1, and Cramer's rule gives $(N^{-1})_{ab} = (-1)^{a+b} \det(N : b, a) / \det(N)$ for invertible N . Taking $a = 1$ and $b = i + 1$ turns the sign into $(-1)^i$ and gives the determinant form. $\square$

The edge weights depend on $\Omega(\sigma, \tau)$, whose definition asks for a search through Cayley permutations. The next proposition computes its coefficients directly from σ and τ , as in Claesson's [10] permutation case. These polynomials are analogous to the correlation polynomials of Guibas and Odlyzko [14].

Proposition 4.3. *Let $\sigma, \tau \in \text{Cay}_+$, put $k = |\sigma|$, $\ell = |\tau|$, $p = \max(\sigma)$, and $q = \max(\tau)$, and fix $1 \leq i < \min(k, \ell)$. Let α be the suffix of σ of length i , and β the prefix of τ of length i :*

$$\alpha = \sigma(k - i + 1) \cdots \sigma(k) \quad \text{and} \quad \beta = \tau(1) \cdots \tau(i).$$

Let r be the number of distinct letters of α . Then there is a $\pi \in \mathcal{O}(\sigma, \tau)$ of length $k + \ell - i$ if and only if both of the following hold:

- (O1) $[k - i + 1, k]$ is an H-interval of σ and $[1, i]$ is an H-interval of τ ;
- (O2) $\alpha = \beta + (p - r)$ or $\beta = \alpha + (q - r)$.

Let $\chi_i(\sigma, \tau)$ be 1 if both (O1) and (O2) hold, and 0 otherwise. Then

$$\Omega(\sigma, \tau) = \sum_{i=1}^{\min(k, \ell)-1} \chi_i(\sigma, \tau) x^{\ell-i}.$$

For example, take $\sigma = 1223$, $\tau = 1123$, and $i = 3$. The shaded columns contain the shared factor 223:

position	1	2	3	4	5
σ	1	2	2	3	
$\tau + 1$		2	2	3	4
π	1	2	2	3	4

The suffix 223 of σ and the prefix 112 of τ are H-intervals, so (O1) holds. Since $p = 3$ and $r = 2$, the equality $223 = 112 + (p - r)$ gives (O2), and the resulting word is $\pi = 12234$.

Proof of Proposition 4.3. Put $n = k + \ell - i$, and suppose $\pi \in \mathcal{O}(\sigma, \tau)$ has length n . Its prefix and suffix occurrences occupy $I_\sigma = [1, k]$ and $I_\tau = [k - i + 1, n]$, with factors $\sigma + c$ and $\tau + d$ for some $c, d \geq 0$. Their intersection is an H-interval of π by Lemma 2.4; applying Lemma 2.1 inside each occurrence gives (O1). The two value intervals are $[c + 1, c + p]$ and $[d + 1, d + q]$. By saturation, they intersect without either containing the other, and their intersection consists of the r values at the shared positions. Since π contains the value 1 and its two occurrences cover all positions, either $c = 0 < d$ or $d = 0 < c$. In the first case the intersection is $[d + 1, p]$, so $r = p - d$ and the equality $\alpha + c = \beta + d$ becomes $\alpha = \beta + (p - r)$. In the second case the intersection is $[c + 1, q]$, so $r = q - c$ and $\beta = \alpha + (q - r)$. This is (O2).

Conversely, assume (O1) and first suppose $\alpha = \beta + \Delta$ with $\Delta = p - r$. The value sets of α and β are intervals of r values. Since the former lies in $[1, p]$ and the latter in $[1, q]$, their equality up to Δ forces them to be $[p - r + 1, p]$ and $[1, r]$, respectively. Glue the patterns by setting

$$\pi = \sigma(1) \cdots \sigma(k) (\tau(i + 1) + \Delta) \cdots (\tau(\ell) + \Delta),$$

whose prefix and suffix factors are σ and $\tau + \Delta$. Their value intervals $[1, p]$ and $[\Delta + 1, \Delta + q]$ intersect, so their union is $[1, \Delta + q]$ and π is a Cayley permutation. By saturation in (O1), letters of τ outside β exceed r , and letters of σ outside α are at most $p - r = \Delta$. After gluing, no value of either occurrence therefore appears outside its positions. Both occurrences are proper, so $\pi \in \mathcal{O}(\sigma, \tau)$. In the other case, put $\Gamma = q - r$. The equality $\beta = \alpha + \Gamma$ forces the value sets of α and β to be $[1, r]$ and $[\Gamma + 1, q]$. Glue $\sigma + \Gamma$ to τ . Their value intervals have union $[1, \Gamma + p]$; letters of σ outside α become greater than q , while letters of τ outside β are at most Γ . Thus the resulting word is a Cayley permutation with two proper occurrences, and belongs to $\mathcal{O}(\sigma, \tau)$.

Finally, $\mathcal{O}(\sigma, \tau)$ has at most one element of each length, as observed above, and an element of length n contributes $x^{n-k} = x^{\ell-i}$ to $\Omega(\sigma, \tau)$. $\square$

The two alternatives in (O2) describe the vertical order of the occurrences. In the first, the shared factor uses the top r values of σ and the bottom r of τ ; in the second, it uses the bottom r of σ and the top r of τ . They cannot both hold under (O1), since a proper H-interval cannot use every value of its word.

