

Finiteness theorems for Cayley permutation classes

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Abstract

We prove that every hereditary Cayley permutation class with finitely many simple members has a finite basis and an algebraic generating function. We first show that every simple Cayley permutation of length at least two contains a simple pattern obtained by deleting one or two letters, and that the only exceptional Cayley permutations are the simple parallel alternations and their symmetries. We then characterize substitution-closed classes by the simplicity of their basis elements and show that classes with finitely many simple members are partially well ordered. Given the simple members and the basis of such a class, we construct a finite system of functional equations and compute an annihilating polynomial.

Keywords: Cayley permutation, simple permutation, hereditary class, substitution closure, algebraic generating function

1 Introduction

A *Cayley permutation* is either empty or a word of positive integers that contains every number between one and its maximum value. In other words, a Cayley permutation is an endofunction ω on $[n] = \{1, 2, \dots, n\}$ such that $\text{im}(\omega) = [m]$ for some $m \leq n$. We write Cay_n for the set of Cayley permutations of length n and $\text{Cay} = \bigcup_{n \geq 0} \text{Cay}_n$. In addition, we write $\text{Cay}_+ = \bigcup_{n \geq 1} \text{Cay}_n$ for the set of nonempty Cayley permutations.

Let ω be a word. The *standardization* of ω is the word $\text{st}(\omega)$ obtained by replacing the smallest letter of ω by 1, the next smallest by 2, and so on, keeping repetitions equal. Note that $\text{st}(\omega)$ is a Cayley permutation.

An occurrence of the classical pattern $\tau \in \text{Cay}_k$ in $\omega \in \text{Cay}_n$ is a subsequence α of ω such that $\text{st}(\alpha) = \tau$. We write $\tau \leq \omega$ to indicate that τ occurs in ω . This makes Cay into a poset. A (hereditary) *Cayley permutation class* is an order ideal $\mathcal{C}$ of Cay ; that is, $\tau \leq \omega \in \mathcal{C} \Rightarrow \tau \in \mathcal{C}$. We say that ω *avoids* τ if $\tau \not\leq \omega$. Further, for a set $X \subseteq \text{Cay}$, we let $\text{Cay}(X)$ denote the class of Cayley permutations avoiding each member of X . The *basis* of a class $\mathcal{C}$ is the set $\mathcal{B}$ of $\leq$ -minimal Cayley permutations not in $\mathcal{C}$; thus $\mathcal{C} = \text{Cay}(\mathcal{B})$.

In the companion paper [4], we define simple and semisimple Cayley permutations and prove a substitution decomposition analogous to that of (ordinary) permutations.

Here we use this decomposition to study hereditary classes, recalling the necessary definitions and results from that paper as they are needed. First, we highlight the relevant results from the literature on permutations.

A permutation is *simple* [1] if it has no proper interval of length at least two, where an interval is a set of consecutive positions whose corresponding values also form a contiguous set of integers. A special case of a general theorem by Schmerl and Trotter [10] on indecomposable relational structures says that every simple permutation of length $n \geq 2$ contains a simple permutation of length $n - 1$ or $n - 2$. In other words, if a permutation is simple then one may obtain a new simple permutation by deleting at most two points. Permutations where a one-point deletion is not sufficient are called *exceptional*. Schmerl and Trotter have classified the critically indecomposable relational structures up to isomorphism of their skeletons. And by reading the classification in terms of permutations, Albert and Atkinson [1] showed that the only exceptional permutations are the simple parallel alternations, i.e. those of the form

$$246 \cdots (2m)135 \cdots (2m-1), \quad m \geq 2,$$

and their symmetries. They also showed that a (hereditary) permutation class containing only finitely many simple permutations is partially well ordered, finitely based, and has a readily computable algebraic generating function.

The main goal of this paper is to establish analogous results for Cayley permutations. In Section 2, we prove a deletion theorem and classify the exceptional Cayley permutations. Section 3 is devoted to substitution-closed classes. We study closure under inflation and the resulting generating functions, with and without the additional restriction that components at repeated quotient values consist of one letter. In Section 4 we remove the hypothesis of substitution closure and study general classes with finitely many simples. We prove that every class with finitely many simple members is partially well ordered (Theorem 4.2), has a finite basis (Theorem 4.3), and has an algebraic generating function (Theorem 4.5). The corresponding assertions for finitely many semisimple members follow as special cases.

2 Exceptional Cayley permutations

Recall that every simple permutation of length $n \geq 2$ contains a simple permutation of length $n - 1$ or $n - 2$, by Schmerl and Trotter [10]. Those for which no one-point deletion is simple are called exceptional. Albert and Atkinson [1] showed that they are precisely the simple parallel alternations and their symmetries.

We wish to prove that both statements carry over to Cayley permutations. First, we recall some definitions from the companion paper [4].

Let ω be a word. We say that β is a *factor* of ω if there are words α and γ such that $\omega = \alpha\beta\gamma$. Let $\omega = w_1 \cdots w_n \in \text{Cay}_n$. If $I = [a, b] \subseteq [n]$ is a nonempty positional interval, write

$$\omega|_I = w_a w_{a+1} \cdots w_b \quad \text{and} \quad \omega(I) = \{w_a, w_{a+1}, \dots, w_b\}$$

for the factor of ω at positions I and for its set of values. More generally, if $S = \{i_1 < \cdots < i_r\} \subseteq [n]$, write $\omega|_S = w_{i_1} \cdots w_{i_r}$ for the corresponding subword and

$\omega(S)$ for its values. We say that I is a *Hertzsprung interval*, or *H-interval*, of ω if there is a value interval $J = [c, d]$ such that

$$\omega(I) = J \quad \text{and} \quad \omega^{-1}(J) = I. \quad (1)$$

A nonempty H-interval occupies consecutive positions and uses every value in a consecutive range, which we call a *value band*. Additionally, it contains every copy in ω of every value in J due to the second equality, the *saturation condition*. Note that $[n]$ is always an H-interval, while a singleton $[i, i] = \{i\}$ is an H-interval precisely when the value $\omega(i)$ occurs nowhere else in ω . We also regard the empty set as an H-interval of every Cayley permutation, with empty value band.

Definition 2.1. A Cayley permutation $\omega \in \text{Cay}_n$ is *semisimple* if every H-interval of ω is empty, a singleton, or $[n]$; these H-intervals are called *trivial*. The word is *primitive* [5] if adjacent letters are distinct. It is *simple* if it is semisimple and primitive, or if $\omega = 11$.

The empty word is semisimple and primitive, hence simple. Both semisimplicity and simplicity restrict to the usual definition of simplicity for permutations. Thus 1, 11, 12, and 21 are simple, whereas 111 is semisimple but not simple. A pair of equal adjacent letters is called a *flat step*.

To start, we take care of semisimple Cayley permutations that are not primitive by showing that removing a flat step from a semisimple Cayley permutation preserves semisimplicity. Given $\omega = w_1 \cdots w_n \in \text{Cay}_n$ and a position $i \in [n]$, let

$$\omega \setminus i = \text{st}(w_1 \cdots w_{i-1} w_{i+1} \cdots w_n)$$

denote the one-point deletion at i .

Proposition 2.2. Let $\omega = w_1 \cdots w_n \in \text{Cay}_n$ be semisimple with $n \geq 2$, and suppose $w_i = w_{i+1}$. Then $\omega \setminus (i + 1)$ is a semisimple Cayley permutation of length $n - 1$.

Proof. Let $\omega' = w_1 \cdots w_i w_{i+2} \cdots w_n$ and consider the order-preserving surjection

$$g: [n] \rightarrow [n - 1], \quad g(j) = \begin{cases} j & \text{if } j \leq i, \\ j - 1 & \text{if } j > i. \end{cases}$$

Since $w_i = w_{i+1}$, we have $\omega(j) = \omega'(g(j))$ for every j . Thus ω and ω' use the same values, so ω' is already a Cayley permutation and $\omega \setminus (i + 1) = \omega'$.

