The species of interval orders

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Abstract

We show that, in the ring of virtual species,

$$\mathcal{I} \;=\; \sum_{m\geq 0} (-1)^m \prod_{i=1}^{m} \big((E^{-1})^i - 1\big),$$

where $\mathcal{I}$ is the species of interval orders and E^{-1} is the multiplicative inverse of the species E of sets. The right-hand side is the virtual species of signed ballot matrices introduced by Claesson and Hannah. They showed that its signed cardinality counts labeled interval orders. We strengthen this to a species identity, which we prove twice: first algebraically and then bijectively, using a natural sign-reversing involution. The cycle index series of $\mathcal{I}$ specializes to the generating series for labeled and unlabeled interval orders. We describe the automorphism group of an interval order as a Young subgroup and prove the identity $\mathcal{I} = \mathcal{R} \circ E_+$, where $\mathcal{R}$ is the species of rigid interval orders. We also show that Glaisher's T-number T_n counts the 24-colored interval orders on $[n]$ in which no isolated element has color 24.

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1 Introduction

An *interval order* is a poset whose elements can be represented by closed real intervals in such a way that $x < y$ if and only if the interval of x lies entirely to the left of the interval of y. By a theorem of Fishburn [12], the finite interval orders are exactly the (2+2)-free posets, those with no induced subposet isomorphic to the disjoint union of two two-element chains. Bousquet-Mélou, Claesson, Dukes and Kitaev [3] proved that the number of unlabeled interval orders on n points is the coefficient of x^n in

$$\sum_{m\geq 0} \prod_{i=1}^{m} \big(1-(1-x)^i\big) \;=\; 1 + x + 2x^2 + 5x^3 + 15x^4 + 53x^5 + \cdots, \tag{1}$$

a series first studied by Zagier [19]. The coefficients are called the *Fishburn numbers* [7] and form sequence A022493 in the OEIS [18]. Claesson, Dukes and Kubitzke [5] derived the generating series for labeled interval orders,

$$\sum_{m\geq 0}\prod_{i=1}^{m}\big(1-e^{-ix}\big) \;=\; 1+x+3\frac{x^2}{2!}+19\frac{x^3}{3!}+207\frac{x^4}{4!}+\cdots \tag{2}$$

The coefficients form sequence A079144 in the OEIS [18]. Claesson and Hannah [6] recast (2) in the language of species. Here e^{-x} is the generating series of the virtual species E^{-1}, the multiplicative inverse of the species E of sets. Thus, the left-hand side of (2) suggests the virtual species

$$\mathrm{BalMat} \;=\; \sum_{m\geq 0}\prod_{i=1}^{m}\big(1-(E^{-1})^i\big) \;=\; \sum_{m\geq 0}(-1)^m\prod_{i=1}^{m}\big((E^{-1})^i-1\big). \tag{3}$$

They represent BalMat by signed ballot matrices, and use a sign-reversing involution to show that the signed cardinality on $[n]=\{1,\dots,n\}$ is the number of labeled interval orders on $[n]$.

Let $\mathcal{I}$ denote the species of interval orders. Thus, $\mathcal{I}[U]$ is the set of (2+2)-free posets on the finite set U, and a bijection $\sigma\colon U\to V$ transports a poset by relabeling. We prove that $\mathrm{BalMat}=\mathcal{I}$ in the ring of virtual species.

Theorem 1.1. *In the ring of virtual species,*

$$\mathcal{I} \;=\; \sum_{m\geq 0}(-1)^m\prod_{i=1}^{m}\big((E^{-1})^i-1\big).$$

Several proofs of (1) use cancellation. Levande [17] gave a sign-reversing involution on Fishburn diagrams. Eriksen and Sjöstrand [11] used the inclusion-exclusion principle to refine the series by statistics on matchings and permutations, and Hannah [13] used it to recover the enumeration of unlabeled interval orders from factorial posets.

Section 2 recalls the language of species and virtual species. We give two proofs of Theorem 1.1 at the level of species. The algebraic proof (Sections 3–4) combines a composition matrix representation of interval orders [5, 8] with inclusion-exclusion over the matrix columns. The bijective proof (Section 5) constructs a natural sign-reversing involution on ballot matrices whose fixed points are identified with composition matrices.

Equations (1) and (2) are direct consequences of Theorem 1.1. An expression for the cycle index series of $\mathcal{I}$ also follows (Section 6):

$$Z_{\mathcal{I}}(x_1,x_2,x_3,\dots) \;=\; \sum_{m\geq 0}\prod_{i=1}^{m}\left(1-\exp\left(-i\left(x_1+\frac{x_2}{2}+\frac{x_3}{3}+\cdots\right)\right)\right).$$

In Section 7 we describe the automorphism group of an interval order and the molecular decomposition of $\mathcal{I}$. In particular, the number of interval orders on $[n]$ fixed by a permutation with c cycles is the number of labeled interval orders on $[c]$, and so depends on c alone. In Section 8 we show that the indistinguishability classes of an interval order form a rigid interval order. This gives $\mathcal{I}=\mathcal{R}\circ E_+$, in which $\mathcal{R}$ is the species of rigid interval orders.

In Section 9 we show that Glaisher's T-numbers count certain colored interval orders.

2 Virtual species

We follow the notation and terminology of Bergeron, Labelle and Leroux [1].

2.1 Species and their series

A species defines both a class of labeled combinatorial structures and how those structures are affected by relabeling. More precisely, a species F associates with each finite set U a finite set $F[U]$, whose elements are called *F-structures* on U, and with each bijection $\sigma\colon U \to V$ a bijection $F[\sigma]\colon F[U] \to F[V]$, called *transport of structure*. These data satisfy

$$F[\mathrm{id}_U] = \mathrm{id}_{F[U]}, \qquad F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$$

for bijections $\sigma\colon U \to V$ and $\tau\colon V \to W$. Equivalently, a species is a functor from $\mathbb{B}$, the category of finite sets and bijections, to the category of finite sets and functions. We write $\sigma \cdot s$ for $F[\sigma](s)$. Two structures $s \in F[U]$ and $t \in F[V]$ are *isomorphic* if $\sigma \cdot s = t$ for some bijection $\sigma\colon U \to V$. Two species F and G are *combinatorially equal*, written $F = G$, if there is a natural isomorphism between them; that is, if there is a family of bijections $F[U] \to G[U]$ commuting with transport. We use the standard species 1 (empty set), X (singletons), E (sets), E_k (sets of cardinality k), E_+ (nonempty sets), L (linear orders), $\mathcal{C}$ (oriented cycles), and the sum and product of species. A species F has three associated series. The *generating series* and the *type generating series* are

$$F(x) = \sum_{n\ge 0} \bigl|F[n]\bigr| \, \frac{x^n}{n!}\,, \qquad \widetilde{F}(x) = \sum_{n\ge 0} \bigl|F[n]/{\sim}\bigr| \; x^n,$$

where $\sim$ is isomorphism of structures. The *cycle index series* is

$$Z_F(x_1, x_2, x_3, \dots) = \sum_{n\ge 0} \frac{1}{n!} \sum_{\sigma\in\mathcal{S}_n} \operatorname{fix} F[\sigma] \; x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \cdots\,,$$

where $\operatorname{fix} F[\sigma]$ is the number of F-structures on $[n]$ fixed by σ and $(\sigma_1, \sigma_2, \dots)$ is the cycle type of σ. All three are additive and multiplicative, and

$$F(x) = Z_F(x, 0, 0, \dots), \qquad \widetilde{F}(x) = Z_F(x, x^2, x^3, \dots).$$

A nonzero species M is *molecular* if any two M-structures are isomorphic. Such an M is concentrated on sets of some fixed cardinality n, and transport makes $M[n]$ a transitive $\mathcal{S}_n$-set. Choosing $s \in M[n]$ with stabilizer, or automorphism group, $H = \{\sigma \in \mathcal{S}_n : \sigma \cdot s = s\}$ identifies the M-structures on $[n]$ with the cosets of H. In this notation we have $M = X^n/H$, and another choice of s replaces H by a conjugate. Familiar examples are

$$X^n = X^n/\{1\}, \qquad E_n = X^n/\mathcal{S}_n, \qquad \mathcal{C}_n = X^n/\langle (1\,2\,\cdots\,n)\rangle,$$

the species of linear orders, of sets, and of oriented cycles on n elements. On three elements, XE_2, a distinguished element together with an unordered pair, is molecular with H generated by the transposition exchanging the pair.

Every species is the sum of its molecular subspecies. Indeed, for each n the orbits of $\mathcal{S}_n$ on $F[n]$, one for each isomorphism type of F-structure, are molecular summands. Collecting isomorphic summands gives the *molecular decomposition*

$$F \;=\; \sum_M f_M\, M \qquad (f_M \in \mathbb{N}),$$

the sum running over isomorphism classes of molecular species. For instance, $E = \sum_n E_n$ and $L = \sum_n X^n$, while the species of graphs on two-element sets decomposes as $2E_2$, since the empty graph and the one-edge graph give two summands of the same molecular type. The decomposition is unique, so $F = G$ if and only if $f_M = g_M$ for every molecular species M.

