

Involution h on Catalan structures

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Abstract

We define an involution h on Catalan structures through an abstract framework, prove an equidistribution theorem for four canonical statistics and present a generating function carrying these. This framework encompasses all combinatorial structures with a decomposition mirroring the first-return decomposition of Dyck paths. The fixed points of h are counted by Catalan numbers. Canonical bijections transport the equidistribution to eight well known concrete families, identifying the canonical statistics with native ones on each. In addition to its primary structure, each Catalan structure has a derived *secondary structure*, and h interchanges primary and secondary structure. The involution factors as $h = \text{rev} \circ \text{corev} \circ \text{rev}$, where rev and corev are two simpler involutions, and the composition $M = h \circ \text{rev}$ coincides with Donaghey's automorphism on plane trees. This yields $M^{-1} = \text{rev} \circ M \circ \text{rev}$ and a period theorem: Iterating the secondary structure construction produces a sequence that repeats with period equal to the order of M . It is an open problem to describe h and the canonical statistics explicitly on most of the more than two hundred known families of Catalan structures.

1 Introduction

The Catalan numbers $\frac{1}{n+1} \binom{2n}{n}$ form one of the most ubiquitous sequences in combinatorics, counting, so far, over two hundred families of objects [21]. Many of these families carry intrinsic algebraic structure (a notion of indecomposability and a way to compose objects), and the standard bijections between them often respect this structure.

In a previous paper [4] the first three authors defined an involution h on $\beta(1, 0)$ -trees; the proof that h is an involution appeared in [5]. Fang [9] subsequently showed that, under natural bijections, h corresponds to map duality on non-separable planar maps

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and to interval duality on synchronized intervals in the Tamari lattice. Here we place the involution h in an abstract setting in which a Catalan structure is specified by a graded set C , a distinguished empty element e , an indecomposability operator λ , and an associative composition \oplus . The defining axiom is that every element decomposes uniquely as e , $\lambda(s)$, or $u \oplus v$ (Definition 2.1). Starting from a primary structure (C, e, λ, \oplus) we derive a secondary structure (C, e, γ, \ominus) , define the involution h that interchanges the primary and secondary structures, characterize its fixed points, and establish an equidistribution theorem on four statistics defined on the abstract structure C .

We then instantiate the framework on eight concrete Catalan families: rooted plane trees, Dyck paths, triangulations of convex n -gons, permutations respectively avoiding the patterns 231 and 321, binary trees, $2 \times n$ standard Young tableaux, and non-crossing partitions. For each family we transport the equidistribution result, identifying the transported statistics with natively defined ones. The resulting equidistribution theorem (Theorem 4.1) shows that, on each family, four natively defined statistics form two equidistributed pairs under h . In the concrete case of Dyck paths this theorem says that the joint distribution of the number of returns to the x -axis and the number of peaks is the same as that for the length of the final run of down-steps and one plus the number of double up-steps. On some families h admits a particularly simple description. On triangulations of the $(n+2)$ -gon, for instance, h is vertex complement, relabeling each vertex i as $n+3-i$.

Many, if not most, known combinatorial structures counted by the Catalan numbers have a decomposition mirroring the well known first-return decomposition of Dyck paths, which automatically makes them an instance of the abstract framework we define here, so that the equidistribution results mentioned above hold for them. What is not trivial, however, is how the involution h , and the corresponding equidistributed statistics, manifest in each instance. Given the vast number of such structures it seems reasonable to expect many of them to have intelligible descriptions of h and the equidistributed statistics, which would be interesting to explore further. Moreover, it seems likely that h will in many instances translate other statistics between Catalan structures in intelligible ways.