Let I' be a nonempty H-interval of ω' with value band J . Its preimage $I = g^{-1}(I')$ is a nonempty positional interval, since g is order-preserving and surjective. Moreover,

$$\omega(I) = \omega'(I') = J, \quad \omega^{-1}(J) = g^{-1}((\omega')^{-1}(J)) = I.$$

Hence I is an H-interval of ω , so semisimplicity makes it a singleton or $[n]$. Taking its image under g makes $I' = g(I)$ a singleton or $[n - 1]$, proving that ω' is semisimple. $\square$

A *relational structure* is a set together with a collection of binary relations on it. Schmerl and Trotter [10] defined *indecomposable* relational structures and proved that a relational structure is indecomposable if and only if it contains no nontrivial *module*. Here, a module is a subset M of the structure such that, for every relation R , every

$i, j \in M$, and every $x \notin M$, we have $iRx \iff jRx$ and $xRi \iff xRj$. A module is nontrivial if it has at least two elements and is a proper subset of the ground set. The empty structure is allowed, and it is indecomposable since it has no nontrivial module.

Given a Cayley permutation $\omega \in \text{Cay}_n$, define the relational structure

$$R(\omega) = ([n], <_1, <_2)$$

on the set of positions $[n]$, carrying two binary relations: the position order $i <_1 j$ if $i < j$, and the value order $i <_2 j$ if $\omega(i) < \omega(j)$. The position order is a linear order, while the value order is a strict weak order whose ties are the positions of equal value. Both are irreflexive.

Lemma 2.3 ([4]). *The modules of $R(\omega)$ are the H -intervals of ω together with its nonempty positional intervals with constant value.*

Theorem 2.4 ([4]). *A Cayley permutation ω is simple if and only if the relational structure $R(\omega)$ is indecomposable.*

Theorem 2.4 lets us use the aforementioned result of Schmerl and Trotter.

Theorem 2.5. *Every simple Cayley permutation of length $n \geq 2$ contains a simple Cayley permutation of length $n - 1$ or $n - 2$, obtained by deleting one or two letters and standardizing.*

Proof. Let $\omega \in \text{Cay}_n$ be simple. For $n = 2$, either deletion leaves 1; for $n = 3, 4$, keeping any two positions and standardizing leaves a Cayley permutation of length two, which is simple. We may therefore assume that $n \geq 5$. By Theorem 2.4, $R(\omega)$ is indecomposable. Since its relations are binary and irreflexive, it follows by Schmerl and Trotter [10, Corollary 2.3] that $R(\omega)$ has an indecomposable induced substructure on $n - 1$ or $n - 2$ points. Standardization preserves both the position order and the value order, so the substructure of $R(\omega)$ induced on a set S of positions is isomorphic to $R(\text{st}(\omega|_S))$; take S with $R(\text{st}(\omega|_S))$ indecomposable and $|S| \in \{n - 1, n - 2\}$. Since $|S| \geq 3$, Theorem 2.4 makes $\text{st}(\omega|_S)$ simple. $\square$

Corollary 2.6. *Every semisimple Cayley permutation of length $n \geq 2$ has a semisimple pattern of length $n - 1$ or $n - 2$.*

Proof. If there is a flat step, apply Proposition 2.2. Otherwise the word is simple, and Theorem 2.5 applies. $\square$

For $n \geq 2$, we call a simple Cayley permutation $\omega \in \text{Cay}_n$ *exceptional* if $\omega \setminus i$ is not simple for each $i \in [n]$. An exceptional word has length at least four, since every word of length at most two is simple. In particular, it is primitive.

Proving that the parallel alternations are the only exceptional Cayley permutations requires a bit more work. Following Schmerl and Trotter [10], we say that two ordered pairs (a, b) and (c, d) of distinct elements of a relational structure have the same *type* if $aR_i b \iff cR_i d$ and $bR_i a \iff dR_i c$ for every relation R_i . The equivalence classes of ordered pairs induced by the type relation form the *type skeleton*.

As an example, consider the Cayley permutation $\omega = 231146561$. In the type skeleton of $R(\omega)$, the type of the arc $i \rightarrow j$ records the comparison of the positions i, j and the comparison of the values $\omega(i), \omega(j)$. There are six possible types:

$$\begin{aligned} A &= (<, <), & B &= (<, =), & C &= (<, >), \\ D &= (>, <), & E &= (>, =), & F &= (>, >), \end{aligned}$$

where the first sign compares i with j and the second compares $\omega(i)$ with $\omega(j)$. The following matrix displays the type skeleton of $R(\omega)$. Its rows and columns are labelled $i_{\omega(i)}$, and the entry in row $i_{\omega(i)}$ and column $j_{\omega(j)}$ is the type of the ordered pair (i, j) :

	1_2	2_3	3_1	4_1	5_4	6_6	7_5	8_6	9_1
1_2	—	A	C	C	A	A	A	A	C
2_3	F	—	C	C	A	A	A	A	C
3_1	D	D	—	B	A	A	A	A	B
4_1	D	D	E	—	A	A	A	A	B
5_4	F	F	F	F	—	A	A	A	C
6_6	F	F	F	F	F	—	C	B	C
7_5	F	F	F	F	F	D	—	A	C
8_6	F	F	F	F	F	E	F	—	C
9_1	D	D	E	E	D	D	D	D	—

For instance, the type of $(1, 2)$ is $A = (<, <)$ since we have positions $1 < 2$ and values $\omega(1) < \omega(2)$. In the matrix, two ordered pairs have the same type exactly when their cells carry the same letter, and the type skeleton is the partition of the off-diagonal cells into the six letter classes. Note also that the matrix makes the modules transparent: a set is a module when its rows agree outside its own columns. For instance, the set $\{1, 2\}$ is a module since rows $1_2, 2_3$ agree outside columns $1_2, 2_3$; it is also an H-interval, occupying consecutive positions and the saturated value band $\{2, 3\}$. Similarly, $[5, 8]$ is a module as well as an H-interval with saturated value band $\{4, 5, 6\}$. These are the maximal proper H-intervals. The set $\{3, 4\}$ is another module: it is a constant positional interval, but not an H-interval since the value 1 occurs again at position 9 (hence saturation fails).

Now, in the language of Schmerl and Trotter, an indecomposable structure is *critically indecomposable* if no one-point deletion of the structure is indecomposable. The two authors [10, Theorem 5.1] have fully classified the type skeletons of critically indecomposable structures. We will only need to count their types to show that no Cayley permutations except for the parallel alternations are exceptional.

Lemma 2.7. *A critically indecomposable type skeleton has at most four types.*

Proof. The type skeleton of a critically indecomposable structure is, for some $r \geq 2$, the type skeleton of one of the nine structures listed by Schmerl and Trotter [10, Section 4]:

$$\mathcal{G}_r, \mathcal{P}_r, \mathcal{P}'_r, \mathcal{B}_r, \mathcal{T}_r^{(1)}, \mathcal{T}_r^{(2)}, \mathcal{T}_r^{(3)}, \mathcal{D}_r, \mathcal{D}'_r.$$

The graph $\mathcal{G}_r$ carries one symmetric relation, so a pair is adjacent or not: two types. The tournaments $\mathcal{T}_r^{(i)}$ carry one relation holding in exactly one direction on each pair: two types. The posets $\mathcal{P}_r$ and $\mathcal{P}'_r$ and the oriented graph $\mathcal{D}_r$ carry one antisymmetric

relation, so a pair is related one way, the other way, or not at all: three types. In $\mathcal{B}_r = (V_r; P_r, P'_r)$ the two orders divide the pairs between them, P_r taking the (a_i, b_j) with $i \geq j$ and P'_r the remaining (a_i, b_j) together with the (a_i, a_j) and the (b_i, b_j) ; each pair is therefore related by exactly one of the two, in exactly one direction: four types. The same holds in $\mathcal{D}'_r = (T_r^{(2)}; F_r, F'_r)$, where F_r and F'_r partition the arcs of the tournament $\mathcal{T}_r^{(2)}$. An isomorphism of type skeletons preserves the number of types, so the bound passes to every critically indecomposable type skeleton. $\square$