2.2 The ring of virtual species

A *virtual species* is an element of

$$\mathrm{Virt} = (\mathrm{Spe} \times \mathrm{Spe})/\!\sim, \qquad (F, G) \sim (H, K) \iff F + K = G + H,$$

where Spe is the semiring of species. The equivalence class of (F, G) is written $F - G$. With the evident addition and multiplication, Virt is a commutative ring, and $F \mapsto F - 0$ embeds Spe into Virt. Suppose that F and G have molecular decompositions

$$F = \sum_M f_M M, \qquad G = \sum_M g_M M.$$

Then their difference has the molecular expansion

$$\Phi = F - G = \sum_M \phi_M M, \qquad \phi_M = f_M - g_M \in \mathbb{Z}.$$

The coefficients ϕ_M depend only on Φ. Consequently, every virtual species has a unique *reduced form*

$$\Phi = \Phi^+ - \Phi^-, \qquad \Phi^+ = \sum_{\phi_M > 0} \phi_M M, \qquad \Phi^- = \sum_{\phi_M < 0} (-\phi_M) M,$$

in which Φ^+ and Φ^- have no common molecular summand. We call Φ *positive* if $\Phi^- = 0$, equivalently, if every ϕ_M is nonnegative. The three associated series extend by differences,

$$\Phi(x) = F(x) - G(x), \qquad \widetilde{\Phi}(x) = \widetilde{F}(x) - \widetilde{G}(x), \qquad Z_\Phi = Z_F - Z_G,$$

and remain additive and multiplicative. In particular, if Φ is positive then $\Phi = \Phi^+$ is a species, so $\Phi(x)$, $\widetilde{\Phi}(x)$ and Z_Φ all have nonnegative coefficients. These three series do not determine a virtual species, and nonnegativity of their coefficients does not imply positivity. Bergeron, Labelle and Leroux give the following example [1, Section 2.6, equations (30)–(31)]. Consider the four molecular species X^3, XE_2, $\mathcal{C}_3$ and E_3. Their fixed-point counts, one representative per cycle type, are

	id	(1 2)	(1 2 3)
X^3	6	0	0
XE_2	3	1	0
$\mathcal{C}_3$	2	0	2
E_3	1	1	1

Summing the relevant rows of the table shows that the species

$$2\,XE_2 + \mathcal{C}_3 \qquad \text{and} \qquad X^3 + 2\,E_3$$

have the same number of fixed structures, namely 8, 2 and 2, for every permutation of [3], and hence equal cycle index series. Yet their molecular decompositions differ, so they are not combinatorially equal. Their difference $2\,XE_2 + \mathcal{C}_3 - X^3 - 2\,E_3$ is therefore a nonzero virtual species whose three series all vanish, and it is not positive. In particular, Theorem 1.1 does not follow from the preceding counting results [3, 5, 6].

We use infinite sums of (virtual) species in the standard *summable* sense: on each finite set, only finitely many summands contribute. For example, the sums in (3) are summable because every factor $(E^{-1})^i - 1$ vanishes on the empty set, so the mth product vanishes on sets of cardinality less than m. Such sums may be regrouped freely. Their generating, type generating, and cycle index series are obtained by summing the corresponding series of the summands coefficientwise.

2.3 Ballot matrices

The multiplicative inverse

$$E^{-1} = (1 + E_+)^{-1} = \sum_{k\geq 0} (-1)^k E_+^k$$

of the set species is a standard example of a virtual species. On a finite set U, an E^{-1}-structure is a *ballot*, a sequence of pairwise disjoint nonempty *blocks* with union U. A ballot with k blocks has sign $(-1)^k$. Thus, the ballots with an even number of blocks make up $(E^{-1})^+$, while those with an odd number make up $(E^{-1})^-$. The three associated series are

$$E^{-1}(x) = e^{-x}, \qquad \widetilde{E^{-1}}(x) = 1 - x, \qquad Z_{E^{-1}} = e^{-q}, \tag{4}$$

where here and throughout

$$q \;=\; x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \cdots, \quad \text{so that} \quad Z_E = e^q.$$

Signed ballot matrices model the virtual species BalMat of (3). We shall use this model in the second proof of Theorem 1.1.

Definition 2.1 (Claesson and Hannah [6])**.** A *ballot matrix* on a finite set U is an upper triangular $m \times m$ array ($m \geq 0$) of ballots on pairwise disjoint sets with union U, such that every row contains at least one element of U. We write $\mathcal{B}[U]$ for the set of ballot matrices on U. Transport along $\sigma\colon U \to V$ replaces every block B by $\sigma(B)$. A ballot matrix of *dimension* m with k blocks in total is *positive* or *negative* according as $k+m$ is even or odd. We write $\mathcal{B}^+[U]$ and $\mathcal{B}^-[U]$ for the corresponding subsets of $\mathcal{B}[U]$.

Note that $m \leq |U|$, so $\mathcal{B}[U]$ is finite. Positivity and negativity are preserved by transport, so $\mathcal{B}^+$ and $\mathcal{B}^-$ are subspecies of $\mathcal{B}$. Here, positive refers to the sign of a ballot matrix, not to positivity in Virt, and $\mathcal{B}^+$ and $\mathcal{B}^-$ are not the two parts of the reduced form of BalMat. In fact, we shall prove that $\mathcal{B}^+ = \mathcal{B}^- + \mathcal{I}$; see Corollary 4.3.

Proposition 2.2 (Claesson and Hannah [6]). *The virtual species* BalMat *defined by* (3) *satisfies*

$$\mathrm{BalMat} = \mathcal{B}^+ - \mathcal{B}^-.$$

Proof. The product $(E^{-1})^i$ is represented by the sequences of i pairwise disjoint ballots, such a sequence with k blocks in total having sign $(-1)^k$. Subtracting 1 removes the sequence in which all i ballots are empty. Take the product over $i = 1, \ldots, m$, matching the factor $(E^{-1})^i - 1$ with the row of length i in an upper triangular $m \times m$ matrix. The mth term of (3) then consists of the ballot matrices of dimension m, each with sign $(-1)^{m+k}$. Summing over m concludes the proof. $\square$

3 Interval orders and composition matrices

By a *poset* on a finite set U we mean a strict partial order $<$ on U, that is, an irreflexive transitive relation. For $x \in U$ write

$$D(x) = \{y : y < x\}$$

for the *strict down-set* of x. The *strict up-set* of x is $\{y : x < y\}$. By Fishburn's theorem, the finite (2+2)-free posets are exactly the finite interval orders; Bogart [2] gives a short proof.

Definition 3.1. The species $\mathcal{I}$ of interval orders assigns to a finite set U the set $\mathcal{I}[U]$ of (2+2)-free posets on U, here viewed as sets of ordered pairs. A bijection $\sigma\colon U \to V$ transports $P \in \mathcal{I}[U]$ to

$$\sigma \cdot P = \{(\sigma x, \sigma y) : (x, y) \in P\} \in \mathcal{I}[V].$$

Since being (2+2)-free is invariant under relabeling, $\mathcal{I}$ is a subspecies of the species of posets.

Claesson, Dukes and Kubitzke [5] define a *composition matrix* on a finite set U to be an upper triangular matrix of subsets of U whose nonempty entries partition U and in which every row and every column has a nonempty entry. For instance,

$$\begin{bmatrix} \{1\} & \emptyset & \{2,5\} \\ & \{3\} & \emptyset \\ & & \{4\} \end{bmatrix}$$

is a 3×3 composition matrix on $[5]$.

An element of U lies in exactly one entry, so an $m \times m$ composition matrix is the same thing as a map recording, for each element, the position of the entry that contains it. That is the form we use.

Definition 3.2. For $m \geq 0$ let $\mathrm{Int}_m = \{(a, b) \in [m] \times [m] : a \leq b\}$. We regard (a, b) both as the position in row a and column b of an upper triangular matrix and as the interval $[a, b] \subseteq [m]$, whence the notation. The species $\mathrm{CM}^{\langle m \rangle}$ of $m \times m$ composition matrices assigns to U the set of maps

$$C\colon U \to \mathrm{Int}_m$$

such that both coordinate maps are surjective onto $[m]$. That is, if we write $C(x) = (a(x), b(x))$, then both $x \mapsto a(x)$ and $x \mapsto b(x)$ are surjective onto $[m]$. The transport of C along a bijection $\sigma\colon U \to V$ is $C \circ \sigma^{-1}$.