For permutations (O1) is implied by (O2). Indeed $r = i$, $p = k$, and $q = \ell$ there, so the common difference is $k - i$ or $-(\ell - i)$; in the first case $\sigma(k - i + j) = \tau(j) + k - i \leq k$ makes the i distinct letters of β lie in $[1, i]$, while the letters of α lie in $[k - i + 1, k]$, and symmetrically in the second case; for a permutation a factor whose values form an interval is an H-interval. The test then reduces to Claesson's [10] overlap test. Repeated values make (O1) a condition of its own: for $\sigma = 121$, $\tau = 212$, and $i = 2$ the two factors are $\alpha = \beta = 21$, so $\alpha = \beta + (p - r)$ with $p - r = 0$, but the value 1 of σ recurs at position 1, so $[2, 3]$ is not an H-interval of σ and (O1) fails. Gluing anyway would give 1212, whose prefix 121 is indeed not saturated, the value 2 recurring at position 4; and $\mathcal{O}(121, 212) = \emptyset$.

Each of the $\min(k, \ell) - 1$ tests costs $O(k + \ell)$ steps, so $\Omega(\sigma, \tau)$ is computed without any search. Condition (O1) also sharpens the bound above: $|\mathcal{O}(\sigma, \tau)|$ is at most the number of proper Hertzprung suffixes of σ , and at most the number of proper Hertzprung prefixes of τ .

Example 4.4. A pattern of length three can overlap itself only with shared length 1 or 2. Applying Proposition 4.3 to the thirteen such patterns gives

$$\chi_i(\tau, \tau) = 1 \text{ for some } i \in \{1, 2\} \iff \tau \in \{123, 321\}.$$

Thus only 123 and 321 are self-overlapping; for example, the intervals $[1, 3]$ and $[2, 4]$ are overlapping occurrences of 123 in 1234. By Corollary 3.5, each of the other eleven patterns satisfies

$$\begin{aligned} \sum_{n \geq 0} |\text{Cay}_n(\tau)| x^n &= \sum_{n \geq 0} B_n (1 - x^2) x^n \\ &= 1 + x + 3x^2 + 12x^3 + 71x^4 + 517x^5 + 4509x^6 + \dots \end{aligned}$$

These eleven patterns fall into four classes under reversal and value complementation, so the coincidence is not explained by symmetry.

For either $\tau = 123$ or $\tau = 321$, both possible overlaps occur, and hence $\Omega(\tau, \tau) = x + x^2$. The overlap digraph has one pattern vertex, so Theorem 4.2 gives

$$C(u; x) = \frac{ux^3}{1 - u(x + x^2)}.$$

Substituting $u = -1$ in the cluster series gives

$$\hat{C}_T(\mathbf{0}; x) = 1 + \frac{C(-1; x)}{x} = \frac{1 + x}{1 + x + x^2},$$

and therefore

$$\begin{aligned} \sum_{n \geq 0} |\text{Cay}_n(123)|x^n &= \sum_{n \geq 0} |\text{Cay}_n(321)|x^n \\ &= \sum_{n \geq 0} B_n \left(\frac{1 + x}{1 + x + x^2} \right) x^n \\ &= 1 + x + 3x^2 + 12x^3 + 72x^4 + 521x^5 + 4532x^6 + \dots \end{aligned}$$

Proposition 4.5. Assume $T \subseteq \{11, 12, 21\}$ and let $c_{11}(u; x) = ux^2$ and $c_{12}(u; x) = c_{21}(u; x) = ux^2/(1 - ux)$. Then

$$C(\mathbf{u}; x) = \sum_{\tau \in T} c_\tau(u_\tau; x) \quad \text{and} \quad \hat{C}_T(\mathbf{0}; x) = 1 - \sum_{\tau \in T} k_\tau(x),$$

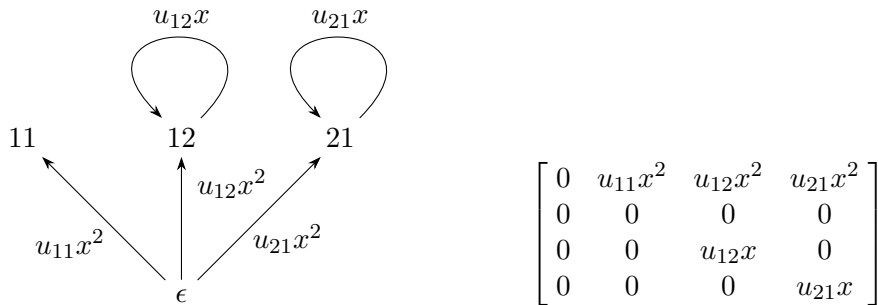
where $k_{11}(x) = x$ and $k_{12}(x) = k_{21}(x) = x/(1 + x)$. Moreover,

$$\sum_{n \geq 0} |\text{Cay}_n(T)|x^n = \sum_{n \geq 0} B_n(\hat{C}_T(\mathbf{0}; x))x^n.$$

Proof. An overlap between two patterns of length two has one shared position. Neither singleton interval of 11 is an H-interval, so (O1) of Proposition 4.3 rules out every overlap involving 11. For 12 and 21, condition (O2) rules out the mixed overlaps: their shared letters are equal, whereas (O2) requires a difference of 1 or -1 . The only nonempty overlap sets are

$$\mathcal{O}(12, 12) = \{123\} \quad \text{and} \quad \mathcal{O}(21, 21) = \{321\},$$

each contributing $\Omega = x$. In the pictures below, parallel edges are combined into one arrow carrying the sum of their weights, and arrows of weight zero are omitted. The digraph D_T for $T = \{11, 12, 21\}$ and its adjacency matrix are shown below.



