# Simple Cayley permutations

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**Abstract**

We propose a notion of simplicity for Cayley permutations that is compatible with inflation. We prove that Cayley permutations admit a substitution decomposition analogous to that of permutations and use it to enumerate simple Cayley permutations, primitive simple Cayley permutations, and simple restricted growth functions. We also prove that every hereditary Cayley permutation class with finitely many simple members has a finite basis and an algebraic generating function.

*Keywords:* Cayley permutation, simple permutation, inflation, substitution decomposition, relational structure

## 1 Introduction

A *Cayley permutation* is either empty or a word of positive integers that contains every number between one and its maximum value. In other words, a Cayley permutation is an endofunction $\omega$ on $[n] = \{1, 2, \ldots, n\}$ such that $\mathrm{im}(\omega) = [m]$ for some $m \leq n$. We write $\mathrm{Cay}_n$ for the set of Cayley permutations of length $n$ and $\mathrm{Cay} = \bigcup_{n\geq 0} \mathrm{Cay}_n$. In addition, we write $\mathrm{Cay}_+ = \bigcup_{n\geq 1} \mathrm{Cay}_n$ for the set of nonempty Cayley permutations. There is a simple one-to-one correspondence between ballots (ordered set partitions) and Cayley permutations: a ballot $\varpi$ corresponds to the Cayley permutation $\omega$ whose $i$th letter equals the unique index $j$ such that $i$ belongs to the $j$th block of $\varpi$. Cayley permutations may also be seen as representatives for equivalence classes of words modulo order isomorphism. For these and many other reasons, recent papers on Cayley permutations have explored their combinatorial and enumerative properties [3, 10, 11, 12, 13], as well as pattern avoidance [8, 9, 15].

The main objective of this paper is to define simple Cayley permutations and develop the surrounding theory. The notion of simplicity on (standard) permutations was introduced by Albert and Atkinson [1]: a permutation is *simple* if it has no proper non-singleton interval, where an interval is a nonempty set of consecutive positions whose corresponding values also form a contiguous set of integers. Every permutation of length $n$ has trivial intervals of length 1 (single entries) and length $n$ (the whole permutation). If these are the only intervals, then the permutation is simple. Simple permutations play an important role in the study of permutation classes. Let us

recall some results that will be relevant in this paper. See the survey by Brignall [6] for a broader introduction.

Albert and Atkinson [1] showed that every permutation may be written as the inflation of a unique simple permutation. This allowed Albert, Atkinson, and Klazar [2] to enumerate simple permutations in terms of the compositional inverse of $f(x) = \sum_{n\geq 1} n!x^n$. A special case of a general theorem by Schmerl and Trotter [22] on indecomposable relational structures says that every simple permutation of length $n \geq 2$ contains a simple permutation of length $n-1$ or $n-2$. In other words, if a permutation is simple then one may obtain a new simple permutation by deleting at most two points. Permutations where a one-point deletion is not sufficient are called *exceptional*. Schmerl and Trotter have classified the critically indecomposable relational structures up to isomorphism of their skeletons. And by reading the classification in terms of permutations, Albert and Atkinson [1] showed that the only exceptional permutations are the simple parallel alternations, i.e. those of the form

$$246\cdots(2m)135\cdots(2m-1), \quad m \geq 2,$$

and their symmetries. Permutation classes containing finitely many simple permutations have also been studied extensively. A classic result in this direction by Albert and Atkinson [1] shows that a class of permutations containing only finitely many simple members has a finite basis and an algebraic generating function.

We propose a notion of simplicity for Cayley permutations that uses Hertzsprung intervals, defined in Section 2. This definition equips Cayley permutations with a substitution decomposition analogous to that of permutations, in which repeated letters are necessarily inflated by the one-letter Cayley permutation. Inflation and the substitution decomposition are defined in Section 3. In Section 4, we derive a bivariate generating function identity for simple Cayley permutations and enumerate them. In Section 5, we view Cayley permutations as relational structures and show that indecomposable structures correspond to Cayley permutations that are both simple and primitive. This approach allows us to prove that the only exceptional Cayley permutations are the simple parallel alternations. In Section 6, we show how inflation works for restricted growth functions and use this to enumerate the simple ones. In Section 7, we establish that a class of Cayley permutations with finitely many simple members has a finite basis and an algebraic generating function.

## 2 Hertzsprung intervals and simple Cayley permutations

Throughout this paper, we will use the following terminology. Let $\omega$ be a word. We say that $\beta$ is a *factor* of $\omega$ if there are words $\alpha$ and $\gamma$ such that $\omega = \alpha\beta\gamma$. If $\alpha$ is empty we also say that $\beta$ is a *prefix* of $\omega$, and if, in addition, $\gamma$ is nonempty then $\beta$ is a *proper prefix* of $\omega$. Similarly, if $\gamma$ is empty we say that $\beta$ is a *suffix* of $\omega$, and that $\beta$ is a *proper suffix* if, in addition, $\alpha$ is nonempty. The *standardization* of $\omega$ is the word $\mathrm{st}(\omega)$ obtained by replacing the smallest letter of $\omega$ by 1, the next smallest by 2, and so on, keeping repetitions equal. Note that $\mathrm{st}(\omega)$ is a Cayley permutation.

The second author [16] calls a permutation $\tau$ of length $k$ a *Hertzsprung factor* of a permutation $\pi$ if there is a nonnegative integer $c$ and a factor $\beta = b_1 b_2 \cdots b_k$ of $\pi$ such that $b_i - \tau(i) = c$ for each $i \in [k]$. That is, the factor $\beta$ is a translate $\tau + c$ of $\tau$.

We shall extend this definition to Cayley permutations. Let $\omega = w_1 \cdots w_n \in \mathrm{Cay}_n$. If $I = [a, b] \subseteq [n]$ is a nonempty positional interval, write

$$\omega|_I = w_a w_{a+1} \cdots w_b \qquad \text{and} \qquad \omega(I) = \{w_a, w_{a+1}, \ldots, w_b\}$$

for the factor of $\omega$ at positions $I$ and for its set of values. More generally, if $S = \{i_1 < \cdots < i_r\} \subseteq [n]$, write $\omega|_S = w_{i_1} \cdots w_{i_r}$ for the corresponding subword and $\omega(S)$ for its values. We say that $I$ is a *Hertzsprung interval*, or *H-interval*, of $\omega$ if there is a value interval $J = [c, d]$ such that

$$\omega(I) = J \quad \text{and} \quad \omega^{-1}(J) = I. \tag{1}$$

Thus, an H-interval occupies consecutive positions and uses every value in a consecutive range, which we sometimes call a *value band*. Additionally, it contains every copy in $\omega$ of every value in $J$ due to the second equality, which we call the *saturation condition*. In particular, $[n]$ is always an H-interval. A singleton $[i, i] = \{i\}$ is an H-interval precisely when the value $\omega(i)$ occurs nowhere else in $\omega$. These two types of singletons (unique/repeated) will play different roles in inflations (see Section 3). Note also that the saturation condition is vacuous for permutations, so the usual definition of permutation interval is obtained as a special case of the one just given. Unlike in a permutation, the same factor may appear more than once in a Cayley permutation. For example, 12 appears in 1212 at both $[1, 2]$ and $[3, 4]$. We therefore specify an occurrence by its positional interval.

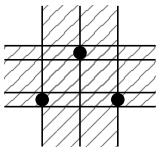
Let $\tau \in \mathrm{Cay}$. An *occurrence* of $\tau$ in $\omega$ is an H-interval $I$ of $\omega$ where $\mathrm{st}(\omega|_I) = \tau$. Equivalently, we have

$$\omega|_I = \tau + c$$

for an integer $c \geq 0$ and no entry of $\omega$ outside $I$ takes a value in $[c + 1, c + \max(\tau)]$. We call $\tau$ a *Hertzsprung factor* of $\omega$. Every H-interval is an occurrence of exactly one pattern, namely $\mathrm{st}(\omega|_I)$, and a singleton H-interval is an occurrence of 1. *Hertzsprung prefixes* and *Hertzsprung suffixes* are defined in the natural way.

*Example* 2.1. In $\omega = 121$ the factor $\omega|_{[1,2]} = 12$ uses consecutive positions and consecutive values, but the value 1 occurs again in the third position. Hence the saturation condition fails, and $[1, 2]$ is not an occurrence of 12. Similarly, $[1, 2]$ is not an occurrence of 11 in 111. The H-intervals of 121 are the singleton $\{2\}$ and the whole word $[1, 3]$, while the only H-interval of 111 is $[1, 3]$.

The reader familiar with mesh patterns [5] and Cayley-mesh patterns [8] may notice that an occurrence of $\tau$ is an occurrence of the Cayley-mesh pattern whose underlying Cayley permutation is $\tau$ and where every box is shaded except for the four corner boxes. For instance, an occurrence of 121 is an occurrence of the Cayley-mesh pattern



and the algebraic condition (1) is a convenient definition of containment in this sense.

Let us briefly motivate the terminology just introduced. For permutations, a block in the sense of Albert and Atkinson [1], also commonly called an interval [6], occupies

consecutive positions and carries a consecutive set of values, and is therefore an H-interval. We avoid both of the established names. "Block" is kept for the blocks of a ballot (where an H-interval is a union of consecutive blocks, see Proposition 2.3). "Interval" is used for positional and value intervals. Interval is also the name under which the permutation patterns literature treats these sets as the modules of a relational structure. This coincidence is special to permutations and does not extend to Cayley permutations, where the modules form a larger family than the H-intervals (see Section 5).

We are now ready to define simplicity on Cayley permutations.

*Definition* 2.2. A Cayley permutation $\omega \in \mathrm{Cay}_n$ is *simple* if every H-interval of $\omega$ is a singleton or $[n]$.

As noted before, for permutations the saturation condition is always satisfied and the definition of simplicity restricts to the usual one. All Cayley permutations of length one and two are simple by definition. Example 2.1 shows that 121 and 111 are simple. An example of a non-simple Cayley permutation is 122: it contains the proper H-interval $[2, 3]$, which is an occurrence of 11 and occupies value band $[2, 2] = \{2\}$. There are three simple Cayley permutations of length three and fifteen of length four, listed below:

$$111, 212, 121;$$

$$1111, 2212, 1121, 2121, 2122, 3132, 2131, 2132,$$
$$3142, 1211, 1212, 1312, 2312, 2313, 2413.$$

Recall from Section 1 that Cayley permutations encode ballots. Namely, to the Cayley permutation $\omega \in \mathrm{Cay}_n$ we associate the ballot of $[n]$ defined by

$$\varpi = F_1 F_2 \cdots F_m, \qquad F_j = \omega^{-1}(j).$$

In this context, the fibres, $F_j$, are called *blocks*. For instance, the Cayley permutation $\omega = 211434125$ is associated with the ballot

$$\varpi = \{2, 3, 7\}\ \{1, 8\}\ \{5\}\ \{4, 6\}\ \{9\}.$$

We now interpret simplicity in terms of ballots, which will make the proofs of some results in the coming sections more convenient to state. To start, we characterize their H-intervals.

**Proposition 2.3.** *Let $\varpi = F_1 F_2 \cdots F_m$ be the ballot associated with $\omega \in \mathrm{Cay}_n$. Then $I \subseteq [n]$ is an H-interval of $\omega$ if and only if $I$ is an interval of $[n]$ and*

$$I = F_c \cup F_{c+1} \cup \cdots \cup F_d$$

*for some $1 \leq c \leq d \leq m$. Furthermore, such an $I$ is an occurrence of $\tau \in \mathrm{Cay}$ if and only if subtracting $\min(I) - 1$ from every element of the factor $F_c \cdots F_d$ turns it into the ballot of $\tau$.*

*Proof.* Let $I$ be an H-interval, with value band $J = [c, d]$ as in (1). Then the saturation condition gives

$$I = \omega^{-1}(J) = \bigcup_{j=c}^{d} \omega^{-1}(j) = F_c \cup F_{c+1} \cup \cdots \cup F_d,$$

as wanted. Conversely, suppose that $I = F_c \cup \cdots \cup F_d$ is an interval of $[n]$, and put $J = [c, d]$. Every block is nonempty, so $\omega(I) = J$, and $\omega^{-1}(J) = F_c \cup \cdots \cup F_d = I$. Hence (1) holds and $I$ is an H-interval.

For the second claim, let $a = \min(I)$. Position $i$ of $\omega|_I$ corresponds to position $a+i-1$ of $\omega$. Since $\omega(I) = [c, d]$, the letter $j$ occurs in $\mathrm{st}(\omega|_I)$ at position $i$ if and only if $\omega(a + i - 1) = c + j - 1$. Thus, the $j$th block of the ballot of $\mathrm{st}(\omega|_I)$ is obtained from $F_{c+j-1}$ by subtracting $a - 1$ from each element. Therefore, subtracting $a - 1$ from every element of $F_c \cdots F_d$ gives the ballot of $\mathrm{st}(\omega|_I)$. This is the ballot of $\tau$ if and only if $I$ is an occurrence of $\tau$. □

In ballot terms, an H-interval is a run of consecutive blocks whose union is an interval of $[n]$. More precisely, an H-interval $I$ covering the value band $\omega(I) = [c, d]$ corresponds to the blocks $F_c \cup F_{c+1} \cup \cdots \cup F_d$, whose union is $I$. Because each block is included in full, saturation is automatic. In particular, a singleton position is an H-interval precisely when it is a singleton block. In the example given before Proposition 2.3, the H-intervals of the Cayley permutation $\omega = 211434125$ correspond to those of the ballot $\varpi = \{2, 3, 7\}\{1, 8\}\{5\}\{4, 6\}\{9\}$ as follows:

$$\begin{aligned}
[1, 9] &\longleftrightarrow \varpi;\\
[1, 8] &\longleftrightarrow \{2, 3, 7\}\{1, 8\}\{5\}\{4, 6\};\\
[4, 6] &\longleftrightarrow \{5\}\{4, 6\};\\
[5, 5] &\longleftrightarrow \{5\};\\
[9, 9] &\longleftrightarrow \{9\}.
\end{aligned}$$

Thus, simplicity for ballots can be stated as follows.