The entry in position $(a,b) \in \mathrm{Int}_m$ is the fiber $C^{-1}(a,b)$. These fibers are pairwise disjoint and have union U. The two surjectivity conditions say that no row and no column is empty. So the $\mathrm{CM}^{\langle m\rangle}$-structures on U are the $m \times m$ composition matrices on U. Since $\mathrm{CM}^{\langle m\rangle}[U] = \emptyset$ when $m > |U|$, only finitely many summands are nonempty on each finite set U, and

$$\mathrm{CM} = \sum_{m\geq 0} \mathrm{CM}^{\langle m\rangle}$$

is a well-defined species.

We may also read a composition matrix directly as a family of intervals. If $C(x) = (a,b)$, then the interval assigned to x is $[a,b]$. Every value in $[m]$ occurs as a left endpoint and as a right endpoint. Thus, composition matrices are the same objects as families of intervals with these two endpoint conditions.

Lemma 3.3 (Fishburn [12]). *A finite poset P is (2+2)-free if and only if its strict down-sets $\{D(x) : x \in P\}$ form a chain under inclusion.*

Proof. Suppose that $D(x)$ and $D(y)$ are incomparable. Choose $a \in D(x) \setminus D(y)$ and $c \in D(y) \setminus D(x)$. Then $a < x$ and $c < y$, while $a \not< y$ and $c \not< x$. Any equality or comparison between the two chains would, by transitivity, give one of these two forbidden relations. Thus, the four elements are distinct and induce a 2+2. Conversely, if $a < b$ and $c < d$ induce a 2+2, then $a \in D(b) \setminus D(d)$ and $c \in D(d) \setminus D(b)$. Thus, $D(b)$ and $D(d)$ are incomparable. $\square$

Let $P \in \mathcal{I}[U]$. By Lemma 3.3, its distinct strict down-sets can be written

$$D_1 \subset \cdots \subset D_m.$$

For $x \in U$, let $\ell(x)$ be its *level*, so that $D_{\ell(x)}$ is its strict down-set, and put

$$r(x) = \max\{k \in [m] : x \notin D_k\}.$$

Finally, define $\Gamma(P) : U \to \mathrm{Int}_m$ by $\Gamma(P)(x) = \big(\ell(x), r(x)\big)$.

Lemma 3.4 (Dukes, Jelínek and Kubitzke [8]). *If $P \in \mathcal{I}[U]$ has m distinct strict down-sets, then $\Gamma(P) \in \mathrm{CM}^{\langle m\rangle}[U]$, and $x < y$ in P if and only if $r(x) < \ell(y)$. Thus, the intervals $[\ell(x), r(x)]$ represent P. The map $\Gamma\colon \mathcal{I}[U] \to \mathrm{CM}[U]$ is a bijection. If $C(x) = (a(x), b(x))$, its inverse is*

$$\Gamma^{-1}(C) = \big\{(x,y) : b(x) < a(y)\big\}.$$

Proof. The result is clear if U is empty. Suppose that U is nonempty and $P \in \mathcal{I}[U]$ has m distinct strict down-sets $D_1 \subset D_2 \subset \cdots \subset D_m$. Here $D_1 = \emptyset$, so the set defining $r(x)$ is nonempty. Since the sets D_k form a chain,

$$x < y \iff x \in D_{\ell(y)} \iff r(x) < \ell(y). \tag{5}$$

Taking $y = x$ and using irreflexivity gives $\ell(x) \leq r(x)$.

The map ℓ is surjective by construction. If $k < m$, then $r(x) = k$ for any $x \in D_{k+1} \setminus D_k$. For $k = m$, choose x with $\ell(x) = m$; then $m = \ell(x) \leq r(x) \leq m$. Thus, r is also surjective, and $\Gamma(P)$ is a composition matrix.

Conversely, let $C \in \mathrm{CM}^{\langle m \rangle}[U]$, with $C(x) = (a(x), b(x))$, and define P_C by $x < y$ if and only if $b(x) < a(y)$. The intervals $[a(x), b(x)]$ represent P_C, so it is an interval order. The strict down-set of y in P_C is $\{x : b(x) < a(y)\}$. Since b is surjective, the sets

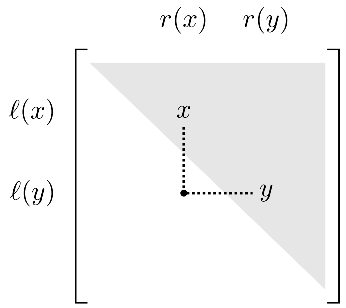
$$D'_k = \{x : b(x) < k\} \qquad (k \in [m])$$

form a chain of m distinct sets. Since a is surjective, each D'_k occurs as a down-set, and the down-set of y is $D'_{a(y)}$. Hence, $\ell(y) = a(y)$. For fixed x, the set D'_k does not contain x precisely when $k \le b(x)$. Thus, $r(x) = b(x)$ and $\Gamma(P_C) = C$. The criterion (5) also gives $P_{\Gamma(P)} = P$, and the two constructions are mutually inverse. $\square$

Equivalently, when the distinct strict up-sets are ordered by reverse inclusion, $r(x)$ is the level of the strict up-set of x.

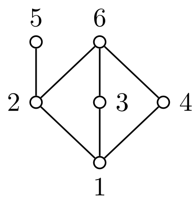
We call $\Gamma(P)$ the *composition matrix of* P.

We read the criterion $r(x) < \ell(y)$ off the matrix as follows: the hook from x down its column to the row of y, and then along that row to y, turns strictly below the diagonal:

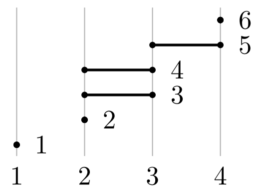


Claesson, Dukes and Kubitzke [5] gave an earlier bijection between composition matrices and (2+2)-free posets on $[n]$, factoring through ascent sequences [3, 10]. Dukes, Jelínek and Kubitzke [8] remark that it agrees with Γ.

Example 3.5. Let P on $U = [6]$ be the poset drawn below. The distinct strict down-sets are $D_1 = \emptyset$, $D_2 = \{1\}$, $D_3 = \{1, 2\}$ and $D_4 = \{1, 2, 3, 4\}$, so $m = 4$ and $(\ell(1), \ldots, \ell(6)) = (1, 2, 2, 2, 3, 4)$. For $x = 1, \ldots, 6$, the largest down-sets not containing x are, respectively, D_1, D_2, D_3, D_3, D_4 and D_4. Hence, $(r(1), \ldots, r(6)) = (1, 2, 3, 3, 4, 4)$. The three equivalent views are shown below.



poset



interval model

$$\begin{bmatrix} \{1\} & \emptyset & \emptyset & \emptyset \\ & \{2\} & \{3,4\} & \emptyset \\ & & \emptyset & \{5\} \\ & & & \{6\} \end{bmatrix}$$

composition matrix

The interval model is read directly from the matrix: for instance, 3 lies in position $(2, 3)$ and is therefore represented by $[2, 3]$. Lemma 3.4 shows that this is the unique interval model of P in which every integer in $[4]$ occurs as both a left and a right endpoint.

Theorem 3.6. *The map* Γ *is a natural isomorphism* $\mathcal{I} = \mathrm{CM}$.

Proof. By Lemma 3.4 the map Γ is a bijection between $\mathcal{I}[U]$ and $\mathrm{CM}[U]$. It remains to prove naturality. Let $\sigma\colon U \to V$ be a bijection. We need to prove that the square

$$\begin{tikzcd} \mathcal{I}[U] \arrow[r, "\Gamma"] \arrow[d, "\mathcal{I}[\sigma]"'] & \mathrm{CM}[U] \arrow[d, "\mathrm{CM}[\sigma]"] \\ \mathcal{I}[V] \arrow[r, "\Gamma"'] & \mathrm{CM}[V] \end{tikzcd}$$

commutes. Let $P \in \mathcal{I}[U]$. Strict down-sets satisfy $D_{\sigma \cdot P}(\sigma x) = \sigma(D_P(x))$. Thus, transport preserves the chain $D_1, \ldots, D_m$, its indices, and membership in its sets. It therefore preserves both ℓ and r:

$$\ell_{\sigma \cdot P}(\sigma x) = \ell_P(x), \qquad r_{\sigma \cdot P}(\sigma x) = r_P(x).$$

Hence, $\Gamma(\sigma \cdot P) = \Gamma(P) \circ \sigma^{-1}$, which is the transport of Definition 3.2. Thus, the square commutes and Γ is a natural isomorphism. $\square$

4 An algebraic proof of Theorem 1.1

A composition matrix asks two things of an upper triangular matrix: that no row be empty and that no column be empty. We shall keep the first condition and impose the second by inclusion-exclusion over the set of columns used. For $m \geq 0$, an $m \times m$ *quasi composition matrix* on U is an upper triangular matrix of subsets of U whose nonempty entries partition U and in which every row has a nonempty entry. It is a composition matrix exactly when every column has a nonempty entry as well.