The abstract framework also sheds light on Donaghey’s automorphism [8] on plane trees, which turns out to be intimately connected to h . Donaghey constructs a size-preserving bijection M on plane trees (equivalently, on each Catalan family) as $L^{-1} \circ R$, where R and L are two maps between general and binary bracketings. Donaghey himself notes “the chaotic nature of M ” [8]. The equivalent map $T \circ R$ on ordered forests, where R and T are Knuth’s conjugate and transpose [12, Ex. 17, §7.2.1.6], has equally intractable cycle structure [2]. While the cycle structure remains open (see Section 6), our framework gives the factorization $M = h \circ \text{rev}$, decomposing Donaghey’s map into two simpler involutions: h interchanges primary and secondary structure, and rev reverses the first-return decomposition. This yields $M^{-1} = \text{rev} \circ M \circ \text{rev}$. Using a result of Pushkarev and Byzov [17], we show that h and rev generate the free product $\mathbb{Z}_2 * \mathbb{Z}_2$, so M has unbounded order. A period theorem complements this: the iterated secondary structures $(\lambda_1, \oplus_1), (\lambda_2, \oplus_2), \dots$ repeat with period equal to the order of M (see Section 5).

The sections of the paper are as follows:

- Section 2: The abstract Catalan framework, where h and all identities live. The map h is defined in Definition 2.6.
- Section 3: Eight concrete Catalan families, their operations, and concrete descriptions of h where available.
- Section 4: Statistics, recursive equations, the equidistribution proof, generating functions, and per-family corollaries obtained by transporting the canonical statistics through canonical bijections, recorded in a dictionary identifying each canonical statistic with a native one.
- Section 5: The factorization $h = \text{rev} \circ \text{corev} \circ \text{rev}$; the map $M = h \circ \text{rev}$, which coincides with Donaghey's automorphism; the group generated by h and rev , which is the free product $\mathbb{Z}_2 * \mathbb{Z}_2$; and iterated secondary structures, whose period equals the order of M .
- Appendix A: Verification that the seven concrete bijections of Section 3.9 are canonical.
- Appendix B: Verification that each entry of the dictionary in Section 4.2 satisfies the canonical recursion.

2 The abstract Catalan framework

Definition 2.1. A *Catalan structure* consists of (C, e, λ, \oplus) , where C is the disjoint union $\sqcup_{n \geq 0} C_n$, with $C_0 = \{e\}$ and

$$\begin{aligned} \lambda: C_n &\rightarrow C_{n+1} \\ \oplus: C_m \times C_n &\rightarrow C_{m+n} \end{aligned}$$

where \oplus is associative with two-sided identity e , and the following unique decomposition property holds: every $t \in C$ can be written in exactly one of three forms:

$$\begin{aligned} t = e & && \text{(empty)} \\ t = \lambda(s) & \text{ for some } s \in C && \text{(indecomposable)} \\ t = u \oplus v & \text{ for some indecomposable } u \text{ and } v \in C \setminus C_0 && \text{(decomposable)} \end{aligned}$$

We denote the size of an object X , usually understood from the context, by $|X|$. In particular, if $t \in C_n$ then $|t| = n$.

By Definition 2.1, every nonempty element factors uniquely as $\lambda(s_1) \oplus \cdots \oplus \lambda(s_k)$ with $k \geq 1$. Peeling from either end gives a pair of two-case decompositions:

- First-return decomposition: every nonempty t is uniquely $\lambda(u) \oplus v$.
- Last-return decomposition: every nonempty t is uniquely $u \oplus \lambda(v)$.

The *free* Catalan structure is realized concretely by the canonical expressions e , $\lambda(t)$, and $\lambda(u) \oplus v$ with $v \neq e$. By Definition 2.1 and the first-return decomposition, every

element corresponds to exactly one such expression, so the elements may be identified with these expressions, built up from e by λ and \oplus . In particular,

$$\begin{aligned} C_0 &= \{ e \}, \\ C_1 &= \{ \lambda(e) \}, \\ C_2 &= \{ \lambda(\lambda(e)), \lambda(e) \oplus \lambda(e) \}, \\ C_3 &= \{ \lambda(\lambda(\lambda(e))), \lambda(\lambda(e) \oplus \lambda(e)), \\ &\quad \lambda(\lambda(e)) \oplus \lambda(e), \lambda(e) \oplus \lambda(\lambda(e)), \lambda(e) \oplus \lambda(e) \oplus \lambda(e) \}. \end{aligned}$$

Note that removing the symbols λ , e , and \oplus from the C_i above we obtain the sets of balanced parentheses well known to be counted by the Catalan numbers. The construction described here is the special case $\lambda = \lambda_1$ of a more general construction on $\beta(1, 0)$ -trees in [4].