**Corollary 2.4.** *Let $\omega \in \mathrm{Cay}_n$ and let $\varpi$ be the corresponding ballot. Then $\omega$ is simple if and only if the only unions of consecutive blocks of $\varpi$ that are intervals of $[n]$ are the singleton blocks and $[n]$ itself.*

For instance, 121 has ballot $\{1, 3\}\{2\}$. The union $\{1, 3\}$ is not an interval of $[3]$, so the only H-intervals are the singleton block $\{2\}$ and $[3]$, and 121 is simple. More generally, in a simple Cayley permutation, no block of size two or more may be an interval of $[n]$ unless it is all of $[n]$.

## 3 Inflation and the substitution decomposition

Given a permutation $\sigma$ of length $k$ and nonempty permutations $\alpha_1, \ldots, \alpha_k$, the inflation of $\sigma$ by $\alpha_1, \ldots, \alpha_k$ is the permutation $\sigma[\alpha_1, \ldots, \alpha_k]$ obtained by replacing each entry $\sigma(i)$ of $\sigma$ with an interval that is order isomorphic to $\alpha_i$, keeping the relative order of the $\sigma(i)$'s.

We shall define the inflation of Cayley permutations similarly, but with one additional constraint: only letters appearing exactly once in the Cayley permutation may be inflated by more than one letter. This prevents positions carrying the same value from being expanded into competing value bands, which would make the process

ambiguous. Formally, let $\sigma \in \mathrm{Cay}_k$. Associate a nonempty Cayley permutation $\alpha_i$ with each position $i$, subject to the restriction

$$|\sigma^{-1}(\sigma(i))| > 1 \quad \Longrightarrow \quad \alpha_i = 1. \tag{2}$$

The *inflation* $\sigma[\alpha_1, \ldots, \alpha_k]$ is the Cayley permutation obtained by replacing each entry $\sigma(i)$ with a factor order isomorphic to $\alpha_i$, with every nonsingleton factor forming an H-interval, while preserving the relative order of the entries of $\sigma$. We call the positional interval occupied by the entries corresponding to $\alpha_i$ the $i$th *part* of the inflation. For example,

$$312143[1, 1, 221, 1, 12, 1] = 4\,1\,332\,1\,56\,4.$$

The two copies of 1 and the two copies of 3 are forced to remain single letters due to (2); the unique copy of 2 is inflated by 221 and becomes the factor 332; and the unique copy of 4 is inflated by 12 to the factor 56. The six parts of the inflation are

$$\{1\}, \{2\}, [3, 5], \{6\}, [7, 8], \{9\}. \tag{3}$$

Now, let $\omega$ be a Cayley permutation. An *H-interval decomposition* of $\omega$ is a partition of its positions into consecutive intervals, each of which is an H-interval or a single position. Contracting the parts of an H-interval decomposition (and standardizing) yields a Cayley permutation, called the *quotient*. Repeated letters in the quotient come only from singleton parts. This distinction matches the two cases of (2): an H-interval part contracts to a letter that occurs once in the quotient, whereas a singleton part whose value occurs elsewhere contracts to a repeated letter. Contracting a Cayley permutation to obtain a quotient is the inverse of inflating the quotient. For instance, an H-interval decomposition of $\omega = 413321564$ is given by the six parts of (3). In general, the H-interval decomposition is not unique. The same $\omega$ admits the H-interval decomposition

$$4\,13321\,56\,4 \quad \longrightarrow \quad \{1\}, [2, 6], [7, 8], \{9\}$$

and contracting its parts yields the quotient 2132. Conversely, we see that

$$2132[1, 13321, 12, 1] = 4\,13321\,56\,4 = \omega.$$

Although H-interval decompositions need not be unique, requiring a simple quotient leaves only a controlled ambiguity: the quotient is unique, and the parts are unique except for the quotients 12 and 21, where an indecomposability condition restores uniqueness. This will be established below. We first record two elementary closure properties of H-intervals.

**Lemma 3.1.** *Let $I$ and $I'$ be two H-intervals of a Cayley permutation $\omega$ such that $I \cap I' \neq \emptyset$.*

1. *The union $I \cup I'$ is an H-interval.*
2. *The intersection $I \cap I'$ is an H-interval with $\omega(I \cap I') = \omega(I) \cap \omega(I')$.*

*Proof.* We use the ballot viewpoint of Proposition 2.3. Each H-interval of $\omega$ is a union of consecutive blocks of the ballot $\varpi$ that form a set of consecutive elements. More precisely, write $\omega(I) = [c, d]$ and $\omega(I') = [c', d']$, so that $I$ and $I'$ are the runs of blocks $F_c \cdots F_d$ and $F_{c'} \cdots F_{d'}$. Suppose that $I$ and $I'$ overlap and let $\ell \in I \cap I'$. Then $F_c \cdots F_d$ and $F_{c'} \cdots F_{d'}$ share the block $F_{\omega(\ell)}$. Hence their union is again a union of consecutive blocks that form a set of consecutive elements, and Proposition 2.3 settles the first claim. Similarly, $I \cap I'$ is the run of blocks indexed by $[c, d] \cap [c', d']$, so Proposition 2.3 applies again and gives the second claim. □

Define direct and skew sums by

$$\alpha \oplus \beta = 12[\alpha, \beta] \quad \text{and} \quad \alpha \ominus \beta = 21[\alpha, \beta].$$

A Cayley permutation is *plus indecomposable* if it cannot be expressed as a direct sum of shorter Cayley permutations. Analogously, it is *minus indecomposable* if it cannot be expressed as a skew sum of shorter Cayley permutations.

The next two lemmas are used in the proof of Theorem 3.4.

**Lemma 3.2.** *Let $I = [1, b]$ be a proper H-interval of $\omega \in \mathrm{Cay}_n$, and suppose that its value band $J$ contains 1 or $\max(\omega)$. Then $[b+1, n]$ is an H-interval of $\omega$. Furthermore, writing $\alpha = \mathrm{st}(\omega|_{[1,b]})$ and $\beta = \mathrm{st}(\omega|_{[b+1,n]})$, we have*

$$\omega = \begin{cases} \alpha \oplus \beta & \text{if } 1 \in J, \\ \alpha \ominus \beta & \text{if } \max(\omega) \in J. \end{cases}$$

*Proof.* Let $m = \max(\omega)$ and write the ballot of $\omega$ as $\varpi = F_1 F_2 \cdots F_m$. Write $J = [c, d]$. By Proposition 2.3,

$$I = F_c \cup F_{c+1} \cup \cdots \cup F_d.$$

Since the blocks partition $[n]$ and $I = [1, b]$, we have

$$[b+1, n] = \begin{cases} F_{d+1} \cup \cdots \cup F_m & \text{if } c = 1, \\ F_1 \cup \cdots \cup F_{c-1} & \text{if } d = m. \end{cases}$$

These are the only cases under the hypothesis, and they are mutually exclusive because $I$ is proper. In either case, $[b+1, n]$ is a positional interval that is the union of consecutive blocks, so it is an H-interval by Proposition 2.3.

If $c = 1$, every value in $I$ is smaller than every value in $[b+1, n]$, and therefore $\omega = \alpha \oplus \beta$. If $d = m$, every value in $I$ is larger than every value in $[b+1, n]$, and therefore $\omega = \alpha \ominus \beta$. □

**Lemma 3.3.** *Let $\omega = \sigma[\alpha_1, \ldots, \alpha_k]$ be an inflation with parts $C_1, \ldots, C_k$, and let $T \subseteq [k]$ be a nonempty positional interval. Let $C = \bigcup_{i \in T} C_i$. Then*

$$C \text{ is an H-interval of } \omega \iff T \text{ is an H-interval of } \sigma.$$

*Proof.* For each value $j$ of $\sigma$, let $J_j$ be the value band of $\omega$ corresponding to $j$. These bands partition $[\max(\omega)]$, with $\max J_j < \min J_{j+1}$ for $1 \le j < \max(\sigma)$, and

$$\omega(C_i) = J_{\sigma(i)}.$$

First suppose that $C$ is an H-interval of $\omega$. Let $a = \min \sigma(T)$ and $b = \max \sigma(T)$. Since $a, b \in \sigma(T)$, we have $J_a \cup J_b \subseteq \omega(C)$. As $\omega(C)$ is a value interval, $J_j \subseteq \omega(C)$ for every $a \leq j \leq b$. If $\sigma(i) \in [a, b]$, it follows that $\omega(C_i) \subseteq \omega(C)$. The saturation of $C$ then gives $C_i \subseteq C$, and hence $i \in T$. The reverse implication $i \in T \Rightarrow \sigma(i) \in [a, b]$ follows from the definitions of $a$ and $b$. Therefore,

$$\sigma^{-1}([a, b]) = T.$$

Since $\sigma$ is a Cayley permutation, every value in $[a, b]$ occurs in $\sigma$ and therefore, by this equality, in $\sigma(T)$. Thus $\sigma(T) = [a, b]$, and $T$ is an H-interval of $\sigma$.

Conversely, suppose that $T$ is an H-interval of $\sigma$, with $\sigma(T) = [a, b]$ and $\sigma^{-1}([a, b]) = T$. Let

$$J = J_a \cup J_{a+1} \cup \cdots \cup J_b.$$

Because the value bands $J_1, \ldots, J_{\max(\sigma)}$ cover $[\max(\omega)]$, the union $J = J_a \cup \cdots \cup J_b$ equals $[\min J_a, \max J_b]$ and is therefore a value interval. Moreover,

$$\omega(C) = \bigcup_{i \in T} \omega(C_i) = \bigcup_{i \in T} J_{\sigma(i)} = J.$$

Every position $p$ of $\omega$ belongs to exactly one part $C_i$, and $\omega(p) \in J$ if and only if $\sigma(i) \in [a, b]$. Since $\sigma^{-1}([a, b]) = T$, this is equivalent to $i \in T$. Hence $\omega^{-1}(J) = C$. Finally, $C$ is a positional interval because it is a union of consecutive parts. Thus $C$ is an H-interval of $\omega$. □

**Theorem 3.4.** *Let $\omega$ be a Cayley permutation of length at least two. Then there is a unique simple Cayley permutation $\sigma$ of length $k \geq 2$ such that*

$$\omega = \sigma[\alpha_1, \ldots, \alpha_k],$$

*for some nonempty Cayley permutations $\alpha_1, \ldots, \alpha_k$ satisfying (2). If the quotient $\sigma \notin \{12, 21\}$, then the components $\alpha_i$ are uniquely determined. If $\sigma = 12$, they are unique if we require $\alpha_1$ to be plus indecomposable. If $\sigma = 21$, they are unique if we require $\alpha_1$ to be minus indecomposable.*

*Proof.* Assume $\omega \in \mathrm{Cay}_n$ with $n \geq 2$. We first prove existence. Among all inflations $\omega = \sigma[\alpha_1, \ldots, \alpha_k]$ with $k \geq 2$, choose one with $k$ minimal. Such an inflation exists, since $\omega = \omega[1, \ldots, 1]$. If $\sigma$ were not simple, it would have an H-interval $T$ with $2 \leq |T| < k$. By Lemma 3.3, the union of the parts indexed by $T$ is an H-interval of $\omega$. Merge these parts into one, standardizing the factor on their union. The merged value band shares no value with any remaining part, so its letter in the new quotient occurs only once; the restrictions at all other repeated quotient letters are preserved. The resulting inflation has $k - |T| + 1$ parts. Since $2 \leq |T| < k$, this number lies between 2 and $k - 1$, contradicting the minimality of $k$. Hence $\sigma$ is simple.

We now prove uniqueness. Suppose first that $\omega$ is plus decomposable. Thus, for some $b \in [n - 1]$, we have $\omega(i) < \omega(j)$ for all positions $i \leq b < j$. Consider any expression of $\omega$ as an inflation of a simple quotient with at least two parts. If $b$ is a boundary between parts, the parts on either side express the quotient as a direct sum. If $b$ lies inside a part, that part has at least two positions, so its quotient letter occurs only once by (2). Its value band contains values from positions on both sides of $b$.

Since the value bands of distinct quotient letters are disjoint, every earlier quotient letter is smaller and every later quotient letter is larger. Separating the quotient immediately before or after this letter, choosing the option that leaves both sides nonempty, expresses it as a direct sum. If the quotient had length at least three, one summand would be a proper H-interval of length at least two. Thus a simple quotient must be 12.

Choose the smallest $b \in [n-1]$ with this property. The standardized prefix is plus indecomposable: otherwise its direct-sum decomposition would give a smaller such value of $b$. For any other $b' > b$ with this property, the standardized prefix ending at $b'$ is plus decomposable, with its first summand ending at $b$. Hence $b$ gives the unique decomposition whose first component is plus indecomposable. Reversing the value comparisons proves that a minus decomposable Cayley permutation has simple quotient 21 and a unique decomposition whose first component is minus indecomposable.

It remains to consider $\omega$ that is neither plus nor minus decomposable. Its maximal proper H-intervals are pairwise disjoint. Indeed, if two distinct ones overlapped, their union would be $[n]$ by Lemma 3.1 and maximality. One would be a proper prefix and the other a proper suffix. Their value bands cover $[\max(\omega)]$, and neither contains both extreme values, since saturation would make the corresponding H-interval equal to $[n]$. The prefix must therefore contain 1 or $\max(\omega)$, so Lemma 3.2 gives a direct-sum or skew-sum decomposition, a contradiction.