For a fixed $m \geq 0$ and $K \subseteq [m]$, let W_K be the species of $m \times m$ quasi composition matrices that use no column outside K. The matrix being upper triangular, row a has

$$\nu_K(a) = \#\{b \in K : b \geq a\}$$

positions at its disposal.

Lemma 4.1. *Let* $m \geq 0$ *and let* $K \subseteq [m]$. *Then*

$$W_K \;=\; \prod_{a=1}^{m} \bigl(E^{\nu_K(a)} - 1\bigr).$$

Proof. In the form of Definition 3.2, W_K is the species of those $C\colon U \to \mathrm{Int}_m$ whose first coordinate map is surjective onto $[m]$ and whose second coordinate map takes its values in K. Both conditions are preserved by transport. The elements in row a form a nonempty set, and each may be placed in any of the $\nu_K(a)$ available positions of that row. For a finite linearly ordered set T, the species of T-valued maps is $E^{|T|}$: the natural isomorphism sends a map to its tuple of fibers. Thus, the species of T-valued maps on a nonempty set is $E^{|T|} - 1$. Multiplying the contributions from the rows gives the claimed formula. $\square$

Lemma 4.2. *Let $m \geq 0$. Then, in* Virt*,*

$$\mathrm{CM}^{\langle m\rangle} = \sum_{k\geq 0} \sum_{\substack{\gamma_1,\dots,\gamma_k\geq 1\\ \gamma_1+\cdots+\gamma_k=m}} (-1)^{m-k} \prod_{i=1}^{k} (E^i - 1)^{\gamma_i}.$$

Proof. Both sides are 1 when $m = 0$, so let $m \geq 1$. Let $W_K^{=}$ be the subspecies of W_K consisting of those quasi composition matrices whose set of nonempty columns is exactly K. Then $W_K = \sum_{J\subseteq K} W_J^{=}$ and $W_{[m]}^{=} = \mathrm{CM}^{\langle m\rangle}$, so the inclusion-exclusion principle gives

$$\mathrm{CM}^{\langle m\rangle} = \sum_{K\subseteq[m]} (-1)^{m-|K|} W_K. \tag{6}$$

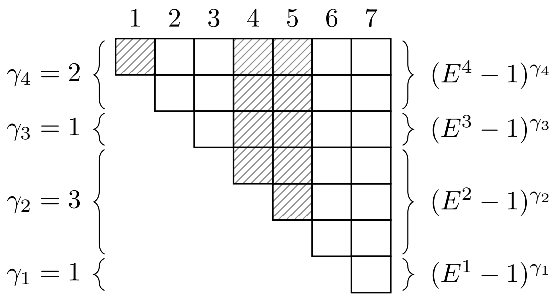
If $m \notin K$, then $\nu_K(m) = 0$, so Lemma 4.1 gives $W_K = 0$. Thus only the subsets containing m contribute. Let $K = \{b_1 < \cdots < b_k = m\}$ and put $b_0 = 0$. If $b_{j-1} < a \leq b_j$, then $\nu_K(a) = k - j + 1$. Thus, on letting

$$\gamma_{k-j+1} = b_j - b_{j-1},$$

we obtain positive integers $\gamma_1, \dots, \gamma_k$ that sum to m, and

$$W_K = \prod_{i=1}^{k} (E^i - 1)^{\gamma_i}.$$

For instance, let $m = 7$ and $K = \{2, 3, 6, 7\}$, so that $k = 4$. The shaded positions below are those in the columns outside K, which must stay empty. The runs of rows with equally many positions are braced on the left and the corresponding factors on the right.



Conversely, a composition $(\gamma_1, \dots, \gamma_k)$ recovers K by

$$b_j = \gamma_k + \gamma_{k-1} + \cdots + \gamma_{k-j+1}.$$

Hence $K \mapsto (\gamma_1, \dots, \gamma_k)$ is a bijection between the k-element subsets of $[m]$ containing m and the compositions of m into k parts. Substitution in (6) proves the result. $\square$

For instance, the compositions of $m = 2$ are 2 and $1 + 1$, so that

$$\mathrm{CM}^{\langle 2\rangle} = -(E - 1)^2 + (E - 1)(E^2 - 1) = E \cdot E_+^2,$$

which agrees with the following direct description. A 2×2 composition matrix has a nonempty entry at $(1, 1)$, an arbitrary entry at $(1, 2)$, and a nonempty entry at $(2, 2)$.

Proof of Theorem 1.1. We have

$$
\begin{aligned}
\mathcal{I} &= \sum_{m\geq 0} \mathrm{CM}^{\langle m\rangle} && \text{(Theorem 3.6)}\\
&= \sum_{m\geq 0}\sum_{k\geq 0}\sum_{\substack{\gamma_1,\dots,\gamma_k\geq 1\\ \gamma_1+\cdots+\gamma_k=m}} (-1)^{m-k}\prod_{i=1}^{k}\bigl(E^i-1\bigr)^{\gamma_i} && \text{(Lemma 4.2)}\\
&= \sum_{k\geq 0}\sum_{\gamma_1,\dots,\gamma_k\geq 1}\prod_{i=1}^{k}(-1)^{\gamma_i-1}\bigl(E^i-1\bigr)^{\gamma_i} && (m-k=\textstyle\sum_{i=1}^{k}(\gamma_i-1))\\
&= \sum_{k\geq 0}\prod_{i=1}^{k}\sum_{\gamma\geq 1}(-1)^{\gamma-1}\bigl(E^i-1\bigr)^{\gamma} && \text{(distributivity)}\\
&= \sum_{k\geq 0}\prod_{i=1}^{k}\Bigl(1-\sum_{\gamma\geq 0}(-1)^{\gamma}\bigl(E^i-1\bigr)^{\gamma}\Bigr)\\
&= \sum_{k\geq 0}\prod_{i=1}^{k}\Bigl(1-\bigl(1+(E^i-1)\bigr)^{-1}\Bigr)\\
&= \sum_{k\geq 0}\prod_{i=1}^{k}\bigl(1-(E^{-1})^i\bigr) = \mathrm{BalMat}
\end{aligned}
$$

□

Corollary 4.3. *The virtual species* BalMat *is positive. In the notation of Proposition 2.2, there is an isomorphism of species*

$$\mathcal{B}^+ \;=\; \mathcal{B}^- + \mathcal{I}.$$

Proof. By Proposition 2.2 and Theorem 1.1, $\mathcal{B}^+ - \mathcal{B}^- = \mathcal{I} - 0$, and the defining equivalence relation for Virt therefore gives $\mathcal{B}^+ = \mathcal{B}^- + \mathcal{I}$. □

Thus, there is a family of bijections $\mathcal{B}^+[U] \to \mathcal{B}^-[U] \sqcup \mathcal{I}[U]$, one for each finite set U, that commutes with transport. The involution η of Claesson and Hannah [6] gives a bijection for each linearly ordered U, but that family does not commute with all bijections. For example, η sends the 1×1 matrix with entry $\{1, 2\}$ to the matrix with entry $\{1\}\{2\}$. The transposition of 1 and 2 fixes the first matrix but not the second. In the next section we construct a natural bijection.

5 A bijective proof of Theorem 1.1

We now prove Corollary 4.3 bijectively. In view of Proposition 2.2, we seek a natural sign-reversing involution on ballot matrices whose fixed points are positive and naturally in bijection with interval orders.

Claesson and Hannah [6] associate an interval order $P(A)$ with every ballot matrix A on U. Let $(\mathrm{row}(x), \mathrm{col}(x))$ be the position of the entry containing $x \in U$. Then

$$x < y \text{ in } P(A) \iff \mathrm{col}(x) < \mathrm{row}(y).$$

Since the matrix is upper triangular, $\mathrm{row}(x) \leq \mathrm{col}(x)$. Let $I_x = [\mathrm{row}(x), \mathrm{col}(x)]$. Since $\max I_x = \mathrm{col}(x)$ and $\min I_y = \mathrm{row}(y)$, the displayed equivalence says precisely that $x < y$ exactly when I_x lies entirely to the left of I_y. Thus, the intervals I_x represent $P(A)$.

Making each nonempty entry of a composition matrix C into a one-block ballot gives a ballot matrix A_C. The formula for Γ^{-1} in Lemma 3.4 then gives $P(A_C) = \Gamma^{-1}(C)$.

5.1 The involution

Fix a ballot matrix A of dimension m. For each row i, let f_i be the column of its first nonempty entry, and let $S \subseteq [m]$ be the set of empty columns. Every row is nonempty, so f_i is well defined. If $m \geq 1$, the only position in row m is (m, m), and that entry is nonempty. Thus, $m \notin S$. Define:

$$\begin{aligned} \text{row } i \text{ is } \textit{splittable} &\iff \text{row } i \text{ contains at least two blocks;} \\ \text{row } i \text{ is } \textit{mergeable} &\iff \text{row } i \text{ contains one block, } i \in S \text{ and } f_{i+1} \geq f_i. \end{aligned}$$

Here $i \in S$ forces $i < m$. No row is both splittable and mergeable.