Lemma 2.8. *Let ω be a primitive Cayley permutation with a repeated value. Then $R(\omega)$ has six types.*

Proof. For $i < j$, the type of (i, j) is determined by whether $\omega(i) < \omega(j)$, $\omega(i) = \omega(j)$, or $\omega(i) > \omega(j)$; reversing the ordered pair gives three corresponding types with the opposite position order. All three value comparisons occur. A repeated value supplies the equality case. If no pair satisfied $\omega(i) < \omega(j)$, then ω would be weakly decreasing and, by primitivity, strictly decreasing, contrary to the repeated value. The same argument with the order reversed supplies a pair with $\omega(i) > \omega(j)$. Thus all six types occur, and they are distinct. $\square$

Theorem 2.9. *The exceptional Cayley permutations are exactly the simple parallel alternations and their symmetries.*

Proof. Let $\omega \in \text{Cay}_n$ be exceptional. Since every Cayley permutation of length at most two is simple, every one-point deletion of a word of length two or three is simple. Thus $n \geq 4$, and ω is primitive by definition. Suppose ω had a repeated value. By Theorem 2.4, $R(\omega)$ is indecomposable, and it is critically indecomposable. Indeed, for each $j \in [n]$, its one-point deletion at j is isomorphic to $R(\omega \setminus j)$, which has $n - 1 \geq 3$ points. By Theorem 2.4, this structure is indecomposable only if $\omega \setminus j$ is simple, and no such deletion is simple. Lemma 2.7 then allows $R(\omega)$ at most four types, while Lemma 2.8 gives it six. So ω has no repeated value. Being a permutation, ω is exceptional in the classical sense, hence a simple parallel alternation or one of its symmetries. Conversely, the simple parallel alternations and their symmetries are permutations with no simple one-point deletion, and so are exceptional here. $\square$

Corollary 2.10. *A simple Cayley permutation with a repeated value has a simple one-point deletion. The exceptional semisimple Cayley permutations of length at least two, defined by the absence of a semisimple one-point deletion, are also exactly the simple parallel alternations and their symmetries.*

Proof. The first assertion follows from Theorem 2.9. For the second, Proposition 2.2 excludes every semisimple word with a flat step. Any remaining exceptional semisimple word is simple and has no simple one-point deletion, so Theorem 2.9 applies. For permutations the two notions of simplicity agree. $\square$

Two further consequences of Schmerl and Trotter's results [10, Theorems 2.2 and 5.9] are useful when retaining a specified pattern during deletion or extension.

Corollary 2.11. *Every simple Cayley permutation of length $n \geq 6$ has a simple pattern of length $n-2$. Moreover, if a set of k positions in a simple Cayley permutation of length n induces a simple pattern, where $3 \leq k \leq n-2$, then it can be extended by two positions to induce a simple pattern of length $k+2$.*

Proof. Let ω be a simple Cayley permutation. For the first assertion with $n \geq 7$ and for the second assertion, apply the cited results to $R(\omega)$ and use Theorem 2.4. Induced substructures correspond to standardized subwords, as in the proof of Theorem 2.5. If $n = 6$, then Theorem 2.5 gives a simple pattern of length four or five. In the latter case, the five-letter pattern is not exceptional by Theorem 2.9, since every parallel alternation has even length. It therefore has a simple one-point deletion of length four, which is also a pattern of ω . $\square$

Since the simple permutation 24153 has no simple pattern of length three the first assertion of the above corollary fails for $n = 5$.

3 Substitution-closed classes with finitely many simples

Theorem 2.5 and Corollary 2.6 supply finite certificates for the finiteness of the sets of simple and semisimple members.

Corollary 3.1. *Let $\mathcal{C}$ be a class of Cayley permutations and $n \geq 1$. If $\mathcal{C}$ has no simple member of length n or $n+1$, then it has no simple member of length greater than n . The same assertion holds with semisimple in place of simple.*

Proof. Suppose, to the contrary, that $\mathcal{C}$ has a simple member of length greater than n , and choose one, ω , of least length ℓ . By hypothesis, $\ell \geq n+2$. By Theorem 2.5 there is a simple pattern $\nu \leq \omega$ of length $\ell-1$ or $\ell-2$. Note that $\nu \in \mathcal{C}$, because $\mathcal{C}$ is a class. Thus $n \leq |\nu| < \ell$. If $|\nu|$ is n or $n+1$, this contradicts the hypothesis. If $|\nu| > n+1$, it contradicts the minimality of ℓ . For semisimple members, the same argument uses Corollary 2.6. $\square$

A class therefore has finitely many simple members if and only if two consecutive lengths lack them. If membership in $\mathcal{C}$ is decidable, for example from a finite basis, we can find all its simple members by searching successive lengths and stopping when two consecutive lengths have none. The search terminates whenever the set of simple members is finite; it does not decide whether that set is finite. The same holds for semisimple members.

We next recall inflation and the two substitution decompositions from the companion paper [4]. The inflation of Cayley permutations is defined analogously to how it is defined for permutations, but only letters appearing exactly once may be inflated by nonconstant words. Otherwise, positions carrying the same value could be expanded into competing value bands, making the process ambiguous. Formally, given $\sigma \in \text{Cay}_k$ we associate a nonempty Cayley permutation α_i with each position i , subject to the restriction

$$|\sigma^{-1}(\sigma(i))| > 1 \implies \alpha_i = 1^{r_i} \text{ for some } r_i \geq 1. \quad (2)$$

The *inflation* $\sigma[\alpha_1, \dots, \alpha_k]$ is the Cayley permutation obtained by replacing each entry $\sigma(i)$ with a factor order isomorphic to α_i , while preserving the relative order of the entries of σ . A singleton quotient fibre is replaced by a consecutive value band; a repeated fibre is replaced by a single value. The positional interval occupied by the entries corresponding to α_i is called the *i*th *part* of the inflation.

The decomposition into H-intervals must satisfy the stronger restriction

$$|\sigma^{-1}(\sigma(i))| > 1 \implies \alpha_i = 1. \quad (3)$$

In this case, every nonsingleton part is an H-interval. The inflations

$$1121 = 121[11, 1, 1], \quad 111 = 11[1, 11]$$

satisfy (2), but not (3).

An *H-interval decomposition* of ω is a partition of its positions into consecutive intervals, each of which is an H-interval or a single position. Contracting its parts and standardizing gives a Cayley permutation, called the *quotient*.

Direct and skew sums are defined by

$$\alpha \oplus \beta = 12[\alpha, \beta] \quad \text{and} \quad \alpha \ominus \beta = 21[\alpha, \beta].$$

A Cayley permutation that cannot be expressed as a direct sum of shorter Cayley permutations is *plus indecomposable*. Analogously, it is *minus indecomposable* if it cannot be expressed as a skew sum of shorter Cayley permutations.

Theorem 3.2 ([4]). *Let ω be a Cayley permutation of length at least two. Then there is a unique semisimple Cayley permutation σ of length $k \geq 2$ such that*

$$\omega = \sigma[\alpha_1, \dots, \alpha_k],$$

for some nonempty Cayley permutations $\alpha_1, \dots, \alpha_k$ satisfying (3). If the quotient $\sigma \notin \{12, 21\}$, then the components α_i are uniquely determined. If $\sigma = 12$, they are unique if we require α_1 to be plus indecomposable. If $\sigma = 21$, they are unique if we require α_1 to be minus indecomposable.

Theorem 3.3 ([4]). *Every Cayley permutation ω of length at least two has a unique simple quotient σ of length $k \geq 2$ such that*

$$\omega = \sigma[\alpha_1, \dots, \alpha_k]$$

for nonempty components satisfying (2). If $\sigma \notin \{11, 12, 21\}$, the components are unique. For $\sigma = 11$ they are unique on requiring $\alpha_1 = 1$; for $\sigma = 12$ or 21 they are unique on requiring α_1 to be plus or minus indecomposable, respectively.