These maximal H-intervals, together with the singleton sets formed by the remaining positions, therefore partition $[n]$. Consider any inflation of $\omega$ with a simple quotient of length at least two. Each nonsingleton part is a proper H-interval and is therefore contained in a maximal one; each singleton part is contained in the unique member of the partition that contains its position. Thus the parts refine this partition. If a maximal H-interval were the union of at least two parts, Lemma 3.3 would give a proper nonsingleton H-interval of the quotient, contradicting its simplicity. Consequently, the parts are exactly the maximal H-intervals and the uncovered singleton positions. Their standardized factors and their quotient are uniquely determined. □

The remaining simple quotient of length two, $\sigma = 11$, causes no ambiguity. Its two positions carry the same value, so (2) forces both components to be 1; its only inflation is the Cayley permutation $11 = 11[1, 1]$ itself.

## 4 Enumeration of simple Cayley permutations

We shall adapt the enumeration of simple permutations [2] to Cayley permutations. The main difference is that a position carrying a repeated letter in the quotient can only be inflated by the one-letter Cayley permutation 1. Given $\omega \in \mathrm{Cay}$, let

$$\mu(\omega) = \#\{j : |\omega^{-1}(j)| = 1\}$$

denote the number of singleton fibres of $\omega$, and define the bivariate ordinary generating function

$$B(x,t) = \sum_{\omega \in \mathrm{Cay}_+} x^{|\omega|} t^{\mu(\omega)} = \sum_{n \geq 1} B_n(t) x^n, \qquad B_n(t) = \sum_{\omega \in \mathrm{Cay}_n} t^{\mu(\omega)}.$$

The ballot viewpoint is convenient to determine an exponential generating function for the polynomials $B_n(t)$.

**Proposition 4.1.** *The polynomials $B_n(t)$ are determined by*

$$1 + \sum_{n \geq 1} B_n(t) \frac{z^n}{n!} = \frac{1}{2 - e^z - (t-1)z}.$$

*Proof.* View a Cayley permutation as a ballot. A singleton block has exponential generating function $tz$, while a block of size at least two has exponential generating function $e^z - 1 - z$. Taking a sequence of such nonempty blocks gives

$$\frac{1}{1 - \left(tz + e^z - 1 - z\right)} = \frac{1}{2 - e^z - (t-1)z}.$$

□

The first few of these polynomials are listed in Table 1.

| $n$ | $B_n(t)$ |
|---|---|
| 1 | $t$ |
| 2 | $1 + 2t^2$ |
| 3 | $1 + 6t + 6t^3$ |
| 4 | $7 + 8t + 36t^2 + 24t^4$ |
| 5 | $21 + 100t + 60t^2 + 240t^3 + 120t^5$ |
| 6 | $141 + 372t + 1170t^2 + 480t^3 + 1800t^4 + 720t^6$ |
| 7 | $743 + 3584t + 5166t^2 + 13440t^3 + 4200t^4 + 15120t^5 + 5040t^7$ |

Table 1: The polynomials $B_n(t) = \sum_{\omega \in \mathrm{Cay}_n} t^{\mu(\omega)}$ for $n \leq 7$.

Let $\left\{ {n \atop k} \right\}$ denote the Stirling number of the second kind, that is, the number of partitions of $[n]$ into $k$ blocks. Let $\left\{ {n \atop k} \right\}_{\geq 2}$ denote the number of such partitions in which every block has size at least two.

**Proposition 4.2.** *For $n \geq 1$,*

$$B_n(t) = \sum_{k=1}^{n} k! \sum_{j=0}^{k} \binom{n}{j} \left\{ {n-j \atop k-j} \right\} (t-1)^j.$$

*Moreover, for $0 \leq j \leq n$,*

$$[t^j]\, B_n(t) = \binom{n}{j} \sum_{i \geq 0} (i+j)! \left\{ {n-j \atop i} \right\}_{\geq 2}.$$

*Proof.* Using

$$(e^z - 1)^r = r! \sum_{m \geq 0} \left\{ {m \atop r} \right\} \frac{z^m}{m!},$$

we can expand the generating function in Proposition 4.1 as

$$\begin{aligned}
1+\sum_{n\geq 1} B_n(t)\frac{z^n}{n!} &= \sum_{k\geq 0}\left(e^z-1+(t-1)z\right)^k \\
&= \sum_{k\geq 0}\sum_{j=0}^{k}\binom{k}{j}(t-1)^j z^j (e^z-1)^{k-j} \\
&= \sum_{k\geq 0}\sum_{j=0}^{k}\binom{k}{j}(t-1)^j z^j (k-j)!\sum_{m\geq 0}\left\{{m \atop k-j}\right\}\frac{z^m}{m!} \\
&= 1+\sum_{n\geq 1}\left(\sum_{k=1}^{n} k!\sum_{j=0}^{k}\binom{n}{j}\left\{{n-j \atop k-j}\right\}(t-1)^j\right)\frac{z^n}{n!}.
\end{aligned}$$

Comparing coefficients of $z^n/n!$ proves the first formula. For the second formula, use

$$(e^z-1-z)^i = i!\sum_{m\geq 0}\left\{{m \atop i}\right\}_{\geq 2}\frac{z^m}{m!}.$$

Extracting the coefficient of $t^j$ from the generating function and expanding gives

$$\begin{aligned}
[t^j]\left(1+\sum_{n\geq 1} B_n(t)\frac{z^n}{n!}\right) &= [t^j]\sum_{r\geq 0}\left(tz+e^z-1-z\right)^r \\
&= z^j\sum_{i\geq 0}\binom{i+j}{j}(e^z-1-z)^i \\
&= z^j\sum_{i\geq 0}\binom{i+j}{j} i!\sum_{m\geq 0}\left\{{m \atop i}\right\}_{\geq 2}\frac{z^m}{m!} \\
&= \sum_{n\geq j}\left(\binom{n}{j}\sum_{i\geq 0}(i+j)!\left\{{n-j \atop i}\right\}_{\geq 2}\right)\frac{z^n}{n!}.
\end{aligned}$$

Comparing coefficients of $z^n/n!$ proves the second formula.

Alternatively, the latter formula follows by a direct combinatorial argument. A ballot of $[n]$ with exactly $j$ singletons is obtained by choosing the $j$ elements in those blocks, partitioning the remaining $n-j$ elements into $i$ blocks of size at least two, and linearly ordering all $i+j$ blocks. For each $i$, these choices give $\binom{n}{j}(i+j)!\left\{{n-j \atop i}\right\}_{\geq 2}$ ballots, and summing over $i$ gives the formula. □

To enumerate simple Cayley permutations and several of their subclasses at once, let $Q$ be a property of nonempty Cayley permutations such that $Q(1)$ holds and, for every inflation,

$$Q(\sigma[\alpha_1,\ldots,\alpha_k]) \quad\Longleftrightarrow\quad Q(\sigma) \text{ and } Q(\alpha_i) \text{ for every } i\in[k]. \tag{4}$$

No closure under classical containment is required. Define

$$F_Q(x,t) = \sum_{\substack{\omega\in \mathrm{Cay}_+ \\ Q(\omega)}} x^{|\omega|}t^{\mu(\omega)}, \qquad H_Q(x,y) = \sum_{\substack{\sigma \text{ simple},\ |\sigma|\geq 3 \\ Q(\sigma)}} x^{|\sigma|-\mu(\sigma)}y^{\mu(\sigma)}.$$

The two variables in $H_Q$ distinguish positions in non-singleton fibres, which can only be inflated trivially, from positions in singleton fibres, which may be inflated by arbitrary components.

**Theorem 4.3.** *For a property $Q$ satisfying $Q(1)$ and (4), let $\chi_\omega$ be 1 if $Q(\omega)$ holds and 0 otherwise, for each word $\omega$. Then*

$$F_Q(x,t) = tx + \chi_{11}x^2 + \frac{(\chi_{12}+\chi_{21})F_Q(x,t)^2}{1+F_Q(x,t)} + H_Q(x, F_Q(x,t)).$$

*Proof.* Write $F = F_Q(x,t)$. We partition the words counted by $F$ according to the unique simple quotient supplied by Theorem 3.4.

The word 1 has weight $tx$. For the quotient 11, the restriction in (2) forces both components to be 1, so this quotient contributes $\chi_{11}x^2$.

Now suppose that $Q(12)$ holds, and let $I_+$ be the generating function for nonempty plus indecomposable words satisfying $Q$. Every word counted by $F$ is uniquely either plus indecomposable or a direct sum $12[\alpha,\beta]$ in which $\alpha$ is plus indecomposable. Moreover, by (4), both components satisfy $Q$. Since the weight is multiplicative under direct sums, this gives

$$F = I_+ + I_+F = I_+(1+F) \quad\Longrightarrow\quad I_+ = \frac{F}{1+F}.$$

Hence the words whose quotient is 12 contribute

$$I_+F = \frac{F^2}{1+F}.$$

If $Q(12)$ does not hold, (4) excludes every nontrivial direct sum. The same argument with skew sums and minus indecomposable words applies to the quotient 21. Thus the quotients 12 and 21 together contribute

$$(\chi_{12}+\chi_{21})\frac{F^2}{1+F}.$$

It remains to consider a simple quotient $\sigma$ of length at least three. If a value occurs more than once in $\sigma$, then (2) forces the component at each of its positions to be 1; each such position contributes $x$. A value that occurs once may instead be inflated by any word counted by $F$, and therefore contributes $F$. There are $|\sigma| - \mu(\sigma)$ positions of the first kind and $\mu(\sigma)$ of the second. Hence the inflations of $\sigma$ contribute

$$x^{|\sigma|-\mu(\sigma)}F^{\mu(\sigma)}.$$

By (4), such inflations satisfy $Q$ exactly when $\sigma$ does. Summing over the eligible quotients gives $H_Q(x,F)$. This substitution is well defined as a formal series: since $F$ has zero constant term, the displayed contribution has degree at least $|\sigma|$. Adding the four contributions proves the formula. □

We now take $Q$ to be the property that holds for every nonempty Cayley permutation. In this unrestricted case, $F_Q = B$ and $\chi_{11} = \chi_{12} = \chi_{21} = 1$. Write $S_{\geq 3} = H_Q$; explicitly,

$$S_{\geq 3}(x,y) = \sum_{\substack{\sigma \text{ simple} \\ |\sigma|\geq 3}} x^{|\sigma|-\mu(\sigma)}y^{\mu(\sigma)}.$$

Theorem 4.3 now specializes as follows.

**Theorem 4.4.** *The generating function for simple Cayley permutations satisfies*

$$B(x,t) = tx + x^2 + \frac{2B(x,t)^2}{1+B(x,t)} + S_{\geq 3}(x, B(x,t)).$$

The identity in Theorem 4.4 rewrites as

$$S_{\geq 3}(x, B(x,t)) = B(x,t) - tx - x^2 - \frac{2B(x,t)^2}{1+B(x,t)}.$$

In their enumeration of simple permutations, Albert, Atkinson, and Klazar [2] use the coefficients of the compositional inverse of $\sum_{n\geq 1} n!x^n$. These coefficients were first studied by Comtet [17, p. 171]. We use the following general inversion lemma.

**Lemma 4.5.** *Let $F(x,t) = tx + \sum_{k\geq 2} F_k(t)x^k$, where $F_k(t) \in \mathbb{Z}[t]$. There is a unique formal power series $V(x) \in \mathbb{Z}[[x]]$ such that $F(x, V(x)) = x$. Its coefficients $v_n = [x^n]V(x)$ are given by $v_0 = 1$ and, for $n \geq 1$, by*

$$v_n = -\sum_{k=2}^{n+1} [x^{n+1-k}]F_k(V(x)),$$

*where the right-hand side depends only on $v_0, \ldots, v_{n-1}$.*

*Proof.* Substituting $t = V(x)$ into the given expansion of $F$ turns the required identity $F(x, V(x)) = x$ into

$$xV(x) + \sum_{k\geq 2} F_k(V(x))x^k = x.$$

Dividing by $x$ and rearranging gives

$$V(x) = 1 - \sum_{k\geq 2} F_k(V(x))x^{k-1}.$$

Its constant term gives $v_0 = 1$. For $n \geq 1$, extracting the coefficient of $x^n$ gives the stated recurrence; terms with $k > n+1$ make no contribution. For each $2 \leq k \leq n+1$, the coefficient $[x^{n+1-k}]F_k(V(x))$ depends only on $v_0, \ldots, v_{n+1-k}$, and $n+1-k \leq n-1$. The recurrence therefore determines $v_n$ from the preceding coefficients. Starting with $v_0 = 1$ constructs a unique solution coefficient by coefficient, and all its coefficients are integers because every $F_k$ has integer coefficients. □

**Corollary 4.6.** *Under the hypotheses of Theorem 4.3, let $S_Q(x)$ count the nonempty simple Cayley permutations satisfying $Q$, and let $V_Q(x)$ be the unique series with $F_Q(x, V_Q(x)) = x$. Then*

$$S_Q(x) = 2x - xV_Q(x) + \frac{(\chi_{12} + \chi_{21})x^3}{1+x}.$$

*Proof.* Since $Q(1)$ holds, the coefficient of $x$ in $F_Q(x,t)$ is $t$, so Lemma 4.5 supplies $V_Q$. Substituting $t = V_Q(x)$ in Theorem 4.3 gives

$$H_Q(x,x) = x - xV_Q(x) - \chi_{11}x^2 - \frac{(\chi_{12} + \chi_{21})x^2}{1+x}.$$

The series $H_Q$ includes only simple words of length at least three. Every Cayley permutation of length one or two is simple, so, using $Q(1)$, we have

$$H_Q(x,x) = S_Q(x) - x - (\chi_{11} + \chi_{12} + \chi_{21})x^2.$$

Combining the two identities proves the formula; the contributions from 11 cancel. □

In the unrestricted case, where $Q$ holds for every nonempty Cayley permutation, write $U = V_Q$. Thus $U$ is the unique series with

$$B(x, U(x)) = x.$$

Write $u_n = [x^n]U(x)$. By the recurrence in Lemma 4.5, for example, $u_1 = -B_2(u_0) = -B_2(1) = -3$. The first terms of $U(x)$ are

$$\begin{aligned} U(x) = 1 - 3x - x^2 - 17x^3 - 95x^4 - 871x^5 - 9189x^6 \\ - 112111x^7 - 1537157x^8 - 23315921x^9 + \cdots \end{aligned}$$

**Corollary 4.7.** *Let $s_n$ be the number of simple Cayley permutations of length $n$ and let $S(x) = \sum_{n\geq 1} s_n x^n$. Then*

$$S(x) = 2x + \frac{2x^3}{1+x} - xU(x).$$

*Proof.* This is Corollary 4.6 with $V_Q = U$, since both 12 and 21 satisfy the unrestricted property. □

Comparing coefficients in Corollary 4.7 gives

$$s_n = -u_{n-1} + 2(-1)^{n-1}$$

for $n \geq 3$. Hence

$$(s_n)_{n\geq 1} = 1, 3, 3, 15, 97, 869, 9191, 112109, 1537159, 23315919, \ldots \tag{5}$$

For reference, Table 2 shows these values, alongside the Fubini numbers $|\mathrm{Cay}_n|$ and the number of primitive simple Cayley permutations $\hat{s}_n$ of Theorem 5.3 below.