For a *split at row* i, let (i, j) be the first nonempty entry and B the first block there. Remove B, insert an empty row and column at index i, and place the one-block ballot B at $(i, j+1)$. The remainder of that entry moves to $(i+1, j+1)$. For a *merge at row* i, let B be its block. Since column i is empty, $f_i > i$. Move B to $(i+1, f_i)$, ahead of any blocks already there, and delete row and column i. See Figure 1.

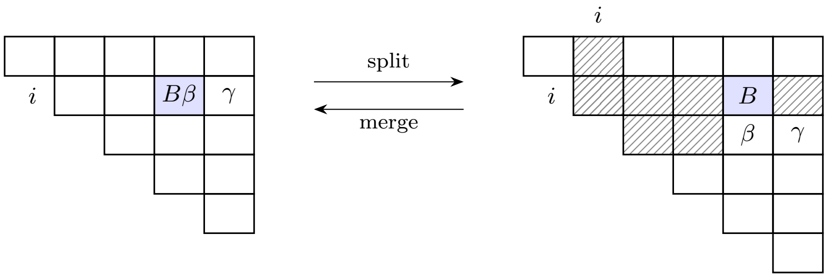
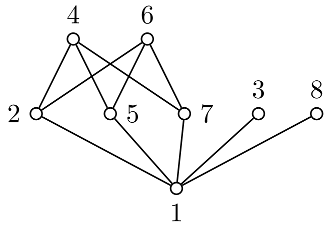


Figure 1: A split and its inverse merge. Here β is the remainder of the source ballot and γ a possible later ballot in the same row. Splittability ensures that at least one is present. The shaded positions are empty.

Example 5.1. Let $U = [8]$. In A, row 2 is the first row admitting a move. Splitting off the block $\{2,5\}$ gives A', and merging row 2 returns A. The gray positions mark the inserted row and column.

$$A = \begin{bmatrix} \{1\} & \emptyset & \emptyset & \emptyset \\ & \emptyset & \{2,5\}\{7\} & \emptyset \\ & & \emptyset & \{3,8\} \\ & & & \{4\}\{6\} \end{bmatrix} \underset{\text{merge}}{\overset{\text{split}}{\rightleftarrows}} \begin{bmatrix} \{1\} & \emptyset & \emptyset & \emptyset & \emptyset \\ & \emptyset & \emptyset & \{2,5\} & \emptyset \\ & & \emptyset & \{7\} & \emptyset \\ & & & \emptyset & \{3,8\} \\ & & & & \{4\}\{6\} \end{bmatrix} = A'.$$

There are six blocks in both matrices, so their signs are positive and negative, respectively. Both matrices determine the interval order



We return to this in Example 5.6 below.

Definition 5.2. For $A \in \mathcal{B}[U]$, let $\theta(A)$ be the result of performing the move (split or merge) at the least row admitting one. If no move is available, let $\theta(A) = A$.

The move conditions, the chosen row and the moves themselves use only occupied positions and block order, not labels. Thus, $\theta(\sigma \cdot A) = \sigma \cdot \theta(A)$ for every bijection σ, and $\mathrm{Fix}(\theta)$ is a subspecies of $\mathcal{B}$.

Theorem 5.3. *For every ballot matrix A, we have $\theta(\theta(A)) = A$ and $P(\theta(A)) = P(A)$. If $\theta(A) \neq A$, then A and $\theta(A)$ have opposite signs.*

Proof. Let A be a ballot matrix of dimension m. Consider a split at row i. Let A' be the result, let B be the block moved by the split, and let $\iota\colon [m] \to [m+1] \setminus \{i\}$ be the increasing bijection, given by $\iota(t) = t$ for $t < i$ and $\iota(t) = t+1$ for $t \geq i$. Assume row, col and $f_1, \ldots, f_m$ are defined as before. Write row$'$, col$'$ and $f'_1, \ldots, f'_{m+1}$ for the corresponding data of A'. Then

$$\mathrm{col}'(x) = \iota(\mathrm{col}(x)), \quad \mathrm{row}'(x) = \begin{cases} \iota(\mathrm{row}(x)), & x \notin B \\ i, & x \in B \end{cases} \tag{7}$$

and, for $1 \leq h \leq i$,

$$f'_h = \iota(f_h). \tag{8}$$

First, row i of A' holds B alone and column i is empty. Row $i+1$, the old row i less B, is nonempty, and its entries lie in columns $\iota(c)$ with $c \geq f_i$. Hence, A' is a ballot matrix and $f'_{i+1} \geq f'_i$. Thus, row i is mergeable and merging restores A. Conversely, consider a merge of B at (i, j). The merge condition gives $f_{i+1} \geq f_i = j$, so the old row $i+1$ has no entry before column j. The inequality $f_i > i$ makes the merged matrix upper triangular. Its merged row contains B, and every other row is nonempty because it comes from a row of A. Hence, the result is a ballot matrix. Its row i contains at least two blocks, and its first nonempty entry begins with B. Thus, row i is splittable, and splitting it restores A.

Second, a split preserves the interval order. Let $x, y \in U$. If $y \notin B$, then (7) and the fact that ι is increasing give

$$\mathrm{col}'(x) < \mathrm{row}'(y) \iff \iota(\mathrm{col}(x)) < \iota(\mathrm{row}(y)) \iff \mathrm{col}(x) < \mathrm{row}(y).$$

If $y \in B$, then $\mathrm{row}'(y) = \mathrm{row}(y) = i$, and

$$\mathrm{col}'(x) < i \iff \iota(\mathrm{col}(x)) < i \iff \mathrm{col}(x) < i.$$

Hence $P(A') = P(A)$. Merges preserve the order as well, since they are inverse to splits.

Third, we track the least row admitting a move. For a split at row i, the number of blocks in row $h < i$ and whether column h is empty do not change. Moreover, $h + 1 \leq i$, so (8) gives

$$f'_{h+1} \geq f'_h \iff \iota(f_{h+1}) \geq \iota(f_h) \iff f_{h+1} \geq f_h.$$

The move conditions therefore show that row h admits the same move, if any, before and after the split. Since every merge is inverse to a split, the same conclusion holds for a merge.

If θ acts first at row i, the inverse move is therefore available at row i of $\theta(A)$, and no earlier row admits a move. Since at most one move is available at a row, $\theta(\theta(A)) = A$.

Finally, each move preserves the number of blocks and changes the dimension by one. Thus, each move reverses the sign. $\square$

5.2 The fixed points

We shall identify the fixed points of θ with composition matrices. First we characterize the ballot matrices fixed by θ.

Proposition 5.4. *A ballot matrix A of dimension m is fixed by θ if and only if every row contains exactly one block and the block positions (i, f_i) satisfy*

$$\textit{column } i \textit{ empty} \implies f_{i+1} < f_i \quad (1 \leq i < m). \tag{9}$$

In particular, every fixed point has m blocks and sign $(-1)^{2m} = +1$.

Proof. Since every row is nonempty, a row is not splittable precisely when it contains one block. Row $i < m$ is mergeable precisely when column i is empty and $f_{i+1} \geq f_i$. Condition (9) excludes exactly these merges. $\square$

We next define a map κ from ballot matrices to composition matrices. The empty ballot matrix is sent to the empty composition matrix. Let A be a nonempty ballot matrix on U of dimension m, let $J \subseteq [m]$ be its set of nonempty columns, and put $d = |J|$. For $i \in [m]$, let

$$\pi(i) = 1 + \#\{j \in J : j < i\}$$

and define $\kappa(A)\colon U \to \mathrm{Int}_d$ by

$$\kappa(A)(x) = \big(\pi(\mathrm{row}_A(x)), \pi(\mathrm{col}_A(x))\big).$$

Thus, κ may send elements from distinct blocks to the same position. For instance,

$$\begin{bmatrix} \emptyset & \{x\} \\ & \{y\} \end{bmatrix} \xrightarrow{\kappa} [\{x, y\}].$$

Here $J = \{2\}$ and $\pi(1) = \pi(2) = 1$. Elements from distinct blocks are not sent to the same position when A is fixed by θ; see Proposition 5.7.

For a composition matrix C, we write $P(C)$ for $P(A_C)$.

Lemma 5.5. *For every ballot matrix A, $\kappa(A)$ is a composition matrix and*

$$P(\kappa(A)) = P(A).$$

Hence, $\kappa(A) = \Gamma(P(A))$. Moreover, the map κ commutes with transport.