Since there are no edges between distinct patterns, no cluster mixes two of the three patterns. For any subset T , including the empty subset, its cluster generating function is therefore the corresponding partial sum of

$$C(\mathbf{u}; x) = u_{11}x^2 + \frac{u_{12}x^2}{1 - u_{12}x} + \frac{u_{21}x^2}{1 - u_{21}x}.$$

The loop at 12 traversed $m - 2$ times is the cluster whose underlying Cayley permutation is the increasing run $12 \cdots m$, with all $m - 1$ adjacent pairs marked. Setting every $u_\tau = -1$ and dividing by x gives the stated expression for $\widehat{C}_T(\mathbf{0}; x)$; Theorem 3.2 then gives the displayed generating function for the numbers $|\text{Cay}_n(T)|$. $\square$

Corollary 4.6. *The joint distribution of the patterns 12 and 21 on primitive Cayley permutations is determined by*

$$\sum_{\substack{\omega \in \text{Cay} \\ \omega \text{ primitive}}} u^{12(\omega)} v^{21(\omega)} x^{|\omega|} = \sum_{n \geq 0} P_n \left(1 + \frac{(u-1)x}{1 - (u-1)x} + \frac{(v-1)x}{1 - (v-1)x} \right) x^n.$$

Consequently, the generating function for primitive Cayley permutations avoiding 12 and 21 is

$$1 + P\left(x, \frac{1-x}{1+x}\right) = 1 + B\left(\frac{x}{1+x}, 1-2x\right). \quad (2)$$

Proof. Apply Corollary 3.4 and the cluster generating functions for 12 and 21 in Proposition 4.5. Setting $u = v = 0$ gives the first expression in (2), and Proposition 2.6 gives the second. $\square$

For the nonempty subsets of $\{11, 12, 21\}$, the substituted arguments in Proposition 4.5 are

$$\begin{aligned} \widehat{C}_{11}(\mathbf{0}; x) &= 1 - x, & \widehat{C}_{12}(\mathbf{0}; x) &= \widehat{C}_{21}(\mathbf{0}; x) = \frac{1}{1+x}, & \widehat{C}_{12,21}(\mathbf{0}; x) &= \frac{1-x}{1+x}, \\ \widehat{C}_{11,12}(\mathbf{0}; x) &= \widehat{C}_{11,21}(\mathbf{0}; x) = \frac{1-x-x^2}{1+x}, & \widehat{C}_{11,12,21}(\mathbf{0}; x) &= \frac{1-2x-x^2}{1+x}. \end{aligned}$$

Complementing values, $\omega \mapsto \max(\omega) + 1 - \omega$, is an involution on Cay_n mapping occurrences of 12 to occurrences of 21 and fixing those of 11; it explains the coincidences $\widehat{C}_{12}(\mathbf{0}; x) = \widehat{C}_{21}(\mathbf{0}; x)$ and $\widehat{C}_{11,12}(\mathbf{0}; x) = \widehat{C}_{11,21}(\mathbf{0}; x)$. Up to this symmetry the seven nonempty subsets fall into five avoidance classes. They are pairwise distinct already at length three, where their counts are 9, 10, 7, 6, and 3 in the order shown in Table 2. Proposition 2.5 determines the polynomials B_n , so all five sequences are computable. As a check, each of 11, 12, and 21 contains itself as a Hertzprung factor, which explains the 0 in row $n = 2$ of the last column.

n	$\{11\}$	$\{12\}$	$\{12, 21\}$	$\{11, 12\}$	$\{11, 12, 21\}$
1	1	1	1	1	1
2	2	2	1	1	0
3	9	10	7	6	3
4	53	56	41	38	27
5	383	400	287	274	193
6	3317	3442	2465	2368	1647
7	33505	34594	24759	23916	16687
8	386713	397770	284369	275924	193067
9	5020995	5149210	3677767	3580754	2509633
10	72432831	74104088	52886201	51630648	36230497

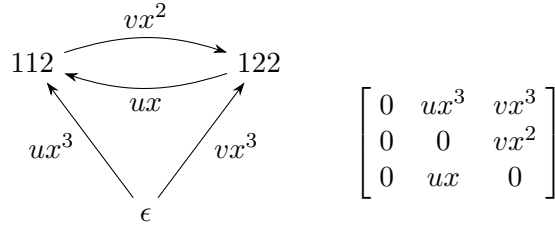
Table 2: The number of Cayley permutations of length n avoiding the given set of Hertzsprung patterns.

At length three, distinct patterns can occur in the same cluster.

Example 4.7. Let $T = \{112, 122\}$, and write $u = u_{112}$ and $v = u_{122}$. Neither pattern is self-overlapping, so $\mathcal{O}(112, 112) = \mathcal{O}(122, 122) = \emptyset$. The mixed overlaps are

$$\mathcal{O}(112, 122) = \{11233\} \quad \text{and} \quad \mathcal{O}(122, 112) = \{1223\},$$

so that $\Omega(112, 122) = x^2$ and $\Omega(122, 112) = x$. In 11233 the prefix 112 and the suffix 233 share a single position, while in 1223 the prefix 122 and the suffix 223 share two. The digraph D_T and its adjacency matrix are shown below.



Theorem 4.2 first gives the denominator-cleared identity

$$(1 - uvx^3)C(\mathbf{u}; x) = x^3(u + v + uvx + uvx^2).$$

Setting $u = v = -1$ and dividing by x similarly gives

$$(1 + x + x^2)\widehat{C}_T(\mathbf{0}; x) = (1 - x)(1 + x)^2.$$

Thus

$$C(\mathbf{u}; x) = \frac{x^3(u + v + uvx + uvx^2)}{1 - uvx^3}, \quad \widehat{C}_T(\mathbf{0}; x) = \frac{(1 - x)(1 + x)^2}{1 + x + x^2}.$$

By Theorem 3.2, the ordinary generating function for Cayley permutations avoiding both patterns is

$$\begin{aligned} \sum_{n \geq 0} |\text{Cay}_n(T)|x^n &= 1 + B(x, \widehat{C}_T(\mathbf{0}; x)) \\ &= 1 + x + 3x^2 + 11x^3 + 68x^4 + 498x^5 + 4365x^6 + \cdots \end{aligned}$$

5 Asymptotics for Cayley permutations

We estimate the numbers of semisimple and simple Cayley permutations by counting those with no H-interval of length two and showing that longer proper H-intervals are rare. Write

$$a_n = |\text{Cay}_n| \quad \text{and} \quad \rho = \log 2.$$

We first need two coefficient estimates for the cluster formulas.