For $\omega = 231146561$ from Section 2, the H-interval decomposition is

$$23 \mid 1 \mid 1 \mid 4656 \mid 1,$$

which by contraction gives the semisimple quotient 21131. Conversely, ω is obtained as the inflation

$$231146561 = 21131[12, 1, 1, 1323, 1].$$

Contracting the constant run in this quotient instead gives the simple quotient 2131 and the inflation

$$231146561 = 2131[12, 11, 1323, 1].$$

A class is *substitution-closed* if $\sigma[\alpha_1, \dots, \alpha_k]$ lies in the class whenever σ and the α_i do. We also consider closure when only inflations satisfying (3) are required to belong to the class. Write $\langle \mathcal{C} \rangle$ for the substitution closure of $\mathcal{C}$, and $\langle \mathcal{C} \rangle_{\text{H}}$ for its closure under this additional restriction. Thus,

$$\mathcal{C} \subseteq \langle \mathcal{C} \rangle_{\text{H}} \subseteq \langle \mathcal{C} \rangle.$$

These closures can differ. For instance, $\text{Cay}(111)$ is closed under the restricted condition, whereas $11[11, 1] = 111$ prevents substitution closure. That is,

$$\text{Cay}(111) = \langle \text{Cay}(111) \rangle_{\text{H}} \subset \langle \text{Cay}(111) \rangle.$$

Albert and Atkinson's [1] characterization of substitution-closed permutation classes, also stated in Brignall's [2] survey, extends to both closure conditions. We first describe how a classical pattern can occur in an inflation.

Lemma 3.4. *Let $\omega = \sigma[\alpha_1, \dots, \alpha_k]$ be an inflation and β a nonempty Cayley permutation. Then $\beta \leq \omega$ if and only if*

$$\beta = \text{st}(\sigma|_T)[\gamma_i : i \in T]$$

for some $\emptyset \neq T \subseteq [k]$ and nonempty $\gamma_i \leq \alpha_i$. At repeated quotient letters the γ_i are constant; if the original inflation satisfies (3), they are 1.

Proof. Choose positions S witnessing an occurrence of β in ω , and let S_i be those in the i th part of the inflation. Put $T = \{i : S_i \neq \emptyset\}$ and $\gamma_i = \text{st}(\omega|_{S_i})$ for $i \in T$. Then $\gamma_i \leq \alpha_i$, and γ_i is constant whenever $\sigma(i)$ is repeated, since the corresponding part is constant. Under (3), that part is a single position and $\gamma_i = 1$. Here T is the set of indices whose corresponding part in the inflation contains at least one entry in the selected occurrence of β ; and γ_i is the standardization of all the entries in S_i taking part in such occurrence. Now, restricting σ to T and inflating $\sigma|_T$ by the γ_i gives the same result as restricting ω to S . Indeed, within a part the selected letters have relative order γ_i . Between parts with distinct values of σ , they compare as those values do. Parts with equal values of σ are constant factors carrying equal letters. Hence

$$\beta = \text{st}(\omega|_S) = \text{st}(\sigma|_T)[\gamma_i : i \in T].$$

This is a valid inflation. Indeed, every repeated letter of $\text{st}(\sigma|_T)$ comes from a repeated letter of σ and is therefore inflated by a constant word.

Conversely, choose an occurrence of each γ_i in the i th part. Their union has the displayed standardization, by the same comparison of values, and thus witnesses an occurrence of β in ω . $\square$

We shall compare a class with its substitution closure, but first we need a lemma.

Lemma 3.5. *Let $\omega = \sigma[\alpha_1, \dots, \alpha_k]$ be an inflation and β a simple Cayley permutation with $\beta \leq \omega$. Then $\beta \leq \sigma$ or $\beta \leq \alpha_i$ for some i . If the inflation satisfies (3), the conclusion holds for every semisimple β .*

Proof. The claim is immediate if β is empty. Otherwise, write $\beta = \text{st}(\sigma|_T)[\gamma_i : i \in T]$ as in Lemma 3.4. If $T = \{i\}$ then $\beta = \gamma_i \leq \alpha_i$ and we are done. Suppose instead that $|T| \geq 2$. A component γ_i of length greater than one would occupy a nontrivial module of $R(\beta)$, which simplicity forbids; so $\gamma_i = 1$ for each i and $\beta = \text{st}(\sigma|_T) \leq \sigma$. Under (3), such a component occupies a proper H-interval of β with more than one position, so semisimplicity suffices to exclude it. $\square$

Proposition 3.6. *Let $\mathcal{C}$ be a class of Cayley permutations.*

1. $\mathcal{C} = \langle \mathcal{C} \rangle$ if and only if each of its basis elements is simple.
2. $\mathcal{C} = \langle \mathcal{C} \rangle_{\text{H}}$ if and only if each basis element is semisimple.

Proof. Let $\mathcal{B}$ be the basis of $\mathcal{C}$. Suppose first that $\mathcal{C}$ is substitution-closed and some $\beta \in \mathcal{B}$ is not simple. By Theorem 3.3, $\beta = \sigma[\alpha_1, \dots, \alpha_k]$ with σ simple of length $2 \leq k < |\beta|$ and the components obeying (2). The quotient and components are proper patterns of β , so they all lie in $\mathcal{C}$ by the minimality of a basis element. Substitution closure then gives $\beta \in \mathcal{C}$, a contradiction.

Conversely, suppose every $\beta \in \mathcal{B}$ is simple, and let $\omega = \sigma[\alpha_1, \dots, \alpha_k]$ with $\sigma, \alpha_i \in \mathcal{C}$. If $\omega \notin \mathcal{C}$ then ω contains some $\beta \in \mathcal{B}$, necessarily simple; by Lemma 3.5, $\beta \leq \alpha_i$ for some i or $\beta \leq \sigma$. Either possibility contradicts $\sigma, \alpha_i \in \mathcal{C}$. Hence $\omega \in \mathcal{C}$.

For closure under (3), the same argument uses semisimple quotients, Theorem 3.2, and the semisimple assertion of Lemma 3.5. $\square$

Lemma 3.7. *Let $\mathcal{C}$ be a class of Cayley permutations.*

1. $\mathcal{C}$ and $\langle \mathcal{C} \rangle$ have the same simple members.
2. $\mathcal{C}$ and $\langle \mathcal{C} \rangle_{\text{H}}$ have the same semisimple members.

Proof. Let $\mathcal{D}$ be the set of all Cayley permutations ω such that every simple Cayley permutation $\sigma \leq \omega$ belongs to $\mathcal{C}$. It is a class containing $\mathcal{C}$, and Lemma 3.5 shows that it is substitution-closed. Thus $\mathcal{C} \subseteq \langle \mathcal{C} \rangle \subseteq \mathcal{D}$. If σ is a simple member of $\langle \mathcal{C} \rangle$, then $\sigma \in \mathcal{D}$ and $\sigma \leq \sigma$, so the defining condition of $\mathcal{D}$ gives $\sigma \in \mathcal{C}$. Conversely, every simple member of $\mathcal{C}$ belongs to $\langle \mathcal{C} \rangle$. Hence the two classes have the same simple members. The same argument applies to semisimple members and $\langle \mathcal{C} \rangle_{\text{H}}$, using the semisimple assertion of Lemma 3.5. $\square$

Corollary 3.8. *If the longest simple members of a class $\mathcal{C}$ have length k , then the basis elements of $\langle \mathcal{C} \rangle$ have length at most $k + 2$. If the longest semisimple members of $\mathcal{C}$ have length k , the same bound holds for the basis elements of $\langle \mathcal{C} \rangle_{\text{H}}$.*

Proof. Let β be a basis element of $\langle \mathcal{C} \rangle$, of length ℓ . As $\langle \mathcal{C} \rangle$ is substitution-closed, Proposition 3.6 makes β simple. Suppose $\ell \geq k + 3$. By Theorem 2.5, β properly contains a simple Cayley permutation σ of length $\ell - 1$ or $\ell - 2$, hence of length greater than k ; so $\sigma \notin \mathcal{C}$ and, by Lemma 3.7, $\sigma \notin \langle \mathcal{C} \rangle$. But every proper pattern of a basis element lies in the class. Hence $\ell \leq k + 2$. For semisimple members and $\langle \mathcal{C} \rangle_{\text{H}}$, use the same argument with Corollary 2.6 in place of Theorem 2.5. $\square$

Given $\omega \in \text{Cay}$, let

$$\mu(\omega) = \#\{j : |\omega^{-1}(j)| = 1\}$$

denote the number of singleton fibres of ω .