A closely related axiomatization was developed by Brak [1], who observes that each Catalan family carries a *free magma structure*: a non-associative binary product with unique factorization and an additive norm. The unique isomorphism between free magmas then gives a “universal bijection” between any two families. The present framework differs in using an associative composition \oplus together with a separate unary operator λ , which gives the three-form decomposition directly and makes the secondary structure (γ, \ominus) and the involution h natural to define. Brak does not consider the secondary structure or the involution h .

Each concrete family (defined in Section 3) is a Catalan structure, and the universal property of the free structure ensures that any identity proved there transfers to every instance. Before introducing the secondary structure, we give two examples.

Let us view Dyck paths as words over the alphabet $\{u, d\}$. Assuming w , w_1 and w_2 are Dyck paths, we define

$$\begin{aligned} \lambda(w) &= uwd && \text{(lift the path } w) \\ w_1 \oplus w_2 &= w_1 w_2 && \text{(concatenate } w_1 \text{ and } w_2). \end{aligned}$$

Secondly, we consider rooted plane trees, represented as lists of subtrees hanging from the root. The single-node tree (a leaf) is the empty list $[\]$. Assuming T , T_1 and T_2 are rooted plane trees, we define

$$\begin{aligned} \lambda(T) &= [T] && \text{(new root with } T \text{ as sole child)} \\ T_1 \oplus T_2 &= [c_1, \dots, c_j, c'_1, \dots, c'_k] \end{aligned}$$

where $T_1 = [c_1, \dots, c_j]$ and $T_2 = [c'_1, \dots, c'_k]$. In other words, \oplus concatenates the child lists of the two roots.

Definition 2.2. Given a Catalan structure (C, e, λ, \oplus) we derive a *secondary structure* (C, e, γ, \ominus) on the same set C . Define

$$\gamma(t) = t \oplus \lambda(e)$$

and define \ominus by recursion on the three-form decomposition of the first argument:

$$\begin{aligned} e \ominus v &= v \\ \lambda(u) \ominus v &= \lambda(u \ominus v) \\ (t \oplus u) \ominus v &= t \oplus (u \ominus v). \end{aligned}$$

A decomposable first argument may factor as $t \oplus u$ with $t, u \neq e$ in more than one way, but $t \oplus (u \ominus v)$ is the same for every such factorization, so \ominus is well-defined and the third clause may be applied to any of them. Alternatively, t may be assumed to be indecomposable. The recursion terminates by induction on the size of the first argument. Indeed, the second and third clauses reduce to the recursive call $u \ominus v$, whose first argument u is strictly smaller than $\lambda(u)$ and than $t \oplus u$ (here $t \neq e$), respectively. The first clause is a base case. These definitions specialize the constructions of [4] to the single- λ setting.

On plane trees, γ appends a leaf as the rightmost child of the root, and $t \ominus v$ replaces the leaf at the end of the rightmost-child chain of t by v (Figure 1). On Dyck paths, $\gamma(w) = wud$ and $w_1 \ominus w_2$ replaces the final ud of w_1 with uw_2d .

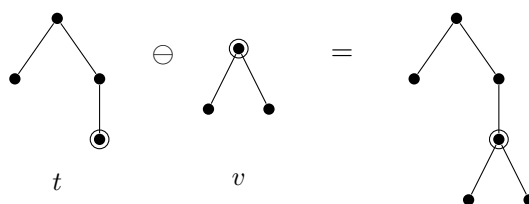


Figure 1: The secondary composition \ominus on plane trees. The rightmost-child chain of t ends at a leaf (circled) and in $t \ominus v$ that leaf is replaced by the root of v (also circled), and the rest of v hangs off below.