Lemma 5.1. *For every fixed $r \geq 0$,*

$$a_m \sim \frac{m!}{2\rho^{m+1}}, \quad B_m^{(r)}(1) \sim a_m \frac{m^r}{2^r}.$$

Proof. Differentiating Proposition 2.5 r times with respect to t gives

$$\sum_{m \geq 0} B_m^{(r)}(1) \frac{x^m}{m!} = \frac{r!x^r}{(2 - e^x)^{r+1}}.$$

The poles of $(2 - e^x)^{-1}$ are $\rho + 2\pi ik$, with $k \in \mathbb{Z}$, so $x = \rho$ is the unique pole of smallest modulus. Since $e^\rho = 2$, the local expansion is

$$\frac{1}{2 - e^x} = \frac{1}{2(\rho - x)} + \frac{1}{4} + O(\rho - x), \quad \text{as } x \rightarrow \rho.$$

For the differentiated generating function, this gives

$$\frac{r!x^r}{(2 - e^x)^{r+1}} = \frac{r!x^r}{2^{r+1}(\rho - x)^{r+1}} + O((\rho - x)^{-r}), \quad \text{as } x \rightarrow \rho.$$

The binomial theorem gives

$$[x^m] \frac{r!x^r}{2^{r+1}(\rho - x)^{r+1}} = \frac{r!}{2^{r+1}\rho^{m+1}} \binom{m}{r}.$$

By the meromorphic coefficient theorem of Flajolet and Sedgewick [11, Theorem IV.10], this term determines the leading coefficient asymptotics. Taking $r = 0$ first, and then general fixed r , gives

$$a_m \sim \frac{m!}{2\rho^{m+1}}, \quad B_m^{(r)}(1) \sim \frac{m!r!}{2^{r+1}\rho^{m+1}} \binom{m}{r}.$$

Since $r! \binom{m}{r} \sim m^r$, the second estimate follows. $\square$

The next lemma handles substitutions into the ordinary generating function $B(x, t)$, whose coefficients grow factorially.

Lemma 5.2. *Suppose that f and g are analytic at the origin and*

$$f(x) = x + cx^2 + O(x^3), \quad g(x) = 1 + dx + O(x^2).$$

Then

$$[x^n]B(f(x), g(x)) \sim a_n \exp\left(\rho c + \frac{\rho d}{2}\right).$$

Proof. Put $h(x) = f(x)/x$ and, for $0 \leq j < n$, define

$$R_{n,j} = \frac{1}{a_n} \sum_{\omega \in \text{Cay}_{n-j}} [x^j] h(x)^{n-j} g(x)^{\mu(\omega)}.$$

Set $R_{n,j} = 0$ for $j \geq n$. Then, for $n \geq 1$,

$$\frac{[x^n] B(f(x), g(x))}{a_n} = \sum_{j \geq 0} R_{n,j}.$$

Write $h(x) = 1 + xA(x)$ and $g(x) = 1 + xD(x)$, where $A(0) = c$ and $D(0) = d$. Expanding both powers and using $\sum_{\omega \in \text{Cay}_m} \binom{\mu(\omega)}{b} = B_m^{(b)}(1)/b!$ gives

$$\sum_{\omega \in \text{Cay}_m} [x^j] h(x)^m g(x)^{\mu(\omega)} = \sum_{\substack{a, b \geq 0 \\ a+b \leq j}} \binom{m}{a} \frac{B_m^{(b)}(1)}{b!} [x^{j-a-b}] A(x)^a D(x)^b.$$

Fix j and put $m = n - j$. Lemma 5.1 gives

$$\frac{a_{n-j}}{a_n} \sim \rho^j n^{-j}.$$

Together with the derivative estimate of the same lemma, this shows that, after division by a_n , the terms with $a + b < j$ tend to zero, and those with $a + b = j$ tend to $\rho^j c^a d^b / (a! 2^b b!)$. Thus

$$\lim_{n \rightarrow \infty} R_{n,j} = \rho^j \sum_{\substack{a, b \geq 0 \\ a+b=j}} \frac{c^a d^b}{a! 2^b b!} = \frac{(\rho(c + d/2))^j}{j!}.$$

It remains to justify summing these limits. Since h and g are analytic at the origin and have constant term 1, there is a constant K such that each is coefficientwise bounded in absolute value by $(1 - Kx)^{-1}$. As $0 \leq \mu(\omega) \leq m$,

$$|[x^j] h(x)^m g(x)^{\mu(\omega)}| \leq K^j \binom{2m + j - 1}{j}.$$

By the first estimate of Lemma 5.1, there are positive constants C_1 and C_2 such that

$$C_1 m! \rho^{-m} \leq a_m \leq C_2 m! \rho^{-m} \quad (m \geq 1).$$

Consequently, with $K_0 = C_2/C_1$,

$$\frac{a_m}{a_n} \leq K_0 \rho^{n-m} \frac{m!}{n!} \quad (1 \leq m \leq n).$$

For $m \geq 1$ and $j \geq 0$,

$$\frac{m!}{(m+j)!} \binom{2m+j-1}{j} = \frac{1}{j!} \prod_{r=0}^{j-1} \frac{2m+r}{m+1+r} \leq \frac{2^j}{j!}.$$

With $m = n - j$, these bounds give

$$|R_{n,j}| \leq K_0 \frac{(2K\rho)^j}{j!}.$$

The same bound holds when $j \geq n$, since $R_{n,j} = 0$. The right-hand side is summable independently of n , so dominated convergence gives

$$\lim_{n \rightarrow \infty} \frac{[x^n]B(f(x), g(x))}{a_n} = \sum_{j \geq 0} \frac{(\rho(c + d/2))^j}{j!} = \exp\left(\rho c + \frac{\rho d}{2}\right). \quad \square$$

To control longer H-intervals, we first count the occurrences of a fixed pattern.