Proof. We first show that κ maps ballot matrices to composition matrices and that the following triangle commutes:

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "\kappa"] \arrow[dr, "P"'] & & \mathrm{CM} \\ & \mathcal{I} \arrow[ur, "\Gamma"'] & \end{tikzcd}$$

Let A be a ballot matrix on U, and let J be its set of nonempty columns. The result is clear when A is empty. Suppose that A has dimension $m \geq 1$, and put $d = |J|$. Its last column is nonempty, so $m \in J$. Thus, π maps $[m]$ onto $[d]$ and restricts to a bijection $J \to [d]$. It follows that $\kappa(A)$ has no empty row or column. Since $i \leq j$ implies $\pi(i) \leq \pi(j)$, it is upper triangular. Its nonempty entries partition the underlying set, so it is a composition matrix.

Let x lie in position (i, j) and y in position (i', j') of A. Since $j \in J$, the definition of π gives

$$\mathrm{col}_A(x) < \mathrm{row}_A(y) \iff j < i' \iff \pi(j) < \pi(i').$$

By the definition of κ, the first and last inequalities are the conditions for $x < y$ in $P(A)$ and $P(\kappa(A))$, respectively. Hence, $P(\kappa(A)) = P(A)$. Lemma 3.4 now gives $\kappa(A) = \Gamma(P(A))$, so the triangle commutes.

Let $\sigma\colon U \to V$ be a bijection. We need to show that the following square commutes:

$$\begin{tikzcd} \mathcal{B}[U] \arrow[r, "\kappa"] \arrow[d, "\mathcal{B}[\sigma]"'] & \mathrm{CM}[U] \arrow[d, "\mathrm{CM}[\sigma]"] \\ \mathcal{B}[V] \arrow[r, "\kappa"'] & \mathrm{CM}[V] \end{tikzcd}$$

Along the top and right sides, A is sent to

$$\mathrm{CM}[\sigma](\kappa(A)) = \kappa(A) \circ \sigma^{-1}.$$

Along the left and bottom sides, the ballot matrix $\mathcal{B}[\sigma](A) = \sigma \cdot A$ has the same occupied positions as A, with every label x replaced by σx. Hence, J and π are unchanged, and, for $x \in U$,

$$(\kappa \circ \mathcal{B}[\sigma])(A)(\sigma x) = \kappa(\sigma \cdot A)(\sigma x) = \big(\pi(\mathrm{row}_A(x)), \pi(\mathrm{col}_A(x))\big) = \kappa(A)(x).$$

Thus, the second path also sends A to $\kappa(A) \circ \sigma^{-1}$, and the square commutes. □

We next construct a map λ that will reverse κ. A fixed point has one block in each row, so each nonempty entry of a composition matrix must acquire a row of its own. We choose its column so that κ returns it to its original position.

Let λ send the empty matrix to the empty ballot matrix. Suppose that C has dimension $d \geq 1$ and m nonempty entries. To satisfy condition (9), list them as $C_1, \ldots, C_m$ by increasing row and, within each row, decreasing column. Write $(a(i), b(i))$ for the position of C_i, and let e_r be the number of entries in rows $1, \ldots, r$. Then

$$0 = e_0 < e_1 < \cdots < e_d = m,$$

and the entries in row r are $C_{e_{r-1}+1}, \ldots, C_{e_r}$.

The entry C_i will occupy row i of $\lambda(C)$. The entries from row r of C have indices $e_{r-1} + 1, \ldots, e_r$. To make π send these indices back to r, we choose e_r as the r-th nonempty column. Thus, to recover the original column $b(i)$, place the one-block ballot C_i in position

$$\bigl(i, e_{b(i)}\bigr). \tag{10}$$

For each $r \in [d]$, column r of C contains some entry C_i, and then $b(i) = r$. Thus, the nonempty columns of $\lambda(C)$ are exactly $e_1, \ldots, e_d$, as required.

Since C_i lies in row $a(i)$ of C, its index in the list satisfies $i \leq e_{a(i)}$. Since C is upper triangular, $a(i) \leq b(i)$, and hence

$$i \leq e_{a(i)} \leq e_{b(i)}.$$

Thus, the chosen position lies in the upper triangle. The entries C_i partition the underlying set, and one entry is placed in each row, so $\lambda(C)$ is a ballot matrix.

Example 5.6. Let P be the interval order in Example 5.1. A fixed point A representing P and its image under κ are

$$A = \begin{bmatrix} \{1\} & \emptyset & \emptyset & \emptyset \\ & \emptyset & \emptyset & \{3,8\} \\ & & \{2,5,7\} & \emptyset \\ & & & \{4,6\} \end{bmatrix} \xrightarrow{\kappa} \begin{bmatrix} \{1\} & \emptyset & \emptyset \\ & \{2,5,7\} & \{3,8\} \\ & & \{4,6\} \end{bmatrix} = C.$$

Here $J = \{1,3,4\}$ and $\pi = (1,2,2,3)$. The four blocks move under κ to the positions

$$(1,1), \quad (2,3), \quad (2,2), \quad (3,3).$$

In the order used to define λ, the entries of C are

$$\{1\}, \quad \{3,8\}, \quad \{2,5,7\}, \quad \{4,6\}.$$

Its rows contain $1, 2, 1$ entries, so $(e_1, e_2, e_3) = (1,3,4)$. Formula (10) places these entries in columns $1, 4, 3, 4$, respectively, and recovers A.

Proposition 5.7. *The maps*

$$\kappa\colon \operatorname{Fix}(\theta) \to \mathrm{CM} \qquad \textit{and} \qquad \lambda\colon \mathrm{CM} \to \operatorname{Fix}(\theta)$$

are mutually inverse natural isomorphisms.

Proof. The result is clear for the empty matrices. Let C be a nonempty composition matrix, with the notation above. By construction, $\lambda(C)$ has one block in each row and nonempty columns $e_1, \dots, e_d$. If column i is empty, then $i < m$, and C_i and C_{i+1} lie in the same row of C. Hence,

$$f_{i+1} = e_{b(i+1)} < e_{b(i)} = f_i,$$

since the entries in that row were listed by decreasing column. Proposition 5.4 shows that $\lambda(C)$ is fixed.

We next compute $\kappa(\lambda(C))$. Let π be the map associated with $\lambda(C)$. Since C_i lies in row $a(i)$ of C,

$$e_{a(i)-1} < i \le e_{a(i)}.$$

The nonempty columns are $e_1 < \cdots < e_d$, so

$$\pi(i) = a(i), \qquad \pi(e_{b(i)}) = b(i).$$

The block C_i lies at $(i, e_{b(i)})$ in $\lambda(C)$. By the definition of κ, this block is sent to

$$\big(\pi(i), \pi(e_{b(i)})\big) = (a(i), b(i)),$$

its original position in C. Thus, $\kappa(\lambda(C)) = C$.

It remains to show that $\lambda(\kappa(A)) = A$. Let A be a nonempty fixed point of dimension m. Write B_i for the block in row i and (i, f_i) for its position. List the nonempty columns as $J = \{t_1 < \cdots < t_d\}$, and put $t_0 = 0$. We have $t_d = m$, and $\pi(i) = r$ precisely when

$$t_{r-1} < i \le t_r.$$

If $t_{r-1} < i < i' \le t_r$, then columns $i, \dots, i'-1$ are empty. Condition (9) therefore gives

$$f_i > f_{i+1} > \cdots > f_{i'}.$$

Since π is increasing on J, these blocks become distinct entries of $\kappa(A)$ in decreasing column order. Thus, the entries of $\kappa(A)$, ordered by increasing row and decreasing column, are $B_1, \dots, B_m$. Its first r rows contain t_r entries.

Put $C = \kappa(A)$. In the notation used to define λ, we have $e_r = t_r$. Moreover, $f_i \in J$, so $t_{\pi(f_i)} = f_i$. Therefore, $\lambda(C)$ places B_i at

$$\big(i, e_{\pi(f_i)}\big) = (i, f_i),$$

and $\lambda(\kappa(A)) = A$. Finally, κ is natural by Lemma 5.5, while the construction of λ uses only positions and transports each entry as a whole. Hence, both maps are natural. $\square$

Theorem 5.8. *The involution θ restricts to a natural isomorphism from the non-fixed points of $\mathcal{B}^+$ to $\mathcal{B}^-$, and $A \mapsto P(A)$ is a natural isomorphism* $\mathrm{Fix}(\theta) \to \mathcal{I}$. *Consequently,*

$$\mathcal{B}^+ \ = \ \mathcal{B}^- + \mathcal{I},$$

and Theorem 1.1 follows again, this time bijectively.