## 5 Modules and primitive Cayley permutations

A *relational structure* is a set together with a collection of binary relations on it. Analogues of simplicity and the substitution decomposition may be defined for any relational structure (e.g., graphs and posets). Schmerl and Trotter [22] defined *indecomposable* relational structures and proved that a relational structure is indecomposable if and only if it contains no nontrivial *module* (sometimes called an interval in the literature). Here, a module is a subset $M$ of the structure such that, for every relation $R$, every $i, j \in M$, and every $x \notin M$,

$$iRx \iff jRx \qquad \text{and} \qquad xRi \iff xRj.$$

| $n$ | $\lvert\mathrm{Cay}_n\rvert$ | $s_n$ | $\hat{s}_n$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 3 | 3 | 2 |
| 3 | 13 | 3 | 2 |
| 4 | 75 | 15 | 10 |
| 5 | 541 | 97 | 70 |
| 6 | 4683 | 869 | 634 |
| 7 | 47293 | 9191 | 6742 |
| 8 | 545835 | 112109 | 82306 |
| 9 | 7087261 | 1537159 | 1126846 |
| 10 | 102247563 | 23315919 | 17050626 |

Table 2: Cayley permutations $\lvert\mathrm{Cay}_n\rvert$, simple Cayley permutations $s_n$, and primitive simple Cayley permutations $\hat{s}_n$.

In other words, every two elements of $M$ are ordered with respect to elements not in $M$ in exactly the same way with respect to all relations. A module is nontrivial if it has at least two elements and is a proper subset of the ground set. Albert and Atkinson [1] encoded a permutation $\pi$ of $[n]$ as the poset on $[n]$ where $x \prec y$ if and only if $x < y$ and $\pi(x) < \pi(y)$, so that $\pi$ is simple exactly when this poset is indecomposable in the relational structure sense. This approach allowed them to use Schmerl and Trotter's [22] characterization of critically indecomposable structures to show that the only exceptional permutations are the parallel alternations and their symmetries, a result we extend to Cayley permutations in Section 5.2.

The presence of repeated entries requires a slightly different approach. To each Cayley permutation, we will associate a relational structure so that indecomposable structures correspond to Cayley permutations that are simple and primitive, where a Cayley permutation is *primitive* [15] if it contains no flat steps, i.e. pairs of equal adjacent letters. Given a Cayley permutation $\omega \in \mathrm{Cay}_n$, define the relational structure

$$R(\omega) = ([n], <_1, <_2)$$

on the set of positions $[n]$, carrying two binary relations: the position order $i <_1 j$ if $i < j$, and the value order $i <_2 j$ if $\omega(i) < \omega(j)$. Both relations are irreflexive; $<_1$ is a linear order, and $<_2$ is a strict weak order whose ties are the positions of equal value. The main difference compared to Albert and Atkinson [1] is that the two orders are kept apart, and not merged into a single relation.

The empty set is a module, since its defining condition is vacuous. The next lemma characterizes the nonempty modules of $R(\omega)$.

**Lemma 5.1.** *The nonempty modules of $R(\omega)$ are the H-intervals of $\omega$ together with its nonempty positional intervals with constant value, which in the ballot are the nonempty positional intervals contained in a single block.*

*Proof.* Let $M$ be a nonempty module. If a position $x \notin M$ lay between two positions of $M$, then $y <_1 x$ for some $y \in M$ and $x <_1 y'$ for some $y' \in M$, contradicting

the module condition. Thus $M$ is a positional interval. If $\omega$ is constant on $M$, then $M$ is contained in a single block. Otherwise, the module condition for $<_2$, in both directions, forces each position outside $M$ to lie in a block below every block intersecting $M$ or above every such block. In particular, no position outside $M$ lies in a block intersecting $M$, so $M$ is a union of whole blocks. Those blocks are consecutive, since a block between two of them would be nonempty and its positions neither below nor above all of $M$. By Proposition 2.3, $M$ is an H-interval. Conversely, an H-interval is a module because every block outside it lies wholly below or wholly above it. A positional interval contained in a single block is also a module. □

Under the encoding employed by Albert and Atkinson [1], where $<_1$ and $<_2$ are merged into one relation, the correspondence between the intervals of a permutation and the modules of the corresponding relational structure is only partial. For instance, the positions 2 and 4 in 1432 form a module of the poset, but not an interval. Keeping $<_1$ and $<_2$ separate restores the exact correspondence in our setting. Indeed, if $\omega$ is a permutation then the only positional intervals with constant value are the singletons, which are H-intervals. Thus for permutations the nonempty modules of $R(\omega)$ are exactly the H-intervals of $\omega$. In general, the correspondence between modules and H-intervals breaks on Cayley permutations: a repeated value makes each single position carrying that value a module that is not an H-interval.

**Proposition 5.2.** *Let $\omega \in \mathrm{Cay}_n$ with $n \geq 3$. The structure $R(\omega)$ is indecomposable if and only if $\omega$ is simple and primitive.*

*Proof.* Suppose $\omega$ is simple and primitive, and let $M$ be a module with $2 \leq |M| < n$. By Lemma 5.1, $M$ is either an H-interval or a constant positional interval. The first is impossible because every H-interval of a simple Cayley permutation is a singleton or $[n]$. The second is impossible because a constant interval of length at least two contains a flat step, which primitivity forbids. No such $M$ exists, so $R(\omega)$ is indecomposable.

Conversely, let $R(\omega)$ be indecomposable. Every H-interval is a module, so every H-interval is a singleton or $[n]$, which is to say that $\omega$ is simple. If $\omega(i) = \omega(i+1)$ for some $i$, then $\{i, i+1\}$ is a constant positional interval and hence a module; it is not a singleton, and it is not all of $[n]$ because $n \geq 3$, contradicting indecomposability. Hence $\omega$ is primitive. □

The hypothesis $n \geq 3$ is needed only for the implication from indecomposability to primitivity, since 11 is simple and indecomposable but not primitive. More generally, a simple Cayley permutation is indecomposable unless it contains a flat step. An alternative definition of simplicity could be obtained by letting a Cayley permutation $\omega$ be simple if $R(\omega)$ has no nontrivial modules. This would make 111 not simple due to the presence of the two modules $[1, 2]$ and $[2, 3]$ (both being positional intervals with constant value). This definition would, however, be incompatible with the substitution decomposition. Contracting either nontrivial module of 111 gives 11, but 111 is not an inflation of 11, since the positions of a repeated value cannot be inflated under (2).

### 5.1 Enumeration of primitive simple Cayley permutations

We now enumerate the primitive simple Cayley permutations, which by Proposition 5.2 are (for length at least three) exactly the Cayley permutations whose associated structure is indecomposable. Primitivity satisfies (4). Indeed, every component of a primitive inflation is primitive because it is a standardized factor. Its quotient is primitive too: adjacent equal quotient letters have forced one-letter components and would give adjacent equal letters in the inflation. Conversely, if the quotient and components are primitive, there are no equal adjacent letters within a component, and adjacent components use disjoint value bands because their quotient letters differ.

Write

$$P(x,t) = \sum_{\substack{\omega \in \mathrm{Cay}_+ \\ \omega \text{ primitive}}} x^{|\omega|} t^{\mu(\omega)} = \sum_{n \geq 1} P_n(t) x^n$$

for the primitive analogue of $B(x,t)$ and let

$$\widehat{S}_{\geq 3}(x,y) = \sum_{\substack{\sigma \text{ primitive simple} \\ |\sigma| \geq 3}} x^{|\sigma| - \mu(\sigma)} y^{\mu(\sigma)}.$$

**Theorem 5.3.** *The generating function for primitive simple Cayley permutations satisfies*

$$\widehat{S}_{\geq 3}(x, P(x,t)) = P(x,t) - tx - \frac{2P(x,t)^2}{1 + P(x,t)}.$$

*Let $W(x)$ be the unique formal power series with $P(x, W(x)) = x$. Then the ordinary generating function $\widehat{S}(x) = \sum_{n \geq 1} \hat{s}_n x^n$ for primitive simple Cayley permutations is*

$$\widehat{S}(x) = 2x + \frac{2x^3}{1+x} - xW(x).$$

*Proof.* Apply the results of Section 4 to the property of being primitive, which satisfies (4) by the argument above. Here $F_Q = P$, the word 11 is excluded, and both 12 and 21 are admitted. Theorem 4.3 gives the first identity, Lemma 4.5 supplies $W$, and Corollary 4.6, with $V_Q = W$, gives the second. □

The formulas for $S$ and $\widehat{S}$ have the same shape because they have the same indicators for 12 and 21; the indicator for 11 cancels in Corollary 4.6.

To compute the coefficients of $\widehat{S}$, we express $P$ in terms of $B$. Define

$$\widetilde{B}(x,t) = \sum_{\omega \in \mathrm{Cay}_+} x^{|\omega| - \mu(\omega)} t^{\mu(\omega)} \quad \text{and} \quad \widetilde{P}(x,t) = \sum_{\substack{\omega \in \mathrm{Cay}_+ \\ \omega \text{ primitive}}} x^{|\omega| - \mu(\omega)} t^{\mu(\omega)}.$$

Here $t$ marks positions in singleton fibres and $x$ marks all other positions. Contracting each maximal constant run to a single letter gives a unique primitive Cayley permutation. Conversely, we recover the original word by expanding each position into its corresponding nonempty constant run. A position in a singleton fibre may expand into a run of length one, contributing $t$, or a run of length at least two, whose

value is then repeated. Summing the weights of the latter runs gives $x^2/(1-x)$, so the possible expansions of that position have total weight

$$t + \frac{x^2}{1-x},$$

whereas the possible expansions of each position carrying a repeated value have total weight $x/(1-x)$. Consequently,

$$\widetilde{B}(x,t) = \widetilde{P}\left(\frac{x}{1-x},\, t + \frac{x^2}{1-x}\right).$$

Since $B(x,t) = \widetilde{B}(x,tx)$ and $P(x,t) = \widetilde{P}(x,tx)$, we have

$$B(x,t) = P\left(\frac{x}{1-x},\, t(1-x) + x\right),$$

and inverting this substitution gives

$$P(x,t) = B\left(\frac{x}{1+x},\, (1+x)t - x\right).$$

Proposition 4.2 determines the coefficient polynomials of $B$, so this identity determines those of $P$. Lemma 4.5 then computes the coefficients of $W$, and Theorem 5.3 gives those of $\widehat{S}$.

The coefficients of $\widehat{S}$ begin

$$(\hat{s}_n)_{n\geq 1} = 1, 2, 2, 10, 70, 634, 6742, 82306, 1126846, 17050626, 282097790, \ldots, \tag{6}$$

agreeing with the number of associated structures $R(\omega)$ that are indecomposable except at $n = 2$, where 11 is indecomposable but not primitive. We have found neither the sequence (6) in the OEIS [20] nor a closed form more explicit than the generating-function identity in Theorem 5.3.

The asymptotics of the numbers $s_n$ and $\hat{s}_n$ are obtained in the companion paper [14], where we show that

$$s_n \sim \frac{|\mathrm{Cay}_n|}{2\sqrt{2}} \sim \frac{n!}{4\sqrt{2}\,(\log 2)^{n+1}}, \qquad \hat{s}_n \sim \frac{|\mathrm{Cay}_n|}{4} \sim \frac{n!}{8(\log 2)^{n+1}}.$$

## 5.2 Deletions and exceptional Cayley permutations

Recall that every simple permutation of length $n \geq 2$ contains a simple permutation of length $n-1$ or $n-2$, by Schmerl and Trotter [22]. Those for which no one-point deletion is simple are called exceptional. Albert and Atkinson [1] showed that they are precisely the simple parallel alternations and their symmetries.

We wish to prove that both statements carry over to Cayley permutations. To start, we take care of Cayley permutations that are not primitive by showing that removing a flat step from a simple Cayley permutation preserves simplicity. On the other hand, by Proposition 5.2 Cayley permutations that are simple and primitive correspond to indecomposable structures, enabling us to use Schmerl and Trotter's result.