Proposition 3.9. *Let $\mathcal{C}$ be a class closed under inflations satisfying (3), containing at least one nonempty word, with only finitely many semisimple members. Let F be the ordinary generating function for its nonempty members, and put*

$$\tilde{\Phi}_{\mathcal{C}}(x, y) = \sum_{\substack{\sigma \in \mathcal{C} \text{ semisimple} \\ |\sigma| \geq 3}} x^{|\sigma| - \mu(\sigma)} y^{\mu(\sigma)}.$$

For a word ω , let δ_{ω} be 1 if $\omega \in \mathcal{C}$ and 0 otherwise. Then

$$\tilde{\Phi}_{\mathcal{C}}(x, F) = F - x - \delta_{11}x^2 - \frac{(\delta_{12} + \delta_{21})F^2}{1 + F}. \quad (4)$$

In particular, the ordinary generating function of $\mathcal{C}$ is algebraic over $\mathbb{Q}(x)$.

Proof. Since $\mathcal{C}$ contains a nonempty word and is closed under classical pattern containment, it contains 1. For an inflation satisfying (3), $\omega = \sigma[\alpha_1, \dots, \alpha_k]$, the quotient σ and each component α_i are classical patterns of ω . Thus, if $\omega \in \mathcal{C}$, then $\sigma, \alpha_1, \dots, \alpha_k \in \mathcal{C}$. Conversely, if the quotient and all components belong to $\mathcal{C}$, the closure condition gives $\omega \in \mathcal{C}$. Lemma 4.3 of the companion paper [4], applied to membership in $\mathcal{C}$ with the singleton-fibre weight set to 1, now gives (4).

To prove algebraicity, note that $\tilde{\Phi}_{\mathcal{C}}$ is a polynomial because $\mathcal{C}$ has only finitely many semisimple members. Multiplying (4) by $1 + F$ and rearranging gives $P(x, F) = 0$, where

$$P(x, y) = (1 + y)(y - x - \delta_{11}x^2 - \tilde{\Phi}_{\mathcal{C}}(x, y)) - (\delta_{12} + \delta_{21})y^2 \in \mathbb{Q}[x, y].$$

Every monomial of $\tilde{\Phi}_{\mathcal{C}}$ has total degree at least three, so the coefficient of x^0y in P is 1. In particular, P is a nonzero polynomial, and $P(x, F) = 0$ proves that F is algebraic over $\mathbb{Q}(x)$. Including the empty word gives the class generating function $1 + F$, which is therefore algebraic as well. $\square$

Proposition 3.10. *Let $\mathcal{C}$ be substitution-closed, contain a nonempty word, and have finitely many simple members. Let F count its nonempty members and put*

$$\Phi_{\mathcal{C}}(x, y) = \sum_{\substack{\sigma \in \mathcal{C} \text{ simple} \\ |\sigma| \geq 3}} x^{|\sigma| - \mu(\sigma)} y^{\mu(\sigma)}.$$

With the indicators δ_{ω} as before,

$$F = x + \delta_{11} \frac{x^2}{1 - x} + \frac{(\delta_{12} + \delta_{21})F^2}{1 + F} + \Phi_{\mathcal{C}}\left(\frac{x}{1 - x}, F\right).$$

In particular, the generating function of $\mathcal{C}$ is algebraic.

Proof. Use the unique decomposition in Theorem 3.3. If $11 \in \mathcal{C}$, closure gives all nonempty constant words; otherwise only the constant word 1 belongs to $\mathcal{C}$. This yields the first two terms. The sum contributions are as in Proposition 3.9. A repeated position of a longer simple quotient may carry any nonempty constant run. Such a quotient contains 11, so all these runs belong to $\mathcal{C}$, and its contribution is obtained by substituting $x/(1-x)$ at repeated positions and F at singleton positions. Finally, $\Phi_{\mathcal{C}}$ is a polynomial. Clearing denominators gives a nonzero polynomial equation for F ; the corresponding equation has coefficient 1 on its linear term in F at the origin. $\square$

For $\mathcal{C} = \text{Cay}(21)$, i.e. the set of weakly increasing Cayley permutations, the polynomial $\Phi_{\mathcal{C}}$ is zero and the equation in Proposition 3.10 reduces to $F = x/(1-x) + F^2/(1+F)$, giving $F = x/(1-2x)$. The same result is obtained by Proposition 3.9, where $\tilde{\Phi}_{\mathcal{C}}(x, y) = x^3/(1-x)$ since the only semisimple members of $\mathcal{C}$ are the constant words.

Example 3.11. For $k \geq 3$, let $\mathcal{C}_k = \text{Cay}(1^k, 21)$. Its members are precisely

$$1^{a_1} 2^{a_2} \dots m^{a_m}, \quad 1 \leq a_i < k.$$

Every constant word 1^{a_1} on the displayed list is semisimple, as is 12. Any other member with a run of length at least two has that run as a proper H-interval, while one with only singleton runs and length at least three begins with the proper H-interval 12. Therefore, the nonempty semisimple members of $\mathcal{C}_k$ are

$$1, 11, \dots, 1^{k-1}, \quad \text{and} \quad 12.$$

Now, in a nonempty Cayley permutation of $\mathcal{C}_k$ the first value occurs between one and $k-1$ times, after which the remaining suffix, standardized, is again in $\mathcal{C}_k$. Thus, writing G for the ordinary generating function of $\mathcal{C}_k$, we have

$$G(x) = 1 + (x + x^2 + \dots + x^{k-1})G(x),$$

and hence

$$G(x) = \frac{1}{1 - x - x^2 - \dots - x^{k-1}}.$$

In particular, $\text{Cay}(111, 21)$ has generating function $1/(1 - x - x^2)$.

We obtain the same answer from Proposition 3.9. The basis elements 1^k and 21 are semisimple, so $\mathcal{C}_k$ is closed under inflations satisfying (3). Here $\delta_{11} = \delta_{12} = 1$, $\delta_{21} = 0$, and $\tilde{\Phi}_{\mathcal{C}_k} = \sum_{j=3}^{k-1} x^j$, since the constant quotients have no singleton fibres. Thus (4) simplifies to $F = (x + x^2 + \dots + x^{k-1})(1 + F)$, as required.

As this example illustrates, semisimple quotients are particularly convenient for multiplicity bounds. Each value occurs at most r times in an inflation exactly when this holds for the quotient and every component. Note that the class $\mathcal{C}_k$ is not substitution-closed, since $11[1, 1^{k-1}] = 1^k$, although the quotient and both components belong to $\mathcal{C}_k$. Thus, Proposition 3.9 applies, whereas Proposition 3.10 does not.

Example 3.12. Let $\mathcal{C}$ be the class avoiding the classical patterns

$$111, \quad 121, \quad 212, \quad 231.$$

The semisimple members of lengths one and two are 1, 11, 12, 21. Direct enumeration finds no semisimple members of length three or four, so Corollary 3.1 shows that these are all the nonempty semisimple members of $\mathcal{C}$.

This class is not closed even under inflations satisfying (3), since $21, 12, 1 \in \mathcal{C}$ but $21[12, 1] = 231 \notin \mathcal{C}$. Thus, Proposition 3.9 does not apply. Instead, we use Theorem 3.2 to partition the nonempty members by their unique semisimple quotient. Each quotient is a classical pattern of the member being decomposed and hence belongs to $\mathcal{C}$, so the only possibilities are 1, 11, 12, 21. Let F be the ordinary generating function for the nonempty members of $\mathcal{C}$. The quotients 1 and 11 contribute x and x^2 , respectively.