Corollary 5.3. *Let $\tau \in \text{Cay}_k$ with $k \geq 2$ and $n \geq k$, and put $m = n - k + 1$. Then*

$$\sum_{\omega \in \text{Cay}_n} \tau(\omega) = B'_m(1) = \frac{m}{2}(a_m + a_{m-1}).$$

The total number of occurrences of τ in the Cayley permutations of length n therefore depends only on the length of τ .

Proof. Apply Theorem 3.2 with $T = \{\tau\}$. Differentiate both sides with respect to u_τ , then set $u_\tau = 1$, so that each word ω contributes $\tau(\omega)x^{|\omega|}$. Since the modified cluster generating function is $1 + C(u_\tau - 1; x)/x$, this evaluates the derivative of $C(u_\tau; x)$ at $u_\tau = 0$. Only the term $u_\tau x^k$ survives, since a cluster with exactly one marked occurrence consists of the pattern itself. Thus

$$\sum_{\omega \in \text{Cay}} \tau(\omega)x^{|\omega|} = x^{k-1} \sum_{j \geq 0} B'_j(1)x^j,$$

which gives the first equality. For the second, write $E(x) = (2 - e^x)^{-1} = \sum_{j \geq 0} a_j x^j / j!$. Differentiating Proposition 2.5 with respect to t gives

$$\sum_{j \geq 1} B'_j(1) \frac{x^j}{j!} = xE(x)^2.$$

Since $E' = 2E^2 - E$, we have $E^2 = (E + E')/2$, and coefficient extraction gives the stated value. $\square$

Lemma 5.4. *The total number of pairs (ω, I) in which $\omega \in \text{Cay}_n$ and I is an H-interval of length between 3 and $n - 1$ is $O(a_n/n)$. Consequently, a uniformly chosen member of Cay_n has such an interval with probability $O(1/n)$.*

Proof. Every H-interval of length k is an occurrence of exactly one pattern in Cay_k . Summing Corollary 5.3 over these patterns, the number of pairs is

$$\sum_{k=3}^{n-1} a_k B'_{n-k+1}(1).$$

Since $B'_m(1) \leq m a_m$, and a_m is bounded above and below by positive constant multiples of $m! \rho^{-m}$, division by a_n bounds this by a constant multiple of

$$\sum_{k=3}^{n-1} \frac{(n - k + 1)k!(n - k + 1)!}{n!}.$$

Put $N = n + 1$. Since $k!(N - k)!/(N - 1)! = N/\binom{N}{k}$, we pair the terms indexed by k and $N - k$ for $3 \leq k \leq N - 3$ and treat $k = N - 2$ separately. This gives

$$\frac{N^2}{2} \sum_{k=3}^{N-3} \binom{N}{k}^{-1} + \frac{2N}{\binom{N}{2}}.$$

For $N \geq 8$, the two end terms have denominator $\binom{N}{3}$, while $\binom{N}{k} \geq \binom{N}{4}$ for $4 \leq k \leq N - 4$. Thus the inner sum is at most

$$\frac{2}{\binom{N}{3}} + \frac{N - 7}{\binom{N}{4}} = O(N^{-3}),$$

so the paired sum is $O(N^{-1})$, proving the first assertion. The number of Cayley permutations with at least one such interval is at most the number of pairs, which proves the probability estimate. $\square$

Theorem 5.5. *Let $\tilde{s}_n$ and s_n be the numbers of semisimple and simple Cayley permutations of length n , respectively. Then*

$$\tilde{s}_n \sim \frac{|\text{Cay}_n|}{2\sqrt{2}} \sim \frac{n!}{4\sqrt{2}(\log 2)^{n+1}}, \quad s_n \sim \frac{|\text{Cay}_n|}{4} \sim \frac{n!}{8(\log 2)^{n+1}}.$$

In particular, $s_n/\tilde{s}_n \rightarrow 1/\sqrt{2}$.

Proof. Let q_n count Cayley permutations with no H-interval of length two. Such an interval is an occurrence of 11, 12, or 21, so Proposition 4.5 gives

$$1 + \sum_{n \geq 1} q_n x^n = 1 + B\left(x, 1 - x - \frac{2x}{1+x}\right).$$

The second argument is $1 - 3x + O(x^2)$, so Lemma 5.2 with $c = 0$ and $d = -3$ gives

$$q_n \sim a_n e^{-3\rho/2} = \frac{a_n}{2\sqrt{2}}.$$

Let $\hat{q}_n$ count primitive Cayley permutations with no H-interval of length two. A primitive word has no adjacent equal letters and therefore avoids 11, so Corollary 4.6 gives

$$1 + \sum_{n \geq 1} \hat{q}_n x^n = 1 + B\left(\frac{x}{1+x}, 1 - 2x\right).$$

Here $x/(1+x) = x - x^2 + O(x^3)$, so Lemma 5.2 with $c = -1$ and $d = -2$ yields

$$\hat{q}_n \sim a_n e^{-2\rho} = \frac{a_n}{4}.$$

For $n \geq 3$, a Cayley permutation counted by q_n is nonsemisimple only if it has an H-interval of length between 3 and $n - 1$. The same holds for a primitive Cayley permutation counted by $\hat{q}_n$, since for $n \geq 3$ simplicity is semisimplicity together with primitivity. Hence Lemma 5.4 gives

$$0 \leq q_n - \tilde{s}_n = O(a_n/n), \quad 0 \leq \hat{q}_n - s_n = O(a_n/n).$$

Since $a_n/n = o(a_n)$, these bounds give the stated equivalents for $\tilde{s}_n$ and s_n . Their explicit forms follow from Lemma 5.1. $\square$

6 Restricted growth functions

A nonempty Cayley permutation $\omega = w_1 \cdots w_n$ is a *restricted growth function*, or RGF, if $w_1 = 1$ and $w_{i+1} \leq \max(w_1, \dots, w_i) + 1$ for $1 \leq i < n$. The empty word is also an RGF. Equivalently, its ballot lists the blocks of a set partition in increasing order of their minima. Write RGF_n for the set of RGFs of length n , $\text{RGF} = \bigcup_{n \geq 0} \text{RGF}_n$, and $\text{RGF}_n(T)$ for the members of RGF_n avoiding T .