Given $\omega = w_1 \cdots w_n \in \mathrm{Cay}_n$ and a position $i \in [n]$, let

$$\omega \setminus i = \mathrm{st}(w_1 \cdots w_{i-1} w_{i+1} \cdots w_n)$$

denote the one-point deletion at $i$.

**Proposition 5.4.** *Let $\omega = w_1 \cdots w_n \in \mathrm{Cay}_n$ be simple with $n \geq 2$, and suppose $w_i = w_{i+1}$. Then $\omega \setminus (i+1)$ is a simple Cayley permutation of length $n-1$.*

*Proof.* Let $\omega' = w_1 \cdots w_i w_{i+2} \cdots w_n$ and consider the order-preserving surjection

$$q\colon [n] \to [n-1], \qquad q(j) = \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{if } j > i. \end{cases}$$

Since $w_i = w_{i+1}$, we have $\omega(j) = \omega'(q(j))$ for every $j$. Thus $\omega$ and $\omega'$ use the same values, so $\omega'$ is already a Cayley permutation and $\omega \setminus (i+1) = \omega'$.

Let $I'$ be an H-interval of $\omega'$ with value band $J$. Its preimage $I = q^{-1}(I')$ is a nonempty positional interval, since $q$ is order-preserving and surjective. Moreover,

$$\omega(I) = \omega'(I') = J, \qquad \omega^{-1}(J) = q^{-1}\big((\omega')^{-1}(J)\big) = I.$$

Hence $I$ is an H-interval of $\omega$, so simplicity makes it a singleton or $[n]$. Taking its image under $q$ makes $I' = q(I)$ a singleton or $[n-1]$, proving that $\omega'$ is simple. □

**Theorem 5.5.** *Every simple Cayley permutation of length $n \geq 2$ contains a simple Cayley permutation of length $n-1$ or $n-2$, obtained by deleting one or two letters and standardizing.*

*Proof.* Let $\omega \in \mathrm{Cay}_n$ be simple. For $n = 2$, either deletion leaves 1; for $n = 3, 4$, keeping any two positions and standardizing leaves a Cayley permutation of length two, which is simple. We may therefore assume that $n \geq 5$. If $\omega$ is not primitive, it has a flat step, and Proposition 5.4 supplies a simple Cayley permutation of length $n-1$. Suppose then that $\omega$ is primitive. By Proposition 5.2, $R(\omega)$ is indecomposable. Since its relations are binary and irreflexive, a result of Schmerl and Trotter [22, Corollary 2.3] shows that $R(\omega)$ has an indecomposable induced substructure on $n-1$ or $n-2$ points. Standardization preserves both the position order and the value order, so the substructure of $R(\omega)$ induced on a set $S$ of positions is isomorphic to $R(\mathrm{st}(\omega|_S))$; take $S$ with $R(\mathrm{st}(\omega|_S))$ indecomposable and $|S| \in \{n-1, n-2\}$. Since $|S| \geq 3$, Proposition 5.2 makes $\mathrm{st}(\omega|_S)$ simple. □

For $n \geq 2$, we call a simple Cayley permutation $\omega \in \mathrm{Cay}_n$ *exceptional* if $\omega \setminus i$ is not simple for each $i \in [n]$. The next result is a corollary of Proposition 5.4.

**Corollary 5.6.** *An exceptional Cayley permutation is primitive.*

Proving that the parallel alternations are the only exceptional Cayley permutations requires a bit more work. Following Schmerl and Trotter [22], we say that two ordered pairs $(a, b)$ and $(c, d)$ of distinct elements of a relational structure have the same *type* if $aR_ib \Leftrightarrow cR_id$ and $bR_ia \Leftrightarrow dR_ic$ for every relation $R_i$. The equivalence classes of ordered pairs induced by the type relation form the *type skeleton*.

As an example, consider the Cayley permutation $\omega = 231146561$. In the type skeleton of $R(\omega)$, the type of the arc $i \to j$ records the comparison of the positions $i, j$ and the comparison of the values $\omega(i), \omega(j)$. There are six possible types:

$$A = (<,<), \qquad B = (<,=), \qquad C = (<,>),$$
$$D = (>,<), \qquad E = (>,=), \qquad F = (>,>),$$

where the first sign compares $i$ with $j$ and the second compares $\omega(i)$ with $\omega(j)$. The following matrix displays the type skeleton of $R(\omega)$. Its rows and columns are labelled $i_{\omega(i)}$, and the entry in row $i_{\omega(i)}$ and column $j_{\omega(j)}$ is the type of the ordered pair $(i, j)$:

| | $1_2$ | $2_3$ | $3_1$ | $4_1$ | $5_4$ | $6_6$ | $7_5$ | $8_6$ | $9_1$ |
|---|---|---|---|---|---|---|---|---|---|
| $1_2$ | $-$ | $A$ | $C$ | $C$ | $A$ | $A$ | $A$ | $A$ | $C$ |
| $2_3$ | $F$ | $-$ | $C$ | $C$ | $A$ | $A$ | $A$ | $A$ | $C$ |
| $3_1$ | $D$ | $D$ | $-$ | $B$ | $A$ | $A$ | $A$ | $A$ | $B$ |
| $4_1$ | $D$ | $D$ | $E$ | $-$ | $A$ | $A$ | $A$ | $A$ | $B$ |
| $5_4$ | $F$ | $F$ | $F$ | $F$ | $-$ | $A$ | $A$ | $A$ | $C$ |
| $6_6$ | $F$ | $F$ | $F$ | $F$ | $F$ | $-$ | $C$ | $B$ | $C$ |
| $7_5$ | $F$ | $F$ | $F$ | $F$ | $F$ | $D$ | $-$ | $A$ | $C$ |
| $8_6$ | $F$ | $F$ | $F$ | $F$ | $F$ | $E$ | $F$ | $-$ | $C$ |
| $9_1$ | $D$ | $D$ | $E$ | $E$ | $D$ | $D$ | $D$ | $D$ | $-$ |

For instance, the type of $(1, 2)$ is $A = (<, <)$ since we have positions $1 < 2$ and values $\omega(1) < \omega(2)$. In the matrix, two ordered pairs have the same type exactly when their cells carry the same letter, and the type skeleton is the partition of the off-diagonal cells into the six letter classes. Note also that the matrix makes the modules transparent: a set is a module when its rows agree outside its own columns. For instance, the set $\{1, 2\}$ is a module since rows $1_2, 2_3$ agree outside columns $1_2, 2_3$; it is also an H-interval, occupying consecutive positions and the saturated value band $\{2, 3\}$. Similarly, $[5, 8]$ is a module as well as an H-interval with saturated value band $\{4, 5, 6\}$. These are the maximal proper H-intervals. The set $\{3, 4\}$ is another module: it is a constant positional interval, but not an H-interval since the value 1 occurs again at position 9 (hence saturation fails). The H-interval decomposition of $\omega$ is therefore

$$23 \mid 1 \mid 1 \mid 4656 \mid 1,$$

which by contraction gives the simple quotient 21131. Conversely, $\omega$ is obtained as the inflation

$$231146561 = 21131[12, 1, 1, 1323, 1].$$

Now, in the language of Schmerl and Trotter, an indecomposable structure is *critically indecomposable* if no one-point deletion of the structure is indecomposable. The two authors [22, Theorem 5.1] have fully classified the type skeletons of critically indecomposable structures. We will only need to count their types to show that no Cayley permutations except for the parallel alternations are exceptional.

**Lemma 5.7.** *A critically indecomposable type skeleton has at most four types.*

*Proof.* The type skeleton of a critically indecomposable structure is, for some $r \geq 2$, the type skeleton of one of the nine structures listed by Schmerl and Trotter [22, Section 4]:

$$\mathcal{G}_r, \mathcal{P}_r, \mathcal{P}'_r, \mathcal{B}_r, \mathcal{T}_r^{(1)}, \mathcal{T}_r^{(2)}, \mathcal{T}_r^{(3)}, \mathcal{D}_r, \mathcal{D}'_r.$$

The graph $\mathcal{G}_r$ carries one symmetric relation, so a pair is adjacent or not: two types. The tournaments $\mathcal{T}_r^{(i)}$ carry one relation holding in exactly one direction on each pair: two types. The posets $\mathcal{P}_r$ and $\mathcal{P}'_r$ and the oriented graph $\mathcal{D}_r$ carry one antisymmetric relation, so a pair is related one way, the other way, or not at all: three types. In

$\mathcal{B}_r = (V_r; P_r, P'_r)$ the two orders divide the pairs between them, $P_r$ taking the $(a_i, b_j)$ with $i \geq j$ and $P'_r$ the remaining $(a_i, b_j)$ together with the $(a_i, a_j)$ and the $(b_i, b_j)$; each pair is therefore related by exactly one of the two, in exactly one direction: four types. The same holds in $\mathcal{D}'_r = (T_r^{(2)}; F_r, F'_r)$, where $F_r$ and $F'_r$ partition the arcs of the tournament $\mathcal{T}_r^{(2)}$. An isomorphism of type skeletons preserves the number of types, so the bound passes to every critically indecomposable type skeleton. □

**Lemma 5.8.** *Let $\omega$ be a primitive Cayley permutation with a repeated value. Then $R(\omega)$ has six types.*

*Proof.* For $i < j$, the type of $(i, j)$ is determined by whether $\omega(i) < \omega(j)$, $\omega(i) = \omega(j)$, or $\omega(i) > \omega(j)$; reversing the ordered pair gives three corresponding types with the opposite position order. All three value comparisons occur. A repeated value supplies the equality case. If no pair satisfied $\omega(i) < \omega(j)$, then $\omega$ would be weakly decreasing and, by primitivity, strictly decreasing, contrary to the repeated value. The same argument with the order reversed supplies a pair with $\omega(i) > \omega(j)$. Thus all six types occur, and they are distinct. □

**Theorem 5.9.** *The exceptional Cayley permutations are exactly the simple parallel alternations and their symmetries.*

*Proof.* Let $\omega \in \mathrm{Cay}_n$ be exceptional. Since every Cayley permutation of length at most two is simple, every one-point deletion of a word of length two or three is simple. Thus $n \geq 4$, and $\omega$ is primitive by Corollary 5.6. Suppose $\omega$ had a repeated value. By Proposition 5.2, $R(\omega)$ is indecomposable, and it is critically indecomposable. Indeed, for each $j \in [n]$, its one-point deletion at $j$ is isomorphic to $R(\omega \setminus j)$, which has $n - 1 \geq 3$ points. By Proposition 5.2, this structure is indecomposable only if $\omega \setminus j$ is simple, and no such deletion is simple. Lemma 5.7 then allows $R(\omega)$ at most four types, while Lemma 5.8 gives it six. So $\omega$ has no repeated value. Being a permutation, $\omega$ is exceptional in the classical sense, hence a simple parallel alternation or one of its symmetries. Conversely, the simple parallel alternations and their symmetries are permutations with no simple one-point deletion, and so are exceptional here. □

## 6 Restricted growth functions

A Cayley permutation $\omega = w_1 \cdots w_n$ is a *restricted growth function*, or RGF, if $w_1 = 1$ and $w_{i+1} \leq \max(w_1, \ldots, w_i) + 1$ for $1 \leq i < n$. Equivalently, in the ballot of $\omega$ the blocks appear in increasing order of their minima. Thus the bijection between Cayley permutations and ballots restricts to a bijection between RGFs and set partitions: a set partition is encoded by ordering its blocks by increasing minima. We inherit the definition of simplicity from Cayley permutations. The enumeration of simple RGFs requires only minor changes: the blocks are unordered, and no RGF is a nontrivial skew sum, since $\omega(1) = 1$.

There is also an intrinsic version of the relational structure of Section 5 for set partitions. If $\rho$ is a set partition of $[n]$, let

$$R(\rho) = ([n], <, \sim_\rho),$$

where $<$ is the natural order and $i \sim_\rho j$ if $i$ and $j$ belong to the same block of $\rho$.

**Proposition 6.1.** *Let $\omega$ be an RGF and $\rho$ its corresponding set partition. A nonempty positional interval is an H-interval of $\omega$ if and only if it is a union of blocks of $\rho$. Moreover, $R(\rho)$ and $R(\omega)$ have the same modules. Consequently, if $|\omega| \geq 3$, then $R(\rho)$ is indecomposable if and only if $\omega$ is simple and primitive.*

*Proof.* Let $I$ be a nonempty positional interval that is a union of blocks. If blocks $A, B, C$ satisfy $A, C \subseteq I$ and $\min A < \min B < \min C$, then $\min B \in I$, hence $B \subseteq I$. The blocks in $I$ are consecutive in their order by minima, so $I$ is an H-interval by Proposition 2.3. The converse follows from the same proposition.