For the quotient 12, the canonical decomposition is $\alpha \oplus \beta$, with α plus indecomposable. The class $\mathcal{C}$ is closed under direct sums because each forbidden pattern is plus indecomposable. If I counts the nonempty plus indecomposable members, then $F = I + IF$, so $I = F/(1 + F)$. Hence this quotient contributes $IF = F^2/(1 + F)$.

For the quotient 21, the canonical decomposition is $\alpha \ominus \beta$, with α minus indecomposable. Since β is nonempty, an increasing pair in α , together with any letter of β , would form 231. Thus α must avoid 12. It is therefore weakly decreasing, and if it used more than one value it would be minus decomposable; hence it is constant. Avoidance of 111 leaves only $\alpha = 1, 11$. Conversely, with $\alpha = 1$ or 11 and $\beta \in \mathcal{C}$ nonempty, none of the forbidden patterns can occur across the two components. Thus the quotient 21 contributes $(x + x^2)F$. Adding the four contributions gives

$$F = x + x^2 + \frac{F^2}{1 + F} + (x + x^2)F.$$

On writing $G = 1 + F$ for the ordinary generating function of $\mathcal{C}$, this simplifies to

$$G = 1 + (x + x^2)G^2.$$

A direct bijection also explains this equation. Avoidance of 111, 121, and 212 says that every value occurs in one contiguous run of length one or two. Contracting the runs gives a 231-avoiding permutation, and conversely each entry of any such permutation may be replaced independently by a run of length one or two. Permutations avoiding 231 are counted by the Catalan numbers; letting Cat denote their generating function, we obtain

$$G(x) = \text{Cat}(x + x^2) = \frac{1 - \sqrt{1 - 4x - 4x^2}}{2x(1 + x)}.$$

The second part of Example 4.6 illustrates the profile construction of Theorem 4.5 for a class with one further simple member.

4 General classes with finitely many simples

We now remove the hypothesis of substitution closure and extend Proposition 3.10 to any class. Namely, we show in Theorem 4.3 and Theorem 4.5 that every class with finitely many simple members has a finite basis and an algebraic generating function. This finiteness hypothesis allows infinitely many semisimple members; for example, $\text{Cay}(21)$ has simple members 1, 11, and 12, but contains every constant word. The first step is to rule out infinite antichains. A partially ordered set is *partially well ordered* if it contains neither an infinite strictly descending chain nor an infinite antichain. For Cayley permutations, a strictly descending chain is necessarily finite, since length decreases at each step.

The *closure* of a set under a family of operations consists of all elements obtained by applying those operations finitely many times, starting with elements of the set. We use the following form of Higman's theorem, stated by Albert and Atkinson [1].

Theorem 4.1 (Higman [7]). *Let $(P, \leq)$ be a partially ordered set and let $f_i : P^{n_i} \rightarrow P$, $1 \leq i \leq r$, be finitely many operations of finite arity. Suppose each f_i is order preserving in every argument; that is, for all $(a_1, \dots, a_{n_i}), (b_1, \dots, b_{n_i}) \in P^{n_i}$,*

$$a_j \leq b_j \text{ for every } 1 \leq j \leq n_i \implies f_i(a_1, \dots, a_{n_i}) \leq f_i(b_1, \dots, b_{n_i}).$$

Suppose also that

$$a_j \leq f_i(a_1, \dots, a_{n_i}) \text{ for every } (a_1, \dots, a_{n_i}) \in P^{n_i} \text{ and } 1 \leq j \leq n_i.$$

Then the closure of any finite subset of P under these operations is partially well ordered.

The proof of the next theorem adapts Albert and Atkinson's [1] approach to our settings. Gustedt [6] proves a related well-quasi-ordering result for substitution compositions of posets.

Theorem 4.2. *A class $\mathcal{C}$ of Cayley permutations with only finitely many simple members is partially well ordered.*

Proof. Let F be the finite set of simple members of $\mathcal{C}$ of length at least two other than 11. Consider finite rooted ordered trees whose nodes have labels in $\{1, *\} \cup F$. The label determines the number of children: a node labelled 1 has no children, a node labelled $*$ has one child, and a node labelled $\sigma \in F$ has $|\sigma|$ children.

We interpret these trees as words recursively. A single leaf represents the word 1. Suppose the root has label $\sigma \in F$, with $k = |\sigma|$, and its child subtrees represent $\alpha_1, \dots, \alpha_k$ in that order. Then the tree represents $\sigma[\alpha_1, \dots, \alpha_k]$, provided the components at repeated quotient letters are constant, as in (2). A tree whose root has label $*$ represents 1^{r+1} if its child subtree represents 1^r . Call a tree *valid* if this interpretation is defined at every node. In particular, 1^r is represented by a chain of $r - 1$ nodes labelled $*$ above a leaf.

Every nonempty member of $\mathcal{C}$ has a valid representation, by induction on length. For length one, use a leaf. For a longer word, Theorem 3.3 gives a simple quotient and shorter components. All are patterns of the original word and hence belong to $\mathcal{C}$. If the quotient is 11, the word is constant and we use the chain above. Otherwise, the quotient belongs to F ; attach a root with this label above the component trees supplied by the induction hypothesis. Figure 1 illustrates the representation for the word 231146561 considered in Section 2.

Now consider all finite ordered trees with these labels and the prescribed numbers of children. We define an embedding order on trees that implies pattern containment of the corresponding words. Recursively, let $S \preceq T$ if and only if at least one of the following conditions holds:

- (i) The roots of S and T have the same label, and their ordered child subtrees $S_1, \dots, S_k$ and $T_1, \dots, T_k$ satisfy $S_i \preceq T_i$ for every i .

- (ii) There is a child of the root of T whose rooted subtree U satisfies $S \preceq U$.

In particular, a leaf embeds in a leaf by (i), since there are no children to compare. Furthermore, repeated use of (ii) allows the root of S to be matched with a node below the root of T .

For each internal label, consider the operation of attaching a root with that label above the prescribed number of trees, in order. Condition (i) makes this operation order preserving, and (ii) ensures that every argument embeds in the resulting tree. These finitely many operations generate all the trees from the single leaf. Theorem 4.1 therefore makes this set of trees partially well ordered. Its subset of valid trees is also partially well ordered.

On valid trees, embedding implies containment of the represented words. We prove this by induction on the number of nodes of T . In case (ii), the word represented by S is a pattern of the word represented by U , by induction. The word represented by U is in turn a pattern of the word represented by T : at an inflation root it is one of the components, and at a root labelled $*$ this follows from $1^r \leq 1^{r+1}$. Transitivity gives the result. In case (i), two leaves both represent 1. If the common root label is $\sigma \in F$, the induction hypothesis gives an occurrence of each component represented at a child of S in the component represented at the corresponding child of T . Combining these occurrences gives an occurrence of the smaller inflation in the larger one. If the common label is $*$, let the child subtrees represent 1^r and 1^s . The induction hypothesis gives $r \leq s$, and hence $1^{r+1} \leq 1^{s+1}$ for the words represented by S and T . Thus any infinite sequence of nonempty members of $\mathcal{C}$ has an earlier member contained in a later one, by choosing their representing trees and applying partial well order there. Adjoining the empty word preserves this property. $\square$

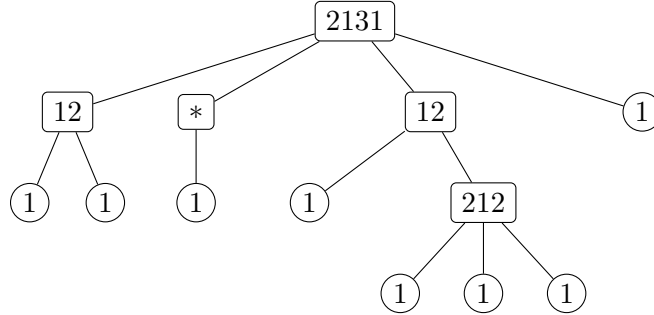


Figure 1: A valid tree representing $231146561 = 2131[12, 11, 1323, 1]$. The $*$ node represents 11, and the right-hand 12 node represents $12[1, 212] = 1323$.