Proof. By Theorem 5.3, θ pairs non-fixed matrices with matrices of opposite sign. Proposition 5.4 shows that the fixed points are positive. Lemma 5.5 and Proposition 5.7 show that $P = \Gamma^{-1} \circ \kappa$ is a natural isomorphism from the fixed points to the interval orders. The two isomorphisms give $\mathcal{B}^+ = \mathcal{B}^- + \mathcal{I}$. Proposition 2.2 now gives $\mathrm{BalMat} = \mathcal{I}$. $\square$

Let w_m count upper triangular 0–1 matrices of all dimensions with m ones and no zero row or column. These numbers form the sequence A138265 in the OEIS [18], which begins $1, 1, 1, 2, 5, 16, 61, 271, 1372, \ldots$

Corollary 5.9. *The fixed-point species is*

$$\mathrm{Fix}(\theta) = \sum_{m \geq 0} w_m \, (E_+)^m. \tag{11}$$

Proof. Proposition 5.7 identifies fixed points with composition matrices. Replacing each nonempty entry by 1 gives one of the w_m matrices above, and its m ones independently carry nonempty sets. □

6 Generating series

Let $q = x_1 + x_2/2 + x_3/3 + \cdots$ as before. Theorem 1.1, together with $E^{-1}(x) = e^{-x}$, $\widetilde{E^{-1}}(x) = 1 - x$, and $Z_{E^{-1}} = e^{-q}$, gives the following three series.

Corollary 6.1.

(a) *The generating series for labeled interval orders is*

$$\mathcal{I}(x) = \sum_{m \geq 0} \prod_{i=1}^{m} \bigl(1 - e^{-ix}\bigr).$$

(b) *The type generating series is*

$$\widetilde{\mathcal{I}}(x) = \sum_{m \geq 0} \prod_{i=1}^{m} \bigl(1 - (1-x)^i\bigr).$$

(c) *The cycle index series is*

$$Z_{\mathcal{I}} = \sum_{m \geq 0} \prod_{i=1}^{m} \bigl(1 - e^{-iq}\bigr) = \mathcal{I}(q).$$

The identities $\mathcal{I}(x) = Z_{\mathcal{I}}(x, 0, 0, \ldots)$ and $\widetilde{\mathcal{I}}(x) = Z_{\mathcal{I}}(x, x^2, x^3, \ldots)$ recover parts (a) and (b) from part (c). Indeed, q becomes x in the first and $-\log(1-x)$ in the second.

7 Automorphisms and the molecular decomposition

We first read the automorphism group from the composition matrix.

Corollary 7.1. *Let P be an interval order on U whose composition matrix has dimension m, and let $C_{ij} = \{x \in U : (\ell(x), r(x)) = (i, j)\}$ be its entries. A bijection $\sigma\colon U \to U$ is an automorphism of P if and only if $\sigma(C_{ij}) = C_{ij}$ for every entry. Thus,* $\mathrm{Aut}(P)$ *is the Young subgroup* $\prod_{(i,j) \in \mathrm{Int}_m} \mathrm{Sym}(C_{ij})$*, where* $\mathrm{Sym}(C)$ *is the symmetric group on the set C.*

Proof. By naturality and the injectivity of Γ,

$$\sigma \cdot P = P \iff \Gamma(P) \circ \sigma^{-1} = \Gamma(P).$$

The last equality holds if and only if σ preserves every fiber C_{ij}. Such a bijection is obtained by choosing a permutation within each entry independently, which gives the stated Young subgroup. $\square$

In Example 3.5, the only nonsingleton entry is $\{3,4\}$. Thus $\mathrm{Aut}(P) = \mathrm{Sym}(\{3,4\})$.

The conclusion is special to interval orders. The automorphism group of 2+2 has order two but is generated by the double transposition that swaps the two chains, whereas a Young subgroup of order two is generated by a transposition.

Corollary 7.2. *A permutation $\sigma \in \mathcal{S}_n$ with c cycles is an automorphism of exactly $|\mathcal{I}[c]|$ interval orders on $[n]$.*

Proof. By Corollary 7.1, σ fixes P precisely when every entry of $\Gamma(P)$ is a union of σ-cycles. Let Ω be the set of these cycles. Replace each entry C_{ij} by

$$\{B \in \Omega : B \subseteq C_{ij}\}.$$

This gives a composition matrix on Ω. Conversely, replace each set of cycles by their union. These constructions are inverse bijections between the composition matrices on $[n]$ fixed by σ and those on Ω. The claim now follows from Theorem 3.6. $\square$

For example, let $\sigma \in \mathcal{S}_6$ have three cycles. Its cycle type may be $(2,2,2)$, $(3,2,1)$ or $(4,1,1)$. In every case, exactly $|\mathcal{I}[3]| = 19$ of the 81663 interval orders in $\mathcal{I}[6]$ are fixed by σ.

For an interval order P whose composition matrix has dimension m and entries C_{ij}, define the integer matrix

$$N(P) = \Big(|C_{ij}|\Big)_{(i,j)\in \mathrm{Int}_m}.$$

Jelínek [15] calls these *Fishburn matrices.* They index the isomorphism types of interval orders and hence the molecules of $\mathcal{I}$.

Corollary 7.3. *The molecular decomposition of $\mathcal{I}$ is*

$$\mathcal{I} \;=\; \sum_{m\geq 0} \sum_{N} \prod_{(i,j)\in \mathrm{Int}_m} E_{N_{ij}},$$

where the inner sum ranges over the upper triangular $m \times m$ matrices N of nonnegative integers with no zero row and no zero column, and factors $E_0 = 1$ are discarded.

Proof. Transport preserves entry sizes, so the interval orders with Fishburn matrix N form a subspecies $\mathcal{I}_N$. By Theorem 3.6, an $\mathcal{I}_N$-structure on U is a family (S_{ij}) of disjoint sets with union U and $|S_{ij}| = N_{ij}$. Hence

$$\mathcal{I}_N = \prod_{(i,j)\in \mathrm{Int}_m} E_{N_{ij}}.$$

Any two such families are isomorphic by matching corresponding entries, so $\mathcal{I}_N$ is molecular. The possible matrices N are precisely those in the statement, and summing over them proves the result. $\square$

To read off the terms of degree n, list the matrices N whose entries sum to n and replace each by the product of the species $E_{N_{ij}}$ over its nonzero entries. In degree 3 the matrices are

$$[3]\,,\quad \begin{bmatrix} 2 & 0 \\ & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 \\ & 2 \end{bmatrix},\quad \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix},$$

with products E_3, XE_2, XE_2, X^3 and X^3, since $E_1 = X$. After we collect equal products and group by degree, the decomposition begins

$$\begin{aligned} \mathcal{I} = 1 + X + \left(E_2 + X^2\right) + \left(E_3 + 2XE_2 + 2X^3\right) \\ + \left(E_4 + 2XE_3 + E_2^2 + 6X^2E_2 + 5X^4\right) + \cdots \end{aligned}$$

The coefficient of each molecular species counts the corresponding isomorphism types. Figure 2 draws them.

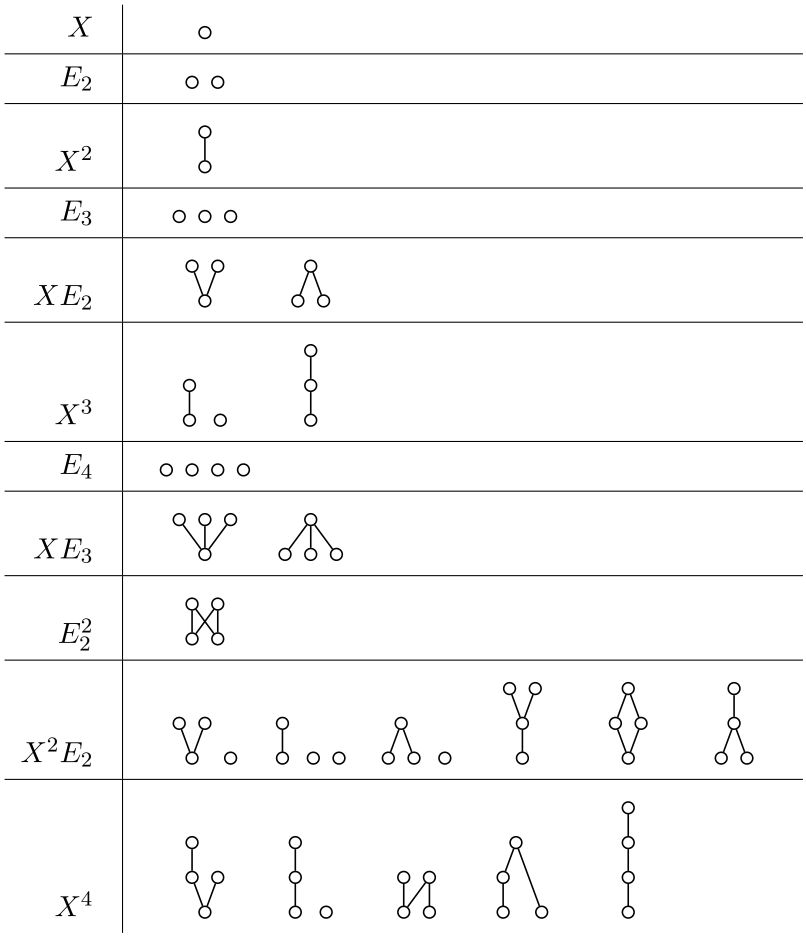


Figure 2: The interval orders in degrees 1 to 4, grouped by molecular species.