The singleton-fibre series for RGFs are

$$B^{\text{RGF}}(x, t) = \sum_{n \geq 1} B_n^{\text{RGF}}(t) x^n, \quad B_n^{\text{RGF}}(t) = \sum_{\omega \in \text{RGF}_n} t^{\mu(\omega)},$$

and

$$P^{\text{RGF}}(x, t) = \sum_{n \geq 1} P_n^{\text{RGF}}(t) x^n, \quad P_n^{\text{RGF}}(t) = \sum_{\substack{\omega \in \text{RGF}_n \\ \omega \text{ primitive}}} t^{\mu(\omega)}.$$

Set $B_0^{\text{RGF}}(t) = P_0^{\text{RGF}}(t) = 1$. We recall their basic formulas from *Simple Cayley permutations* [7].

Proposition 6.1 ([7]). *The RGF singleton-fibre series satisfy*

$$1 + \sum_{n \geq 1} B_n^{\text{RGF}}(t) \frac{x^n}{n!} = \exp(tx + e^x - 1 - x)$$

and

$$P^{\text{RGF}}(x, t) = B^{\text{RGF}}\left(\frac{x}{1+x}, (1+x)t - x\right).$$

We first check that occurrences and clusters of RGF patterns remain within the family of RGFs.

Lemma 6.2. *Every pattern occurring in an RGF is an RGF. Conversely, the underlying Cayley permutation of a cluster of RGF patterns is an RGF.*

Proof. A Cayley permutation is an RGF if and only if, for each pair of consecutive values, the first appearance of the smaller value precedes that of the larger one. Let I be an occurrence in an RGF ω . Its value band is an interval, and saturation puts every position carrying a value from that band inside I . Thus $\text{st}(\omega|_I)$ satisfies the same condition and is an RGF.

Conversely, let (ω, M) be a cluster whose marked patterns are RGFs. The value bands $\omega(I)$, for $I \in M$, cover all values of ω . They form a connected family of intervals, since the marks are connected by intersections and intersecting marks have a value in common. Hence, for every $v < \max(\omega)$, some value band contains both v and $v + 1$; otherwise the bands on the two sides of the cut between these values would disconnect the family. In a mark containing both values, the RGF pattern puts the first appearance of v before that of $v + 1$. By saturation, these are also their first appearances in ω . Thus ω is an RGF. $\square$

Thus patterns that are not RGFs may be discarded when studying occurrences in RGFs.

An inflation is an RGF if and only if its quotient and all its components are RGFs [7]. Together with Lemma 6.2, this lets us restrict the cluster formula to RGFs.

Corollary 6.3. *Let T be a finite set of RGF patterns, each of length at least two. Then*

$$\sum_{\omega \in \text{RGF}} \mathbf{u}^{T(\omega)} x^{|\omega|} = \sum_{n \geq 0} B_n^{\text{RGF}}(\hat{C}_T(\mathbf{u}; x)) x^n.$$

In particular,

$$\sum_{n \geq 0} |\text{RGF}_n(T)| x^n = \sum_{n \geq 0} B_n^{\text{RGF}}(\hat{C}_T(\mathbf{0}; x)) x^n.$$

Proof. The RGF property is compatible with inflation, and every T -cluster is an RGF by Lemma 6.2. Corollary 3.3 therefore applies with $\mathcal{R} = \text{RGF}$, for which $B_n^{\mathcal{R}}(t) = B_n^{\text{RGF}}(t)$. $\square$

Since factors of primitive words are primitive, only primitive RGF patterns can occur in a primitive RGF. The cluster formula restricts to this family as well.

Corollary 6.4. *Let T be a finite set of primitive RGF patterns, each of length at least two. Then*

$$\sum_{\substack{\omega \in \text{RGF} \\ \omega \text{ primitive}}} \mathbf{u}^{T(\omega)} x^{|\omega|} = \sum_{n \geq 0} P_n^{\text{RGF}}(\hat{C}_T(\mathbf{u}; x)) x^n.$$

In particular,

$$\sum_{n \geq 0} \#\{\omega \in \text{RGF}_n(T) : \omega \text{ is primitive}\} x^n = \sum_{n \geq 0} P_n^{\text{RGF}}(\hat{C}_T(\mathbf{0}; x)) x^n.$$

Proof. Both properties are compatible with inflation, and every T -cluster is primitive and an RGF. Corollary 3.3 therefore applies with $\mathcal{R}$ equal to the set of primitive RGFs, for which $B_n^{\mathcal{R}}(t) = P_n^{\text{RGF}}(t)$. $\square$

Applying these formulas to the length-two patterns gives the generating functions for RGFs with no H-interval of length two.