Theorem 4.3. *A class $\mathcal{C}$ of Cayley permutations with only finitely many simple members has a finite basis.*

Proof. If $\mathcal{C}$ contains no nonempty word, its basis consists of the empty word or the word 1. Otherwise, let k be the greatest length of a simple member of $\mathcal{C}$. By Lemma 3.7 and Theorem 4.2, the substitution closure $\langle \mathcal{C} \rangle$ is partially well ordered.

The basis elements of $\mathcal{C}$ that lie in $\langle \mathcal{C} \rangle$ form an antichain there, so there are finitely many. Any remaining basis element is also a basis element of $\langle \mathcal{C} \rangle$: it lies outside $\langle \mathcal{C} \rangle$,

while all its proper patterns lie in $\mathcal{C} \subseteq \langle \mathcal{C} \rangle$. Corollary 3.8 bounds its length by $k + 2$, leaving only finitely many possibilities. $\square$

It remains to find the generating function. As in Example 3.12, we must keep track of avoidance when inflating a quotient. A finite basis makes this possible with finitely many generating functions by recording which patterns below a basis element occur in each component. This is analogous to the query-complete property sets used by Brignall, Huczynska, and Vatter [3] for ordinary permutations: the properties satisfied by an inflation are determined by its quotient and the properties satisfied by its components. Before giving the construction, we state the algebraic fact needed to pass from the equations to the result.

Lemma 4.4. *Let $y_j = P_j(x, y_1, \dots, y_m)$, $1 \leq j \leq m$, be a proper algebraic system over $\mathbb{Q}$; that is, $P_1, \dots, P_m \in \mathbb{Q}[x, y_1, \dots, y_m]$ satisfy*

$$\begin{aligned} P_j(0, \dots, 0) &= 0, & 1 \leq j \leq m, \\ \frac{\partial P_j}{\partial y_\ell}(0, \dots, 0) &= 0, & 1 \leq j, \ell \leq m. \end{aligned}$$

Then the system

$$y_j = P_j(x, y_1, \dots, y_m), \quad 1 \leq j \leq m,$$

has a unique solution $(f_1, \dots, f_m)$ with every $f_j \in x\mathbb{Q}[[x]]$, and each f_j is algebraic over $\mathbb{Q}(x)$. Moreover, given $H \in \mathbb{Q}[x, y_1, \dots, y_m]$, a nonzero polynomial in $\mathbb{Q}[x, T]$ annihilating $H(x, f_1, \dots, f_m)$ can be computed from H and the P_j .

Proof. Existence and uniqueness of the solution with zero constant terms, and algebraicity of its components, follow from Stanley's [11, Proposition 6.6.3 and Theorem 6.6.10] results on proper algebraic systems.

Panholzer [9] gives an algorithm computing an annihilating polynomial for any component of a proper algebraic system. To apply it to H , write $y = (y_1, \dots, y_m)$, put $c = H(0, \dots, 0)$, and append the equation

$$z = H(x, P_1(x, y), \dots, P_m(x, y)) - c.$$

Its right-hand side vanishes at the origin, as do all its derivatives with respect to the unknowns, so the enlarged system is still proper. Its new component is $g = H(x, f_1, \dots, f_m) - c$. Panholzer's algorithm therefore computes a nonzero $q(x, T) \in \mathbb{Q}[x, T]$ with $q(x, g) = 0$. Then $q(x, T - c)$ is the required polynomial annihilating $H(x, f_1, \dots, f_m)$. $\square$

Theorem 4.5. *Let $\mathcal{C}$ be a class of Cayley permutations with only finitely many simple members. Its ordinary generating function $G(x) = \sum_{\omega \in \mathcal{C}} x^{|\omega|}$ is algebraic over $\mathbb{Q}(x)$. Given the simple members and the basis of $\mathcal{C}$, one can compute a nonzero polynomial $q(x, T) \in \mathbb{Q}[x, T]$ such that $q(x, G(x)) = 0$.*

Proof. If $\mathcal{C}$ contains no nonempty word, then $G(x)$ is 0 or 1, and the result is immediate. Otherwise, both the empty word and 1 belong to $\mathcal{C}$.

We construct a proper algebraic system that counts the nonempty members of $\langle \mathcal{C} \rangle$ according to the patterns they contain. We can then recover G by selecting the members that avoid the basis of $\mathcal{C}$. Let $\mathcal{B}$ be the basis of $\mathcal{C}$, finite by Theorem 4.3, and let Γ consist of all patterns contained in a member of $\mathcal{B}$. This is a finite set, closed downward. For a word ω , let

$$\mathcal{P}(\omega) = \{\gamma \in \Gamma : \gamma \leq \omega\}$$

be the set of patterns in Γ contained in ω , called its *profile*. This is analogous to the set of satisfied properties recorded by Brignall, Huczynska, and Vatter [3], with the properties here being containment of the individual patterns in Γ . Since $\mathcal{B} \subseteq \Gamma$, a word $\omega \in \langle \mathcal{C} \rangle$ belongs to $\mathcal{C}$ if and only if $\mathcal{P}(\omega) \cap \mathcal{B} = \emptyset$.

The set $\mathcal{P}(\omega)$ is determined by the quotient of ω and the corresponding sets for its components. To see this, let $\omega = \sigma[\alpha_1, \dots, \alpha_k]$. By Lemma 3.4, a nonempty $\beta \in \Gamma$ occurs in ω if and only if

$$\beta = \text{st}(\sigma|_T)[\gamma_i : i \in T]$$

for nonempty $T \subseteq [k]$ and nonempty $\gamma_i \leq \alpha_i$, with γ_i constant whenever $\sigma(i)$ is repeated. Each γ_i is a pattern of β , so it lies in Γ . Thus the condition $\gamma_i \leq \alpha_i$ can be read from $\mathcal{P}(\alpha_i)$. Enumerating the finitely many choices of T and the $\gamma_i \in \Gamma$ determines whether each $\beta \in \Gamma$ occurs, and hence determines $\mathcal{P}(\omega)$. The empty pattern, if in Γ , belongs to every $\mathcal{P}(\omega)$.

For each subset $Q \subseteq \Gamma$, define the ordinary generating function

$$g_Q(x) = \sum_{\substack{\omega \in \langle \mathcal{C} \rangle \cap \text{Cay}_+ \\ \mathcal{P}(\omega) = Q}} x^{|\omega|}.$$

Define $g_Q^+(x)$ and $g_Q^-(x)$ by restricting this sum to plus indecomposable and minus indecomposable words, respectively. Since Γ is finite, this gives a finite list of generating functions. We also need the series c_Q for the nonempty constant words in the closure with profile Q . These are rational: if L is the greatest length in Γ , taking $L = 1$ if Γ is empty, then $\mathcal{P}(1^r)$ is constant for $r \geq L$. Hence each c_Q is a finite sum of powers of x , possibly together with $x^L/(1-x)$. If 11 is absent from the closure, only 1 is available instead.

We obtain equations for the g_Q, g_Q^+, g_Q^- from the unique decomposition in Theorem 3.3. Constant words are counted by c_Q and contribute to all three series, since they are both plus and minus indecomposable. By Lemma 3.7, the available nonconstant simple quotients are exactly those in $\mathcal{C}$. For such a quotient σ of length $k \geq 2$, choose component profiles $Q_1, \dots, Q_k$. At a repeated quotient letter use the factor c_{Q_i} ; at a singleton fibre use g_{Q_i} , with two exceptions: the first factor is $g_{Q_1}^+$ when $\sigma = 12$ and $g_{Q_1}^-$ when $\sigma = 21$. These are exactly the restrictions that make the decomposition unique.

The product of these k factors counts the inflations with the chosen property sets. Compute the resulting set $Q = \mathcal{P}(\omega)$ as above, and include this product in the equation for g_Q . To obtain the equation for g_Q^+ , omit the quotient 12; to obtain that for g_Q^- , omit 21. Indeed, these quotients characterize plus and minus decomposability, respectively. There are finitely many quotients and profiles, so this gives a finite system polynomial in the unknown series, with rational functions of x as coefficients

regular at the origin. Each right-hand side vanishes at the origin and has zero Jacobian there. Indeed, every inflation contribution has at least two nonempty parts, and all the factors counting such parts have zero constant term.