8 Rigid interval orders

An interval order is *rigid* if its only automorphism is the identity. Two elements are *indistinguishable* if they have the same strict down-set and the same strict up-set.

Dukes, Jelínek and Kubitzke [8] prove the following as part of their Lemma 8. We give a short proof from Corollary 7.1.

Lemma 8.1 (Dukes, Jelínek and Kubitzke [8])**.** *The nonempty entries of the composition matrix of an interval order are precisely its indistinguishability classes.*

Proof. The transposition exchanging x and y is an automorphism precisely when x and y are indistinguishable. By Corollary 7.1, it is an automorphism precisely when x and y lie in the same entry. $\square$

For instance, 3 and 4 are the only indistinguishable elements in Example 3.5. Both have strict down-set $\{1\}$ and strict up-set $\{6\}$, and they form the entry $\{3, 4\}$.

Corollary 7.1 shows that an interval order is rigid if and only if every nonempty entry of its composition matrix is a singleton.

Theorem 8.2. *Let $\mathcal{R} \subseteq \mathcal{I}$ be the subspecies of rigid interval orders. Partitioning an interval order into indistinguishability classes gives a natural isomorphism*

$$\mathcal{I} = \mathcal{R} \circ E_+.$$

Proof. Fix a finite set U. An $(\mathcal{R} \circ E_+)$-structure on U is a partition β of U together with a rigid interval order on β. Let $P \in \mathcal{I}[U]$, and let β be its partition into indistinguishability classes. Define a relation $<_Q$ on β by

$$B <_Q B' \quad \iff \quad x <_P y \quad \text{for some } x \in B \text{ and } y \in B'.$$

Since elements in the same block have the same strict down-sets and strict up-sets, "some" may be replaced by "every". Thus, Q is well defined. Choosing one element from each block identifies Q with an induced subposet of P, so Q is an interval order. The strict down-set and up-set of $x \in B$ in P are, respectively,

$$\{y \in U : y <_P x\} = \bigcup_{A <_Q B} A, \qquad \{y \in U : x <_P y\} = \bigcup_{B <_Q A} A.$$

Thus, if two blocks were indistinguishable in Q, their elements would be indistinguishable in P and hence lie in the same block of β. The elements of Q are therefore pairwise distinguishable. By Lemma 8.1, every nonempty entry of its composition matrix is a singleton, so Corollary 7.1 shows that Q is rigid.

Conversely, let β be a partition of U and let $Q \in \mathcal{R}[\beta]$. Define a relation on U by $x <_P y$ if and only if $B <_Q B'$, where $B, B' \in \beta$ are the blocks containing x, y. Take an interval representation of Q and give every $x \in B$ the interval of B. This represents P, so P is an interval order. Since the elements of Q are pairwise distinguishable, the same description of down-sets and up-sets shows that the indistinguishability classes of P are precisely the blocks of β. The two constructions are inverse and commute with relabeling. Hence, they give the claimed natural isomorphism. $\square$

Corollary 8.3. *With w_m as above,*

$$\mathcal{R} = \sum_{m \geq 0} w_m \, X^m.$$

Proof. By Corollary 7.1, an interval order is rigid if and only if every nonempty entry of its composition matrix is a singleton. Thus, in the molecular decomposition of Corollary 7.3, the rigid interval orders are indexed by the Fishburn matrices with entries zero or one. A matrix with m ones contributes the molecule X^m, and there are w_m such matrices by definition. □

Combining Theorem 8.2 and Corollary 8.3 gives

$$\mathcal{I} = \mathcal{R} \circ E_+ = \sum_{m \geq 0} w_m \, (E_+)^m.$$

By Theorem 5.8, this recovers the fixed-point identity (11) of Corollary 5.9. More explicitly, the nonempty entries of $\kappa(A)$ are the indistinguishability classes of $P(A)$. Replacing each nonempty entry C_{ij} by $\{C_{ij}\}$ gives the composition matrix of the rigid interval order on these classes.

Taking type generating series gives

$$\widetilde{\mathcal{R}}(x) = \sum_{m \geq 0} w_m \, x^m, \qquad \widetilde{\mathcal{I}}(x) = \sum_{m \geq 0} w_m \Big(\frac{x}{1-x}\Big)^m.$$

Thus, w_m is the number of rigid interval orders on m points up to isomorphism. Substituting $x/(1+x)$ for x in the second identity gives

$$\sum_{m \geq 0} w_m \, x^m = \widetilde{\mathcal{I}}\Big(\frac{x}{1+x}\Big).$$

The enumerative content here is known. Brightwell and Keller [4] use the inflation of rigid orders in their asymptotic analysis. Khamis [16] and Dukes, Kitaev, Remmel and Steingrímsson [9] obtain this series for rigid orders, or equivalently for upper triangular 0–1 matrices with no zero row or column.

9 Glaisher's T-numbers

Zagier's [19] asymptotic analysis of the Fishburn numbers rests on the identity

$$e^{-t/24} \sum_{m \geq 0} \prod_{i=1}^{m} \big(1 - e^{-it}\big) \;=\; \sum_{n \geq 0} \frac{T_n}{n!} \Big(\frac{t}{24}\Big)^n, \tag{12}$$

where $T_0, T_1, T_2, \ldots$ are *Glaisher's T-numbers*, listed as A002439 in the OEIS [18]. They begin $1, 23, 1681, 257543, 67637281, \ldots$ and are defined by

$$\sum_{n \geq 0} \frac{T_n}{(2n+1)!} \, x^{2n+1} \;=\; \frac{\sin 2x}{2 \cos 3x}.$$

We give a combinatorial interpretation of T_n in terms of colored interval orders.

After the rescaling $t = 24x$, the left-hand side of (12) is $e^{-x}\mathcal{I}(24x)$. Call an element of a poset *isolated* if it is comparable to no other element. We shall see that $e^{-x}\mathcal{I}(24x)$ is the generating series of a species: the 24-colored interval orders in which no isolated element has color 24.

For a positive integer c, a *c-colored F-structure* on U is an F-structure on U together with a map $U \to [c]$. These form the species $F(cX)$, the usual substitution of cX into F. In particular, $E(cX) = E^c$ and $F(cX)(x) = F(cx)$.

Definition 9.1. Let $\mathcal{T}$ be the species of 24-colored interval orders in which no isolated element has color 24.

Theorem 9.2. *There is a natural isomorphism* $\mathcal{I}(24X) = E \cdot \mathcal{T}$ *and, consequently,*

$$\mathcal{T} \;=\; E^{-1}\mathcal{I}(24X)$$

in Virt. *Moreover,* $|\mathcal{T}[n]| = T_n$*.*

Proof. A 24-colored interval order is uniquely the disjoint union of its isolated elements of color 24 and a $\mathcal{T}$-structure. This decomposition is natural and proves $\mathcal{I}(24X) = E \cdot \mathcal{T}$. Taking generating series gives $\mathcal{T}(x) = e^{-x}\mathcal{I}(24x)$. Comparing with (12) at $t = 24x$ yields $|\mathcal{T}[n]| = T_n$. □

This interpretation by colored interval orders seems to be new. The OEIS [18] records the generating function obtained from (12) by setting $t = 24x$, but gives no combinatorial interpretation. Hoffman [14] shows that $2T_n$ counts the ER_{2n+1}-snakes, a class of 3-signed alternating permutations.

From $\mathcal{T} = E^{-1}\mathcal{I}(24X)$ we can also calculate the cycle index series of $\mathcal{T}$.

Corollary 9.3. *With* $q = x_1 + x_2/2 + \cdots$ *as before, the cycle index series of* $\mathcal{T}$ *is*

$$Z_{\mathcal{T}} = e^{-q} \sum_{m \geq 0} \prod_{i=1}^{m} \big(1 - e^{-24iq}\big) = \mathcal{T}(q).$$

Here $\mathcal{T}(x) = \sum_{n \geq 0} T_n x^n/n!$ *as above. Its type generating series is*

$$\widetilde{\mathcal{T}}(x) = (1-x) \sum_{m \geq 0} \prod_{i=1}^{m} \big(1 - (1-x)^{24i}\big).$$

The coefficients of $\widetilde{\mathcal{T}}(x)$ begin

$$1, 23, 852, 43772, 2883382, 231864234, 22020936532, \ldots$$

and this sequence is not in the OEIS [18].

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