The natural order forces every nonempty module $M$ of $R(\rho)$ to be a positional interval. If $M$ meets at least two blocks, the relation $\sim_\rho$ forces every block meeting $M$ to be contained in $M$, so the first assertion makes $M$ an H-interval. Otherwise $\omega$ is constant on $M$. Conversely, both kinds of interval are modules of $R(\rho)$. By Lemma 5.1, these are exactly the nonempty modules of $R(\omega)$; the empty set is a module of both structures. The final assertion now follows from Proposition 5.2. □

**Lemma 6.2.** *Assuming $\sigma \in \mathrm{Cay}_k$ and $\alpha_1, \ldots, \alpha_k \in \mathrm{Cay}$ are nonempty, an inflation $\sigma[\alpha_1, \ldots, \alpha_k]$ is an RGF if and only if $\sigma, \alpha_1, \ldots, \alpha_k$ are all RGFs.*

*Proof.* A Cayley permutation is an RGF precisely when, for each value $v$ below its maximum, the first occurrence of $v$ precedes the first occurrence of $v+1$. In $\omega = \sigma[\alpha_1, \ldots, \alpha_k]$ the values split into consecutive bands indexed, in increasing order, by the values of $\sigma$. Thus the RGF condition splits into conditions within the bands and between consecutive bands. The band indexed by $j$ is a single value if the fibre $\sigma^{-1}(j)$ is not a singleton, in which case the corresponding components are forced to be 1. If $\sigma^{-1}(j) = \{i\}$, the values in the band first occur in the order in which they first occur in $\alpha_i$. The conditions within the bands therefore say precisely that every $\alpha_i$ is an RGF. Suppose they hold. For either kind of fibre, the largest value of the $j$th band first occurs in the part indexed by $\min \sigma^{-1}(j)$. Since every $\alpha_i$ begins with 1, the smallest value of the $(j+1)$st band first occurs at the first position of the part indexed by $\min \sigma^{-1}(j+1)$. The parts are disjoint intervals occurring in index order, so the conditions between consecutive bands say precisely that $\min \sigma^{-1}(j) < \min \sigma^{-1}(j+1)$ for every $j$, which is to say that $\sigma$ is an RGF. □

**Theorem 6.3.** *Let $B_n^{\mathrm{RGF}}(t) = \sum_\omega t^{\mu(\omega)}$, the sum being over the RGFs $\omega$ of length $n$, let $B^{\mathrm{RGF}}(x,t) = \sum_{n\geq 1} B_n^{\mathrm{RGF}}(t)x^n$, and let $S_{\geq 3}^{\mathrm{RGF}}$ and $S^{\mathrm{RGF}}$ be defined as $S_{\geq 3}$ and $S$ with "simple" replaced by "simple RGF". Then*

$$1 + \sum_{n\geq 1} B_n^{\mathrm{RGF}}(t)\frac{z^n}{n!} = \exp\bigl(tz + e^z - 1 - z\bigr),$$

*and, for $n \geq 1$,*

$$B_n^{\mathrm{RGF}}(t) = \sum_{k=1}^{n}\sum_{j=0}^{k}\binom{n}{j}\left\{{n-j \atop k-j}\right\}(t-1)^j.$$

*Moreover, for $n \geq 1$ and $0 \leq j \leq n$,*

$$[t^j]\, B_n^{\mathrm{RGF}}(t) = \binom{n}{j}\sum_{i\geq 0}\left\{{n-j \atop i}\right\}_{\geq 2}.$$

*Furthermore,*

$$S^{\mathrm{RGF}}_{\geq 3}(x, B^{\mathrm{RGF}}(x,t)) = B^{\mathrm{RGF}}(x,t) - tx - x^2 - \frac{B^{\mathrm{RGF}}(x,t)^2}{1 + B^{\mathrm{RGF}}(x,t)},$$

*and, letting* $U^{\mathrm{RGF}}(x)$ *be the unique formal power series with* $B^{\mathrm{RGF}}(x, U^{\mathrm{RGF}}(x)) = x$,

$$S^{\mathrm{RGF}}(x) = 2x + \frac{x^3}{1+x} - xU^{\mathrm{RGF}}(x).$$

*Proof.* The blocks of a set partition are unordered, so the sequence of blocks in the proof of Proposition 4.1 is replaced by a set of blocks and $1/(1-u)$ becomes $e^u$; in the two counts of Proposition 4.2 the same change removes the factors $(i+j)!$ and $k!$.

Lemma 6.2 verifies (4) for the RGF property, which holds for 1, 11, and 12, but not for 21. The remaining identities and the existence and uniqueness of $U^{\mathrm{RGF}}$ follow from Theorem 4.3, Lemma 4.5, and Corollary 4.6, with $F_Q = B^{\mathrm{RGF}}$ and $V_Q = U^{\mathrm{RGF}}$. □

Writing $u^{\mathrm{RGF}}_n = [x^n]U^{\mathrm{RGF}}(x)$, the coefficients $s^{\mathrm{RGF}}_n = [x^n]S^{\mathrm{RGF}}(x)$ satisfy

$$s^{\mathrm{RGF}}_n = -u^{\mathrm{RGF}}_{n-1} + (-1)^{n-1}$$

for $n \geq 3$, and the sequence $(s^{\mathrm{RGF}}_n)_{n\geq 1}$ begins

$$(s^{\mathrm{RGF}}_n)_{n\geq 1} = 1, 2, 2, 4, 13, 51, 228, 1129, 6093, 35351, 218467, \ldots$$

We could not find this sequence in the OEIS [20] either.

For primitive simple RGFs, we use the same method as for the coefficients $\hat{s}_n$ in Theorem 5.3. Define

$$P^{\mathrm{RGF}}(x,t) = \sum_{\substack{\omega \in \mathrm{Cay}_+ \\ \omega \text{ primitive RGF}}} x^{|\omega|} t^{\mu(\omega)},$$

and let $W^{\mathrm{RGF}}$ satisfy $P^{\mathrm{RGF}}(x, W^{\mathrm{RGF}}(x)) = x$. Run expansion and Corollary 4.6 give

$$P^{\mathrm{RGF}}(x,t) = B^{\mathrm{RGF}}\left(\frac{x}{1+x}, (1+x)t - x\right),$$
$$\widehat{S}^{\mathrm{RGF}}(x) = 2x + \frac{x^3}{1+x} - xW^{\mathrm{RGF}}(x).$$

Thus, writing $\hat{s}^{\mathrm{RGF}}_n = [x^n]\widehat{S}^{\mathrm{RGF}}(x)$, we obtain

$$(\hat{s}^{\mathrm{RGF}}_n)_{n\geq 1} = 1, 1, 1, 1, 5, 16, 69, 316, 1591, 8614, \ldots$$

These counts are related to OEIS A099947, the numbers $t_n$ of topologically connected set partitions [20]. These partitions have no proper subinterval that is a union of blocks; Beissinger [4] calls them irreducible, and Dykema [18] calls them connected. By Proposition 6.1, for $n \geq 2$ these partitions correspond to simple RGFs with no singleton fibres. Hence their generating function is

$$T(x) = \sum_{n\geq 1} t_n x^n = x + x^2 + S^{\mathrm{RGF}}_{\geq 3}(x, 0).$$

Combining run expansion with the deletion of marked singleton blocks gives

$$P^{\mathrm{RGF}}(x,t) = x + (1+x)B^{\mathrm{RGF}}(x,(1+x)(t-1))\,.$$

Substituting $t = W^{\mathrm{RGF}}(x)$ in this identity and using Theorem 6.3 gives

$$T(x) = (1+x)\widehat{S}^{\mathrm{RGF}}(x) - x^2 - x^3.$$

Thus $t_n = \hat{s}_n^{\mathrm{RGF}} + \hat{s}_{n-1}^{\mathrm{RGF}}$ for $n \geq 4$. The asymptotics are established in the companion paper [14]. Writing $|\mathrm{RGF}_n|$ for the number of restricted growth functions of length $n$ (the $n$th Bell number), we have

$$s_n^{\mathrm{RGF}} \sim |\mathrm{RGF}_n|, \qquad \hat{s}_n^{\mathrm{RGF}} \sim |\mathrm{RGF}_{n-1}|, \qquad \frac{\hat{s}_n^{\mathrm{RGF}}}{s_n^{\mathrm{RGF}}} \sim \frac{\log n}{n}.$$

Thus almost every RGF is simple, and almost every primitive RGF is simple.

## 7 Classes with finitely many simple members

Albert and Atkinson [1] showed that a (hereditary) permutation class containing only finitely many simple permutations has a readily computable algebraic generating function. Is the analogous statement true for Cayley permutations? The answer turns out to be yes, which we now detail.

An occurrence of the classical pattern $\tau \in \mathrm{Cay}_k$ in $\omega \in \mathrm{Cay}_n$ is a subsequence $\alpha$ of $\omega$ such that $\mathrm{st}(\alpha) = \tau$. We write $\tau \leq \omega$ to indicate that $\tau$ occurs in $\omega$. This makes Cay into a poset. A (hereditary) *Cayley permutation class* is an order ideal $\mathcal{C}$ of Cay; that is, $\tau \leq \omega \in \mathcal{C} \Rightarrow \tau \in \mathcal{C}$. We say that $\omega$ *avoids* $\tau$ if $\tau \not\leq \omega$. Further, for a set $X \subseteq \mathrm{Cay}$, we let $\mathrm{Av}(X)$ denote the class of Cayley permutations avoiding each member of $X$. The *basis* of a class $\mathcal{C}$ is the set $\mathcal{B}$ of $\leq$-minimal Cayley permutations not in $\mathcal{C}$; thus $\mathcal{C} = \mathrm{Av}(\mathcal{B})$.

Theorem 5.5 supplies a finite certificate for the finiteness of the set of simple members.

**Corollary 7.1.** *Let $\mathcal{C}$ be a class of Cayley permutations with no simple member of length $n$ or of length $n+1$. Then $\mathcal{C}$ has no simple member of length greater than $n$.*

*Proof.* Suppose, to the contrary, that $\mathcal{C}$ has a simple member of length greater than $n$, and choose one, $\omega$, of least length $\ell$. By hypothesis, $\ell \geq n+2$. By Theorem 5.5 there is a simple pattern $\nu \leq \omega$ of length $\ell-1$ or $\ell-2$. Note that $\nu \in \mathcal{C}$, because $\mathcal{C}$ is a class. Thus $n \leq |\nu| < \ell$. If $|\nu|$ is $n$ or $n+1$, this contradicts the hypothesis. If $|\nu| > n+1$, it contradicts the minimality of $\ell$. □

A class therefore has finitely many simple members if and only if two consecutive lengths lack them. If membership in $\mathcal{C}$ is decidable, for example from a finite basis, we can find all its simple members by searching successive lengths and stopping when two consecutive lengths have none. The search terminates whenever the set of simple members is finite; it does not decide whether that set is finite.

## 7.1 Substitution-closed classes

A class is *substitution-closed* if $\sigma[\alpha_1, \ldots, \alpha_k]$ lies in the class whenever $\sigma$ and the $\alpha_i$ do, subject to (2). Albert and Atkinson's [1] characterization of substitution-closed permutation classes, also stated in Brignall's [6] survey, extends to Cayley permutations. We first describe how a classical pattern can occur in an inflation.

**Lemma 7.2.** *Let $\omega = \sigma[\alpha_1, \ldots, \alpha_k]$ be an inflation and $\beta$ a nonempty Cayley permutation. Then $\beta \leq \omega$ if and only if*

$$\beta = \mathrm{st}(\sigma|_T)[\gamma_i : i \in T]$$

*for some $\emptyset \neq T \subseteq [k]$ and nonempty $\gamma_i \leq \alpha_i$ with $\gamma_i = 1$ whenever $|\sigma^{-1}(\sigma(i))| > 1$.*

*Proof.* Choose positions $S$ witnessing an occurrence of $\beta$ in $\omega$, and let $S_i$ be those in the $i$th part of the inflation. Put $T = \{i : S_i \neq \emptyset\}$ and $\gamma_i = \mathrm{st}(\omega|_{S_i})$ for $i \in T$. Then $\gamma_i \leq \alpha_i$, and $\gamma_i = 1$ whenever $\sigma(i)$ is repeated, since the corresponding part is a single position. Here $T$ is the set of indices whose corresponding part in the inflation contains at least one entry in the selected occurrence of $\beta$; and $\gamma_i$ is the standardization of all the entries in $S_i$ taking part in such occurrence. Now, restricting $\sigma$ to $T$ and inflating $\sigma|_T$ by the $\gamma_i$ gives the same result as restricting $\omega$ to $S$. Indeed, within a part the selected letters have relative order $\gamma_i$. Between parts with distinct values of $\sigma$, they compare as those values do. Parts with equal values of $\sigma$ are single positions carrying equal letters. Hence

$$\beta = \mathrm{st}(\omega|_S) = \mathrm{st}(\sigma|_T)[\gamma_i : i \in T].$$

This is a valid inflation: every repeated letter of $\mathrm{st}(\sigma|_T)$ comes from a repeated letter of $\sigma$ and is therefore inflated by 1.

Conversely, choose an occurrence of each $\gamma_i$ in the $i$th part. Their union has the displayed standardization, by the same comparison of values, and thus witnesses an occurrence of $\beta$ in $\omega$. □

**Lemma 7.3.** *Let $\omega = \sigma[\alpha_1, \ldots, \alpha_k]$ be an inflation and $\beta$ a simple Cayley permutation with $\beta \leq \omega$. Then $\beta \leq \sigma$ or $\beta \leq \alpha_i$ for some $i$.*

*Proof.* The claim is immediate if $\beta$ is empty. Otherwise, write $\beta = \mathrm{st}(\sigma|_T)[\gamma_i : i \in T]$ as in Lemma 7.2. If $T = \{i\}$ then $\beta = \gamma_i \leq \alpha_i$ and we are done. Suppose instead that $|T| \geq 2$. A component $\gamma_i$ of length greater than one would occupy a proper H-interval of $\beta$ with more than one position, which simplicity forbids; so $\gamma_i = 1$ for each $i$ and $\beta = \mathrm{st}(\sigma|_T) \leq \sigma$. □

**Proposition 7.4.** *A class of Cayley permutations is substitution-closed if and only if each of its basis elements is simple.*

*Proof.* Let $\mathcal{B}$ be the basis of $\mathcal{C}$. Suppose first that $\mathcal{C}$ is substitution-closed and some $\beta \in \mathcal{B}$ is not simple. By Theorem 3.4, $\beta = \sigma[\alpha_1, \ldots, \alpha_k]$ with $\sigma$ simple of length $2 \leq k < |\beta|$ and the components obeying (2). The quotient and components are proper patterns of $\beta$, so they all lie in $\mathcal{C}$ by the minimality of a basis element. Substitution closure then gives $\beta \in \mathcal{C}$, a contradiction.