Clear denominators, normalized to have constant term 1, and move their nonconstant multiples of the unknowns to the right. This gives a proper polynomial system. Lemma 4.4 therefore applies, and the counting series form its unique solution with zero constant terms. The ordinary generating function of $\mathcal{C}$ is

$$G(x) = 1 + \sum_{\substack{Q \subseteq \Gamma \\ Q \cap \mathcal{B} = \emptyset}} g_Q(x),$$

so it is algebraic. Finally, Γ and all the polynomial equations are computable from the basis and simple members of $\mathcal{C}$ by finite enumeration. Applying the last assertion of Lemma 4.4 to $1 + \sum_{Q \cap \mathcal{B} = \emptyset} g_Q(x)$ produces the required polynomial $q(x, T)$. $\square$

Every simple word is semisimple, so Theorem 4.2, Theorem 4.3 and Theorem 4.5 apply in particular to classes with finitely many semisimple members. Their simple members are obtained by retaining the primitive words and 11. In this case, the profile construction can also impose (3) and count $\langle \mathcal{C} \rangle_{\mathcal{H}}$. Use its finitely many semisimple quotients of length at least two, and at every repeated quotient letter require the component profile $\mathcal{P}(1)$ and use the factor x . The remaining word 1 contributes x to each of the three series with profile $\mathcal{P}(1)$; the quotient 11, when present, is counted with the other quotients. The profile transitions and the normalizations at 12 and 21 are unchanged. This gives a proper polynomial system directly.

Example 4.6. Let $\mathcal{C} = \text{Cay}(112, 212, 231)$. Direct enumeration finds no simple members of lengths four or five, so Corollary 3.1 shows that its nonempty simple members are

$$1, \quad 11, \quad 12, \quad 21, \quad 121. \quad (5)$$

By Lemma 3.7, these are also the nonempty simple members of the closure $\langle \mathcal{C} \rangle$. However, $\mathcal{C}$ contains infinitely many nonconstant semisimple members as every word 121^b , with $b \geq 1$, is semisimple and lies in $\mathcal{C}$. The class is not substitution-closed, since $21, 12, 1 \in \mathcal{C}$ but $21[12, 1] = 231 \notin \mathcal{C}$. Nevertheless, every hereditary subclass of $\mathcal{C}$ has finitely many simple members and therefore has a finite basis and an algebraic generating function by Theorems 4.3 and 4.5.

Write $\mathcal{B} = \{112, 212, 231\}$ and let $\Gamma = \{1, 11, 12, 21, 112, 212, 231\}$ be its downward closure, omitting the empty pattern. The simple basis element 212 does not belong to $\langle \mathcal{C} \rangle$, so every member of $\langle \mathcal{C} \rangle$ avoids it. Thus $\omega \in \langle \mathcal{C} \rangle$ belongs to $\mathcal{C}$ if and only if $112, 231 \notin \mathcal{P}(\omega)$. We use the profile construction in the proof of Theorem 4.5, summing over the sets Q disjoint from $\mathcal{B}$ and distinguishing whether $11 \in Q$ and whether $12 \in Q$. Write

$$A = \sum_{\substack{Q \subseteq \Gamma, \\ Q \cap \mathcal{B} = \emptyset \\ 12 \notin Q}} g_Q, \quad B = \sum_{\substack{Q \subseteq \Gamma, \\ Q \cap \mathcal{B} = \emptyset \\ 12 \in Q}} g_Q, \quad I = \sum_{\substack{Q \subseteq \Gamma, \\ Q \cap \mathcal{B} = \emptyset \\ 11 \notin Q}} g_Q^+.$$

Thus A counts the nonempty members avoiding 12, B counts those containing 12, and I counts the plus indecomposable members avoiding 11. Every word counted by A is weakly decreasing and hence plus indecomposable. Its minus indecomposable

members are the nonempty constant words, counted by $X = x/(1 - x)$. The words counted by I are precisely the plus indecomposable 231-avoiding permutations, each of which consists of its maximum followed by an arbitrary 231-avoiding permutation. Thus $I = x \text{Cat}(x)$ and, using that $\text{Cat}(x) = 1 + x \text{Cat}(x)^2$,

$$I = x \text{Cat}(x) = x(1 + x \text{Cat}(x)^2) = x + x^2 \text{Cat}(x)^2 = x + I^2.$$

We now read the contributions quotient by quotient. The nonempty constant words contribute X to A . For the quotient 12, the first component is plus indecomposable and must avoid 11, or else two equal letters there would form 112 together with any letter of the second component. It is therefore counted by I . Since 231 is plus indecomposable, an inflation of 12 avoids 231 precisely when both components do. With the first component avoiding 11, no occurrence of 112 crosses the components either. Every such inflation contains 12, giving the contribution $I(A + B)$ to B .

For the quotient 21, an occurrence of 231 crosses the two components precisely when the first contains 12. This component must also be minus indecomposable, so it is constant. No occurrence of 112 crosses these components. The resulting inflation contains 12 if and only if the second component does, giving XA to A and XB to B .

For the quotient 121, the two positions in the repeated fibre are inflated by arbitrary nonempty constant words 1^a and 1^b , whereas the middle component α is free. Avoidance of 112 forces $a = 1$. An occurrence of 231 crosses the components of $121[1^a, \alpha, 1^b]$ precisely when α contains 12. Thus α is counted by A , and these inflations contribute xXA to B .

Collecting the contributions, we obtain the system

$$\begin{aligned} A &= X + XA, & X &= \frac{x}{1 - x}, \\ B &= I(A + B) + XB + xXA, \\ I &= x + I^2. \end{aligned}$$

Each right-hand side vanishes at the origin and has no term linear in an unknown alone. Clearing denominators gives a proper algebraic system, as in the proof of the theorem.

To eliminate the auxiliary series, put $G = 1 + A + B$. The first equation gives $A = X/(1 - X)$, and adding the first two gives $G - 1 = X + (I + X)(G - 1) + xXA$. Hence

$$G = 1 + \frac{X + xX^2/(1 - X)}{1 - X - I}, \quad I = \frac{1 - \sqrt{1 - 4x}}{2}.$$

This generating function is algebraic but not rational, since I can be recovered rationally from G . Eliminating I and clearing denominators produces the annihilating polynomial

$$\begin{aligned} q(x, T) &= x^2(1 - 2x)^2(T - 1)^2 \\ &\quad - (1 - x)^2(1 - 2x)(1 - 3x)(T - 1) + x(1 - x)^4. \end{aligned}$$

The solution with constant term 1 begins

$$G(x) = 1 + x + 3x^2 + 10x^3 + 33x^4 + 109x^5 + 363x^6 + 1223x^7 + \cdots$$

For comparison, let $\mathcal{D} = \text{Cay}(111, 112, 212, 231)$. Direct enumeration and Corollary 3.1 show that its nonempty semisimple members are exactly the five simple words listed in (5). The same calculation now has $X = x + x^2$, since the nonempty constant words are 1 and 11, and the quotient 121 contributes x^2A , since both outer components must be 1. The series I is unchanged. Thus the generating function of $\mathcal{D}$ is

$$G_{\mathcal{D}} = 1 + \frac{(x + x^2)(1 - x)}{(1 - x - x^2)(1 - x - x^2 - I)},$$

and begins

$$G_{\mathcal{D}}(x) = 1 + x + 3x^2 + 9x^3 + 28x^4 + 88x^5 + 284x^6 + 937x^7 + \dots$$

Neither counting sequence appears in the OEIS [8].

An exhaustive computation over the $2^{13} - 1 = 8191$ nonempty bases $\mathcal{B} \subseteq \text{Cay}_3$ finds exactly 7380 for which $\text{Cay}(\mathcal{B})$ has no simple member of length six or seven. By Corollary 3.1, all simple members of these classes have length at most five. By Theorems 4.3 and 4.5, every hereditary subclass of any of these classes has a finite basis and an algebraic generating function.

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