Corollary 6.5. *Let q_n^{RGF} and $\hat{q}_n^{\text{RGF}}$ count, respectively, the RGFs and the primitive RGFs of length n with no H-interval of length two. Then*

$$\begin{aligned} \sum_{n \geq 0} q_n^{\text{RGF}} x^n &= \sum_{n \geq 0} B_n^{\text{RGF}} \left(1 - x - \frac{x}{1+x} \right) x^n \\ &= 1 + x + 2x^3 + 8x^4 + 25x^5 + 101x^6 + 454x^7 + 2185x^8 + \dots \end{aligned}$$

and

$$\begin{aligned} \sum_{n \geq 0} \hat{q}_n^{\text{RGF}} x^n &= \sum_{n \geq 0} P_n^{\text{RGF}} \left(\frac{1}{1+x} \right) x^n \\ &= 1 + B^{\text{RGF}} \left(\frac{x}{1+x}, 1-x \right) \\ &= 1 + x + x^3 + 3x^4 + 9x^5 + 33x^6 + 132x^7 + 585x^8 + \dots \end{aligned}$$

Proof. By Lemma 6.2, the patterns of length two that can occur in an RGF are 11 and 12. Proposition 4.5 gives the first substituted argument. A primitive RGF automatically avoids 11, leaving only 12 and the argument $1 - x/(1+x) = 1/(1+x)$. The final equality follows from Proposition 6.1. $\square$

We next estimate how often an RGF has a proper H-interval. Write $b_n = |\text{RGF}_n|$ for the n th Bell number, with $b_0 = 1$. We use two standard properties of these numbers. First, they are log-convex: $b_j^2 \leq b_{j-1}b_{j+1}$ for $j \geq 1$. This follows from Dobiński's formula

$$b_j = \frac{1}{e} \sum_{\ell \geq 0} \frac{\ell^j}{\ell!}$$

and the Cauchy–Schwarz inequality, with $0^0 = 1$. Second, for every fixed $d \geq 0$,

$$\frac{b_{n-d}}{b_n} \sim \left(\frac{\log n}{n} \right)^d. \quad (3)$$

For example, this follows from the consecutive-ratio estimate of Grunwald and Serafin [13].

Theorem 6.6. *Let $\tilde{s}_n^{\text{RGF}}$ and s_n^{RGF} be the numbers of semisimple and simple RGFs of length n , respectively. Then*

$$\frac{\tilde{s}_n^{\text{RGF}}}{|\text{RGF}_n|} = 1 + O\left(\frac{(\log n)^2}{n}\right), \quad \frac{s_n^{\text{RGF}}}{|\text{RGF}_{n-1}|} = 1 + O\left(\frac{(\log n)^2}{n}\right).$$

In particular,

$$\tilde{s}_n^{\text{RGF}} \sim |\text{RGF}_n|, \quad s_n^{\text{RGF}} \sim |\text{RGF}_{n-1}|, \quad \frac{s_n^{\text{RGF}}}{\tilde{s}_n^{\text{RGF}}} \sim \frac{\log n}{n}.$$

Thus almost every RGF is semisimple and almost every primitive RGF is simple, whereas the proportion of all RGFs that are simple tends to zero.

Proof. A positional interval of an RGF is an H-interval precisely when it is a union of blocks [7]. For a fixed interval of length k , we may partition the interval and its complement independently, giving $b_k b_{n-k}$ RGFs in which it is an H-interval. Thus the total number of RGFs with a marked H-interval of length two is $2(n-1)b_{n-2}$. By (3),

$$0 \leq 1 - \frac{q_n^{\text{RGF}}}{b_n} \leq \frac{2(n-1)b_{n-2}}{b_n} = O\left(\frac{(\log n)^2}{n}\right).$$

To pass from length-two avoidance to semisimplicity, we must also count longer proper H-intervals. Their total number is

$$L_n = \sum_{k=3}^{n-1} (n-k+1)b_k b_{n-k}.$$

Put $\eta_n = (\log n)/n$. By (3), the terms with $k \in \{3, n-3, n-2, n-1\}$ contribute $O(b_n(n\eta_n^3 + \eta_n)) = O(b_n\eta_n)$, since $n\eta_n^2 \rightarrow 0$. Log-convexity implies that, for $4 \leq k \leq n-4$,

$$b_k b_{n-k} \leq b_4 b_{n-4}.$$

The remaining terms contribute at most $n^2 b_4 b_{n-4} = O(b_n n^2 \eta_n^4) = o(b_n \eta_n)$, since $n^2 \eta_n^3 \rightarrow 0$. Hence

$$L_n = O\left(b_n \frac{\log n}{n}\right).$$

An RGF counted by q_n^{RGF} is nonsemisimple only if it has one of these longer intervals, so $0 \leq q_n^{\text{RGF}} - \tilde{s}_n^{\text{RGF}} \leq L_n$. Together with the preceding estimate, this proves the first assertion.

For primitive RGFs, we use a bijection with partitions of $[n-1]$. Represent a partition of $[n]$ by arcs (i, j) , with $i < j$, joining successive elements of each block. Each vertex has at most one incoming and one outgoing arc, and conversely any such collection of arcs determines a partition. Primitivity means that no arc is of the form $(i, i+1)$. Replacing each (i, j) by $(i, j-1)$ then gives a partition of $[n-1]$, with inverse $(i, j) \mapsto (i, j+1)$. Thus there are b_{n-1} primitive RGFs of length n .

This bijection also sends each proper H-interval $[a, b]$ of length at least three to a proper H-interval $[a, b-1]$. Indeed, the original interval is a union of blocks, so no arc has exactly one endpoint in it. If a shifted arc had exactly one endpoint in $[a, b-1]$, its original arc would have exactly one endpoint in $[a, b]$. Thus $[a, b-1]$ is a union of blocks, of length between 2 and $n-2$.