Conversely, suppose every $\beta \in \mathcal{B}$ is simple, and let $\omega = \sigma[\alpha_1, \ldots, \alpha_k]$ with $\sigma, \alpha_i \in \mathcal{C}$. If $\omega \notin \mathcal{C}$ then $\omega$ contains some $\beta \in \mathcal{B}$, necessarily simple; by Lemma 7.3, $\beta \leq \alpha_i$ for some $i$ or $\beta \leq \sigma$. Either possibility contradicts $\sigma, \alpha_i \in \mathcal{C}$. Hence $\omega \in \mathcal{C}$. □

We next compare a class with its substitution closure: the smallest substitution-closed class $\overline{\mathcal{C}}$ containing it.

**Lemma 7.5.** *A class $\mathcal{C}$ and its substitution closure $\overline{\mathcal{C}}$ have the same simple members.*

*Proof.* Let $\mathcal{D}$ be the set of all Cayley permutations $\omega$ such that every simple Cayley permutation $\sigma \leq \omega$ belongs to $\mathcal{C}$. It is a class containing $\mathcal{C}$, and Lemma 7.3 shows that it is substitution-closed. Thus $\mathcal{C} \subseteq \overline{\mathcal{C}} \subseteq \mathcal{D}$. If $\sigma$ is a simple member of $\overline{\mathcal{C}}$, then $\sigma \in \mathcal{D}$ and $\sigma \leq \sigma$, so the defining condition of $\mathcal{D}$ gives $\sigma \in \mathcal{C}$. Conversely, every simple member of $\mathcal{C}$ belongs to $\overline{\mathcal{C}}$. Hence the two classes have the same simple members. □

**Corollary 7.6.** *If the longest simple members of a class $\mathcal{C}$ have length $k$, then the basis elements of $\overline{\mathcal{C}}$ have length at most $k + 2$.*

*Proof.* Let $\beta$ be a basis element of $\overline{\mathcal{C}}$, of length $\ell$. As $\overline{\mathcal{C}}$ is substitution-closed, Proposition 7.4 makes $\beta$ simple. Suppose $\ell \geq k + 3$. By Theorem 5.5, $\beta$ properly contains a simple Cayley permutation $\sigma$ of length $\ell - 1$ or $\ell - 2$, hence of length greater than $k$; so $\sigma \notin \mathcal{C}$ and, by Lemma 7.5, $\sigma \notin \overline{\mathcal{C}}$. But every proper pattern of a basis element lies in the class. Hence $\ell \leq k + 2$. □

For a substitution-closed class we can now apply Theorem 4.3. When there are only finitely many simple members, the sum over simple quotients is a polynomial.

**Proposition 7.7.** *Let $\mathcal{C}$ be a substitution-closed class of Cayley permutations containing at least one nonempty word, with only finitely many simple members. Let $F$ be the ordinary generating function for its nonempty members, and put*

$$\Phi_{\mathcal{C}}(x, y) = \sum_{\substack{\sigma \in \mathcal{C} \text{ simple} \\ |\sigma| \geq 3}} x^{|\sigma| - \mu(\sigma)} y^{\mu(\sigma)}.$$

*For a word $\omega$, let $\delta_\omega$ be 1 if $\omega \in \mathcal{C}$ and 0 otherwise. Then*

$$\Phi_{\mathcal{C}}(x, F) = F - x - \delta_{11} x^2 - \frac{(\delta_{12} + \delta_{21}) F^2}{1 + F}. \tag{7}$$

*In particular, the ordinary generating function of $\mathcal{C}$ is algebraic over $\mathbb{Q}(x)$.*

*Proof.* We apply Theorem 4.3 to the property $Q(\omega) \iff \omega \in \mathcal{C}$, first verifying its hypotheses. Since $\mathcal{C}$ contains a nonempty word and is closed under classical pattern containment, it contains 1. For an inflation $\omega = \sigma[\alpha_1, \ldots, \alpha_k]$, the quotient $\sigma$ and each component $\alpha_i$ are classical patterns of $\omega$. Thus, if $\omega \in \mathcal{C}$, then $\sigma, \alpha_1, \ldots, \alpha_k \in \mathcal{C}$. Conversely, if the quotient and all components belong to $\mathcal{C}$, substitution closure gives $\omega \in \mathcal{C}$. This verifies (4). For this property, $F_Q(x, 1) = F$, $H_Q = \Phi_{\mathcal{C}}$, and $\chi_\omega = \delta_\omega$. Setting $t = 1$ in Theorem 4.3 and rearranging gives (7).

To prove algebraicity, note that $\Phi_{\mathcal{C}}$ is a polynomial because $\mathcal{C}$ has only finitely many simple members. Multiplying (7) by $1+F$ and rearranging gives $P(x,F)=0$, where

$$P(x,y)=(1+y)(y-x-\delta_{11}x^2-\Phi_{\mathcal{C}}(x,y))-(\delta_{12}+\delta_{21})y^2\in\mathbb{Q}[x,y].$$

Every monomial of $\Phi_{\mathcal{C}}$ has total degree at least three, so the coefficient of $x^0y$ in $P$ is 1. In particular, $P$ is a nonzero polynomial, and $P(x,F)=0$ proves that $F$ is algebraic over $\mathbb{Q}(x)$. Including the empty word gives the class generating function $1+F$, which is therefore algebraic as well. □

*Example* 7.8. For $k\geq 3$, let $\mathcal{C}_k=\mathrm{Av}(1^k,21)$. Its members are precisely

$$1^{a_1}2^{a_2}\cdots m^{a_m},\qquad 1\leq a_i<k.$$

Every constant word $1^{a_1}$ on the displayed list is simple, as is 12. Any other member with a run of length at least two has that run as a proper H-interval, while one with only singleton runs and length at least three begins with the proper H-interval 12. Therefore, the nonempty simple members of $\mathcal{C}_k$ are

$$1,11,\ldots,1^{k-1},\quad\text{and}\quad 12.$$

Now, in a nonempty Cayley permutation of $\mathcal{C}_k$ the first value occurs between one and $k-1$ times, after which the remaining suffix, standardized, is again in $\mathcal{C}_k$. Thus, writing $G$ for the ordinary generating function of $\mathcal{C}_k$, we have

$$G(x)=1+(x+x^2+\cdots+x^{k-1})G(x),$$

and hence

$$G(x)=\frac{1}{1-x-x^2-\cdots-x^{k-1}}.$$

In particular, $\mathrm{Av}(111,21)$ has generating function $1/(1-x-x^2)$.

We obtain the same answer from Proposition 7.7. The basis elements $1^k$ and 21 are simple, so $\mathcal{C}_k$ is substitution-closed. Here $\delta_{11}=\delta_{12}=1$, $\delta_{21}=0$, and $\Phi_{\mathcal{C}_k}=\sum_{j=3}^{k-1}x^j$, since the constant quotients have no singleton fibres. Thus (7) simplifies to $F=(x+x^2+\cdots+x^{k-1})(1+F)$, as required.

*Example* 7.9. Let $\mathcal{C}$ be the class avoiding the classical patterns

$$111,\quad 121,\quad 212,\quad 231.$$

The simple members of lengths one and two are $1,11,12,21$. Direct enumeration finds no simple members of length three or four, so Corollary 7.1 shows that these are all the nonempty simple members of $\mathcal{C}$.

This class is not substitution-closed, since its basis element 231 is not simple. Thus Proposition 7.7 does not apply. Instead, we use Theorem 3.4 to partition the nonempty members by their unique simple quotient. Each quotient is a classical pattern of the member being decomposed and hence belongs to $\mathcal{C}$, so the only possibilities are $1,11,12,21$. Let $F$ be the ordinary generating function for the nonempty members of $\mathcal{C}$. The quotients 1 and 11 contribute $x$ and $x^2$, respectively.

For the quotient 12, the canonical decomposition is $\alpha\oplus\beta$, with $\alpha$ plus indecomposable. The class $\mathcal{C}$ is closed under direct sums because each forbidden pattern is plus

indecomposable. If $I$ counts the nonempty plus indecomposable members, then $F = I + IF$, so $I = F/(1+F)$. Hence this quotient contributes $IF = F^2/(1+F)$.

For the quotient 21, the canonical decomposition is $\alpha \ominus \beta$, with $\alpha$ minus indecomposable. Since $\beta$ is nonempty, an increasing pair in $\alpha$, together with any letter of $\beta$, would form 231. Thus $\alpha$ must avoid 12. It is therefore weakly decreasing, and if it used more than one value it would be minus decomposable; hence it is constant. Avoidance of 111 leaves only $\alpha = 1, 11$. Conversely, with $\alpha = 1$ or 11 and $\beta \in \mathcal{C}$ nonempty, none of the forbidden patterns can occur across the two components. Thus the quotient 21 contributes $(x + x^2)F$. Adding the four contributions gives

$$F = x + x^2 + \frac{F^2}{1+F} + (x + x^2)F.$$

On writing $G = 1 + F$ for the ordinary generating function of $\mathcal{C}$, this simplifies to

$$G(x) = 1 + (x + x^2)G(x)^2.$$

A direct bijection also explains this equation. Avoidance of 111, 121, and 212 says that every value occurs in one contiguous run of length one or two. Contracting the runs gives a 231-avoiding permutation, and conversely each entry of any such permutation may be replaced independently by a run of length one or two. Thus, if Cat is the Catalan generating function,

$$G(x) = \mathrm{Cat}(x + x^2) = \frac{1 - \sqrt{1 - 4x - 4x^2}}{2x(1+x)}.$$

Example 7.15 illustrates the finite system constructed in the proof of Theorem 7.14 for a class with one further simple member.

## 7.2 General classes

We now remove the hypothesis of substitution closure and extend Proposition 7.7 to any class. Namely, we show in Theorem 7.12 and Theorem 7.14 that every class with finitely many simple members has a finite basis and an algebraic generating function. The first step is to rule out infinite antichains. A partially ordered set is *partially well ordered* if it contains neither an infinite strictly descending chain nor an infinite antichain. For Cayley permutations, a strictly descending chain is necessarily finite, since length decreases at each step.

The *closure* of a set under a family of operations consists of all elements obtained by applying those operations finitely many times, starting with elements of the set. We use the following form of Higman's theorem, stated by Albert and Atkinson [1].

**Theorem 7.10** (Higman [19])**.** *Let $(P, \leq)$ be a partially ordered set and let $f_i : P^{n_i} \to P$, $1 \leq i \leq r$, be finitely many operations of finite arity. Suppose each $f_i$ is order preserving in every argument; that is, for all $(a_1, \ldots, a_{n_i}), (b_1, \ldots, b_{n_i}) \in P^{n_i}$,*

$$a_j \leq b_j \text{ for every } 1 \leq j \leq n_i \quad \Longrightarrow \quad f_i(a_1, \ldots, a_{n_i}) \leq f_i(b_1, \ldots, b_{n_i}).$$

*Suppose also that*

$$a_j \leq f_i(a_1, \ldots, a_{n_i}) \quad \text{for every } (a_1, \ldots, a_{n_i}) \in P^{n_i} \text{ and } 1 \leq j \leq n_i.$$

*Then the closure of any finite subset of $P$ under these operations is partially well ordered.*

The proof of the next proposition adapts Albert and Atkinson's approach [1, Corollary 8] to our settings.

**Proposition 7.11.** *A class $\mathcal{C}$ of Cayley permutations with only finitely many simple members is partially well ordered.*

*Proof.* Let $\mathcal{S}$ be the finite set of nonempty simple members of $\mathcal{C}$. For $\sigma \in \mathcal{S}$ of length $k \geq 2$ with $\mu(\sigma) > 0$, define an operation on nonempty Cayley permutations by inflating the singleton fibres of $\sigma$. More precisely, let $i_1 < \cdots < i_{\mu(\sigma)}$ be the positions whose values occur only once in $\sigma$. Then, define

$$f_\sigma \colon (\mathrm{Cay}_+)^{\mu(\sigma)} \to \mathrm{Cay}_+, \qquad f_\sigma(\gamma_1, \ldots, \gamma_{\mu(\sigma)}) = \sigma[\alpha_1, \ldots, \alpha_k],$$

where $\alpha_{i_j} = \gamma_j$ for $1 \leq j \leq \mu(\sigma)$ and $\alpha_i = 1$ at all remaining positions. For instance, let $\sigma = 3132$, with $\mu(\sigma) = 2$, and let $(\gamma_1, \gamma_2) = (121, 11)$. Then

$$f_\sigma(\gamma_1, \gamma_2) = 3132[1, 121, 1, 11],$$

where the $\mu(\sigma)$ singleton fibres of $\sigma$ are inflated by $\gamma_1$ and $\gamma_2$ and the others are inflated by 1; that is, $\alpha_1 = \alpha_3 = 1$, $\alpha_2 = 121$, and $\alpha_4 = 11$. Lemma 7.2 shows that this operation is order preserving in each argument and that every argument is a pattern of the result. Thus the hypotheses of Higman's theorem hold.

To generate every nonempty member of $\mathcal{C}$, we include 1 and every simple member with no singleton fibres: such a member cannot be obtained by nontrivial inflation of a shorter quotient. Let

$$\mathcal{G} = \{1\} \cup \{\sigma \in \mathcal{S} : \mu(\sigma) = 0\}.$$

This set is finite. We show by induction on $|\omega|$ that every nonempty member $\omega$ of $\mathcal{C}$ belongs to its closure under the operations $f_\sigma$. If $|\omega| = 1$, then $\omega = 1 \in \mathcal{G}$. Otherwise, Theorem 3.4 gives $\omega = \sigma[\alpha_1, \ldots, \alpha_k]$, with $\sigma$ simple and $k \geq 2$. The quotient and components are classical patterns of $\omega$, so they belong to $\mathcal{C}$; in particular, $\sigma \in \mathcal{S}$. If $\mu(\sigma) = 0$, every component is forced to be 1, and hence $\omega = \sigma \in \mathcal{G}$. If $\mu(\sigma) > 0$, each component at a singleton fibre has length less than $|\omega|$ and therefore belongs to the closure by induction. Applying $f_\sigma$ to these components produces $\omega$, since all remaining components are forced to be 1.

By Theorem 7.10, the closure of the finite set $\mathcal{G}$ is partially well ordered. Its subset of nonempty members of $\mathcal{C}$ is therefore partially well ordered as well. Adjoining the empty word creates neither an infinite descending chain nor an infinite antichain, so $\mathcal{C}$ is partially well ordered. □

**Theorem 7.12.** *A class of Cayley permutations with only finitely many simple members has a finite basis.*

*Proof.* If $\mathcal{C}$ contains no nonempty word, its basis consists of the empty word or the word 1. Otherwise, let $k$ be the greatest length of a simple member of $\mathcal{C}$. By Lemma 7.5 and Proposition 7.11, the substitution closure $\overline{\mathcal{C}}$ is partially well ordered.

The basis elements of $\mathcal{C}$ that lie in $\overline{\mathcal{C}}$ form an antichain there, so there are finitely many. Any remaining basis element is also a basis element of $\overline{\mathcal{C}}$: it lies outside $\overline{\mathcal{C}}$, while all its proper patterns lie in $\mathcal{C} \subseteq \overline{\mathcal{C}}$. Corollary 7.6 bounds its length by $k + 2$, leaving only finitely many possibilities. □

It remains to find the generating function. As in Example 7.9, we must keep track of avoidance when inflating a simple quotient. A finite basis makes this possible with finitely many generating functions by recording which patterns below a basis element occur in each component. This is analogous to the query-complete property sets used by Brignall, Huczynska, and Vatter [7] for ordinary permutations: the properties satisfied by an inflation are determined by its quotient and the properties satisfied by its components. Before giving the construction, we state the algebraic fact needed to pass from the equations to the result.

**Lemma 7.13.** *Let $y_j = P_j(x, y_1, \dots, y_m)$, $1 \le j \le m$, be a* proper algebraic system *over $\mathbb{Q}$; that is, $P_1, \dots, P_m \in \mathbb{Q}[x, y_1, \dots, y_m]$ satisfy*

$$\begin{aligned} P_j(0, \dots, 0) &= 0, && 1 \le j \le m, \\ \frac{\partial P_j}{\partial y_\ell}(0, \dots, 0) &= 0, && 1 \le j, \ell \le m. \end{aligned}$$

*Then the system*

$$y_j = P_j(x, y_1, \dots, y_m), \qquad 1 \le j \le m,$$

*has a unique solution $(f_1, \dots, f_m)$ with every $f_j \in x\mathbb{Q}[[x]]$, and each $f_j$ is algebraic over $\mathbb{Q}(x)$. Moreover, given $H \in \mathbb{Q}[x, y_1, \dots, y_m]$, a nonzero polynomial in $\mathbb{Q}[x, T]$ annihilating $H(x, f_1, \dots, f_m)$ can be computed from $H$ and the $P_j$.*

*Proof.* Existence and uniqueness of the solution with zero constant terms, and algebraicity of its components, follow from Stanley's [23, Proposition 6.6.3 and Theorem 6.6.10] results on proper algebraic systems.

Panholzer [21] gives an algorithm computing an annihilating polynomial for any component of a proper algebraic system. To apply it to $H$, write $y = (y_1, \dots, y_m)$, put $c = H(0, \dots, 0)$, and append the equation

$$z = H(x, P_1(x, y), \dots, P_m(x, y)) - c.$$

Its right-hand side vanishes at the origin, as do all its derivatives with respect to the unknowns, so the enlarged system is still proper. Its new component is $g = H(x, f_1, \dots, f_m) - c$. Panholzer's algorithm therefore computes a nonzero $q(x, T) \in \mathbb{Q}[x, T]$ with $q(x, g) = 0$. Then $q(x, T - c)$ is the required polynomial annihilating $H(x, f_1, \dots, f_m)$. □

**Theorem 7.14.** *Let $\mathcal{C}$ be a class of Cayley permutations with only finitely many simple members. Its ordinary generating function $G(x) = \sum_{\omega \in \mathcal{C}} x^{|\omega|}$ is algebraic over $\mathbb{Q}(x)$. Given the simple members and the basis of $\mathcal{C}$, one can compute a nonzero polynomial $q(x, T) \in \mathbb{Q}[x, T]$ such that $q(x, G(x)) = 0$.*

*Proof.* If $\mathcal{C}$ contains no nonempty word, then $G(x)$ is 0 or 1, and the result is immediate. Otherwise, both the empty word and 1 belong to $\mathcal{C}$.

We construct a proper algebraic system that counts the nonempty members of $\overline{\mathcal{C}}$ according to the patterns they contain. We can then recover $G$ by selecting the members that avoid the basis of $\mathcal{C}$. Let $\mathcal{B}$ be the basis of $\mathcal{C}$, finite by Theorem 7.12,

and let $\Gamma$ consist of all patterns contained in a member of $\mathcal{B}$. This is a finite set, closed downward. For a word $\omega$, let

$$\mathcal{P}(\omega) = \{\gamma \in \Gamma : \gamma \leq \omega\}$$

be the set of patterns in $\Gamma$ contained in $\omega$. This is analogous to the set of satisfied properties recorded by Brignall, Huczynska, and Vatter [7], with the properties here being containment of the individual patterns in $\Gamma$. Since $\mathcal{B} \subseteq \Gamma$, a word $\omega \in \overline{\mathcal{C}}$ belongs to $\mathcal{C}$ if and only if $\mathcal{P}(\omega) \cap \mathcal{B} = \emptyset$.

The set $\mathcal{P}(\omega)$ is determined by the quotient of $\omega$ and the corresponding sets for its components. To see this, let $\omega = \sigma[\alpha_1, \ldots, \alpha_k]$. By Lemma 7.2, a nonempty $\beta \in \Gamma$ occurs in $\omega$ if and only if

$$\beta = \mathrm{st}(\sigma|_T)[\gamma_i : i \in T]$$

for nonempty $T \subseteq [k]$ and nonempty $\gamma_i \leq \alpha_i$, with $\gamma_i = 1$ whenever $\sigma(i)$ is repeated. Each $\gamma_i$ is a pattern of $\beta$, so it lies in $\Gamma$. Thus the condition $\gamma_i \leq \alpha_i$ can be read from $\mathcal{P}(\alpha_i)$. Enumerating the finitely many choices of $T$ and the $\gamma_i \in \Gamma$ determines whether each $\beta \in \Gamma$ occurs, and hence determines $\mathcal{P}(\omega)$. The empty pattern, if in $\Gamma$, belongs to every $\mathcal{P}(\omega)$.

For each subset $Q \subseteq \Gamma$, define the ordinary generating function

$$g_Q(x) = \sum_{\substack{\omega \in \overline{\mathcal{C}} \cap \mathrm{Cay}_+ \\ \mathcal{P}(\omega) = Q}} x^{|\omega|}.$$

Define $g_Q^+(x)$ and $g_Q^-(x)$ by restricting this sum to plus indecomposable and minus indecomposable words, respectively. Since $\Gamma$ is finite, this gives a finite list of generating functions. We obtain equations for these series from the unique decomposition in Theorem 3.4. By Lemma 7.5, the available simple quotients are exactly the nonempty simple members of $\mathcal{C}$. Fix such a quotient $\sigma$ of length $k \geq 2$, and a tuple of $k$ property sets $(Q_1, \ldots, Q_k)$. At every position where $\sigma(i)$ is repeated, require $Q_i = \mathcal{P}(1)$ and use the factor $x$. At a singleton fibre use $g_{Q_i}$, with two exceptions: the first factor is $g_{Q_1}^+$ when $\sigma = 12$ and $g_{Q_1}^-$ when $\sigma = 21$. These are exactly the restrictions that make the decomposition unique.

The product of these $k$ factors counts the inflations with the chosen property sets. Compute the resulting set $Q = \mathcal{P}(\omega)$ as above, and include this product in the equation for $g_Q$. Summing over all quotients of length at least two and all tuples of property sets gives the contributions from words of length at least two. The remaining word 1 contributes $x$ to the equation for $g_{\mathcal{P}(1)}$. To obtain the equation for $g_Q^+$, omit the quotient 12; to obtain that for $g_Q^-$, omit 21. Indeed, these quotients characterize plus and minus decomposability, respectively. The word 1 contributes $x$ to all three equations with $Q = \mathcal{P}(1)$.

Numbering the series as $y_1, \ldots, y_m$ gives a finite polynomial system

$$y_j = P_j(x, y_1, \ldots, y_m), \qquad 1 \leq j \leq m.$$

Each contribution on the right is either $x$ or a product of at least two factors drawn from $x$ and the unknowns. Consequently, every $P_j$ vanishes at the origin and has no term linear in an unknown alone, hence the system is proper. Lemma 7.13 therefore

applies, and the counting series form its unique solution with zero constant terms. The ordinary generating function of $\mathcal{C}$ is

$$G(x) = 1 + \sum_{\substack{Q \subseteq \Gamma \\ Q \cap \mathcal{B} = \emptyset}} g_Q(x),$$

so it is algebraic. Finally, $\Gamma$ and all the polynomial equations are computable from the basis and simple members of $\mathcal{C}$ by finite enumeration. Applying the last assertion of Lemma 7.13 to $1 + \sum_{Q \cap \mathcal{B} = \emptyset} g_Q(x)$ produces the required polynomial $q(x, T)$. □

*Example* 7.15. Let $\mathcal{C} = \mathrm{Av}(111, 212, 231, 1312)$. Direct enumeration and Corollary 7.1 show that its nonempty simple members are

$$1, \quad 11, \quad 12, \quad 21, \quad 121.$$

By Lemma 7.5, these are also the nonempty simple members of the substitution closure $\overline{\mathcal{C}}$. Write $\mathcal{B} = \{111, 212, 231, 1312\}$ and let

$$\Gamma = \{1, 11, 12, 21, 111, 112, 121, 132, 212, 231, 312, 1312\}$$

be its downward closure, omitting the empty pattern. The simple basis elements $111, 212, 1312$ do not belong to $\overline{\mathcal{C}}$, so every member of $\overline{\mathcal{C}}$ avoids them. Thus $\omega \in \overline{\mathcal{C}}$ belongs to $\mathcal{C}$ if and only if $231 \notin \mathcal{P}(\omega)$. We sum the equations in the proof of Theorem 7.14 over the sets $Q$ disjoint from $\mathcal{B}$, distinguishing whether $12 \in Q$. Write

$$A = \sum_{\substack{Q \subseteq \Gamma,\ Q \cap \mathcal{B} = \emptyset \\ 12 \notin Q}} g_Q, \qquad B = \sum_{\substack{Q \subseteq \Gamma,\ Q \cap \mathcal{B} = \emptyset \\ 12 \in Q}} g_Q, \qquad J = \sum_{\substack{Q \subseteq \Gamma,\ Q \cap \mathcal{B} = \emptyset \\ 12 \in Q}} g_Q^+.$$

Thus $A$ counts the nonempty members avoiding 12, $B$ counts those containing 12, and $J$ counts the plus indecomposable members counted by $B$. Every word counted by $A$ is weakly decreasing and hence plus indecomposable. Its minus indecomposable members are the constant words 1 and 11, counted by $X = x + x^2$.

We now read the contributions quotient by quotient. The word 1 and the quotient 11 contribute $X$ to $A$. For the quotient 12, the first component is plus indecomposable, counted by $A + J$. Since 231 is plus indecomposable, an inflation of 12 avoids 231 precisely when both components do. Every such inflation contains 12, giving the contribution $(A + J)(A + B)$ to $B$.

For the quotient 21, an occurrence of 231 crosses the two components precisely when the first contains 12. This component must also be minus indecomposable, so it is 1 or 11. The resulting inflation contains 12 if and only if the second component does, giving $XA$ to $A$ and $XB$ to $B$.

For the quotient 121, the two positions in the repeated fibre are forced to have component 1, whereas the middle component $\alpha$ is free. An occurrence of 231 crosses the components of $121[1, \alpha, 1]$ precisely when $\alpha$ contains 12. Thus $\alpha$ is counted by $A$, and these inflations contribute $x^2 A$ to $B$.

Finally, omitting the quotient 12 gives the equation for $J$. We obtain the proper algebraic system

$$\begin{aligned} A &= X + XA, \quad X = x + x^2, \\ B &= (A + J)(A + B) + XB + x^2 A, \\ J &= XB + x^2 A. \end{aligned}$$

Each right-hand side vanishes at the origin and has no term linear in an unknown alone, as required in the proof of the theorem.

To eliminate the auxiliary series, put $G = 1 + A + B$. The first equation gives $A = X/(1 - X)$, and the last two give $G - 1 = (A + J)G$. Since

$$A + J = A + X(G - 1 - A) + x^2 A = XG + x^2 A,$$

we obtain

$$G = 1 + XG^2 + \frac{x^2 X}{1 - X} G.$$

Hence the construction produces the annihilating polynomial

$$q(x, T) = (1 - X)(XT^2 - T + 1) + x^2 XT, \qquad X = x + x^2.$$

The solution with constant term 1 begins

$$G(x) = 1 + x + 3x^2 + 10x^3 + 36x^4 + 135x^5 + 527x^6 + 2120x^7 + \cdots$$

and the counting sequence does not appear in the OEIS [20].

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