It remains to count length-two H-intervals in primitive RGFs. Such an interval consists of two singleton blocks. Assume $n \geq 4$. If the interval is at either end, deleting it leaves a primitive RGF of length $n-2$, with b_{n-3} choices. At each of the $n-3$ internal locations, deletion can create a flat step only where the remaining parts join. There are b_{n-3} choices without a flat step and b_{n-4} with one, counted by contracting that step. In either case, inserting two singleton blocks reverses the deletion. The total number of primitive RGFs with a marked length-two H-interval is therefore

$$(n-1)b_{n-3} + (n-3)b_{n-4}.$$

Consequently,

$$0 \leq 1 - \frac{\hat{q}_n^{\text{RGF}}}{b_{n-1}} \leq \frac{(n-1)b_{n-3} + (n-3)b_{n-4}}{b_{n-1}} = O\left(\frac{(\log n)^2}{n}\right).$$

If a primitive RGF counted by $\hat{q}_n^{\text{RGF}}$ is nonsemisimple, it has an H-interval of length at least three. Its image under the arc bijection is then nonsemisimple, so

$$0 \leq \hat{q}_n^{\text{RGF}} - s_n^{\text{RGF}} \leq b_{n-1} - \tilde{s}_{n-1}^{\text{RGF}} = O\left(b_{n-1} \frac{(\log n)^2}{n}\right).$$

This proves the second assertion, and the ratio follows from (3). $\square$

7 Further directions

Theorem 5.5 gives only the first-order asymptotics of $\tilde{s}_n$ and s_n . For permutations, Albert, Atkinson, and Klazar [2] obtain the asymptotic expansion

$$\frac{n!}{e^2} \left(1 - \frac{4}{n} + \frac{2}{n(n-1)} + O(n^{-3})\right).$$

What are the corresponding asymptotic expansions for $\tilde{s}_n$ and s_n ?

Claesson [10] asks how many Wilf-classes there are among single Hertzprung patterns of a given length. The same question makes sense for Cayley permutations. For $\tau \in \text{Cay}_k$ with $k \geq 2$, the cluster formula gives

$$\sum_{n \geq 0} |\text{Cay}_n(\tau)| x^n = \sum_{n \geq 0} B_n \left(1 - \frac{x^{k-1}}{1 + \Omega(\tau, \tau)} \right) x^n.$$

Thus, patterns of the same length with the same self-overlap polynomial are Wilf-equivalent. Do Cayley permutations introduce any self-overlap polynomials beyond those realized by ordinary permutations? More precisely, does

$$\{\Omega(\tau, \tau) : \tau \in \text{Cay}_k\} = \{\Omega(\pi, \pi) : \pi \in \mathcal{S}_k\}$$

hold for every $k \geq 3$? An exhaustive computation using direct-sum decompositions verifies this equality for $3 \leq k \leq 27$. At length two, the Cayley permutation 11 gives the additional polynomial 0. For $2 \leq k \leq 27$, the computation gives 1,578 self-overlap polynomials in total, counted separately at each length.

We also ask whether the increasing pattern has the most avoiders, as conjectured for permutations by Claesson [10]. Writing $\iota_k = 12 \cdots k$, do we have

$$|\text{Cay}_n(\tau)| \leq |\text{Cay}_n(\iota_k)| \quad \text{for every } \tau \in \text{Cay}_k \text{ and all } k \geq 2 \text{ and } n \geq 0?$$

By applying the cluster formula to the 1,578 self-overlap polynomials from the preceding paragraph, we have verified the inequality for every $\tau \in \text{Cay}_k$ with $2 \leq k \leq 27$ and every $n \leq 1000$.

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References

- [1] M. H. Albert and M. D. Atkinson, *Simple permutations and pattern restricted permutations*, Discrete Math. 300 (2005), pp. 1–15.
- [2] M. H. Albert, M. D. Atkinson, and M. Klazar, *The enumeration of simple permutations*, J. Integer Seq. 6 (2003), Article 03.4.4.
- [3] P. Brändén and A. Claesson, *Mesh patterns and the expansion of permutation statistics as sums of permutation patterns*, Electron. J. Combin. 18(2) (2011), Paper P5.
- [4] R. Brignall, *A survey of simple permutations*, In *Permutation patterns*, S. Linton, N. Ruškuc, and V. Vatter (eds.), London Math. Soc. Lecture Note Ser., vol. 376, Cambridge University Press, Cambridge, 2010, pp. 41–66.

- [5] G. Cerbai, *Sorting Cayley permutations with pattern-avoiding machines*, Australas. J. Combin. 80(3) (2021), pp. 322–341.
- [6] G. Cerbai, *Pattern-avoiding modified ascent sequences*, Electron. J. Combin. 32(3) (2025), Paper P3.6.
- [7] G. Cerbai and A. Claesson, *Simple Cayley permutations*, preprint, 2026, <https://arxiv.org/abs/2609.19078>.
- [8] G. Cerbai and A. Claesson, *Finiteness theorems for Cayley permutation classes*, in preparation.
- [9] G. Cerbai, A. Claesson, D. C. Ernst, and H. Golab, *Pattern-avoiding Cayley permutations via combinatorial species*, Australas. J. Combin. 94 (2026), pp. 267–299.
- [10] A. Claesson, *From Hertzprung’s problem to pattern-rewriting systems*, Algebr. Comb. 5 (2022), no. 6, pp. 1257–1277.
- [11] P. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge University Press, Cambridge, 2009.
- [12] I. P. Goulden and D. M. Jackson, *Combinatorial enumeration*, John Wiley & Sons, New York, 1983.
- [13] J. Grunwald and G. Serafin, *Explicit bounds for Bell numbers and their ratios*, J. Math. Anal. Appl. 549 (2025), no. 2, 129527.
- [14] L. J. Guibas and A. M. Odlyzko, *String overlaps, pattern matching, and non-transitive games*, J. Combin. Theory Ser. A 30(2) (1981), pp. 183–208.
- [15] S. Hertzprung, *En kombinationsopgave*, Tidsskrift for Mathematik 5 (1887), pp. 13–17.
- [16] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, published electronically at <https://oeis.org>.