Lemma 2.3. *For all $t, u, v \in C$,*

$$(t \oplus \lambda(u)) \ominus v = t \oplus \lambda(u \ominus v).$$

In particular, $\gamma(t) \ominus v = t \oplus \lambda(v)$.

Proof. Applying the third and then the second defining clause of \ominus we get

$$(t \oplus \lambda(u)) \ominus v = t \oplus (\lambda(u) \ominus v) = t \oplus \lambda(u \ominus v).$$

Setting $u = e$ and using $\gamma(t) = t \oplus \lambda(e)$ gives the particular case. \square

Proposition 2.4. *The derived quadruple (C, e, γ, \ominus) is itself a Catalan structure.*

Proof. We verify the axioms of Definition 2.1. The grading is preserved: $\gamma: C_n \rightarrow C_{n+1}$ since $\gamma(t) = t \oplus \lambda(e)$ adds one to the size, and $\ominus: C_m \times C_n \rightarrow C_{m+n}$ since each clause of the recursion preserves the total size.

Identity. $e \ominus v = v$ is the first clause of the definition. The right identity, $t \ominus e = t$ follows by induction on the three-form decomposition of t : if $t = e$ it is immediate; if $t = \lambda(u)$ then $\lambda(u) \ominus e = \lambda(u \ominus e) = \lambda(u)$, and if $t = s \oplus u$ then $(s \oplus u) \ominus e = s \oplus (u \ominus e) = s \oplus u$.

Associativity. $(t \ominus u) \ominus v = t \ominus (u \ominus v)$ follows by induction on the three-form decomposition of t . If $t = e$ then both sides equal $u \ominus v$. If $t = \lambda(s)$ then

$$(\lambda(s) \ominus u) \ominus v = \lambda(s \ominus u) \ominus v = \lambda((s \ominus u) \ominus v) = \lambda(s \ominus (u \ominus v)) = \lambda(s) \ominus (u \ominus v),$$

using the inductive hypothesis on s . If $t = s \oplus w$ then

$$\begin{aligned} ((s \oplus w) \ominus u) \ominus v &= (s \oplus (w \ominus u)) \ominus v = s \oplus ((w \ominus u) \ominus v) = \\ &= s \oplus (w \ominus (u \ominus v)) = (s \oplus w) \ominus (u \ominus v), \end{aligned}$$

using the inductive hypothesis on w .

Unique decomposition. Every nonempty element has a unique last-return decomposition $t = u \oplus \lambda(v)$. By Lemma 2.3, $\gamma(u) \ominus v = u \oplus \lambda(v)$, so $t = \gamma(u) \ominus v$. If $v = e$ then $t = \gamma(u)$. If $v \neq e$ then $t = \gamma(u) \ominus v$ with $\gamma(u)$ and v both nonempty. Conversely, from $t = \gamma(u) \ominus v$ we recover $u \oplus \lambda(v)$, so the secondary decomposition is unique. \square

Lemma 2.5. $\gamma(u \oplus v) = u \oplus \gamma(v)$.

Proof. $\gamma(u \oplus v) = (u \oplus v) \oplus \lambda(e) = u \oplus (v \oplus \lambda(e)) = u \oplus \gamma(v)$. \square

Conversely, λ and \oplus can be recovered from γ and \ominus :

$$\lambda(t) = \gamma(e) \ominus t$$

and

$$\begin{aligned} t \oplus e &= t \\ t \oplus \gamma(u) &= \gamma(t \oplus u) \\ t \oplus (u \ominus v) &= (t \oplus u) \ominus v \quad (u \neq e). \end{aligned}$$

Thus, given a primary Catalan structure (C, e, λ, \oplus) we derive a secondary structure (C, e, γ, \ominus) as above (or vice versa).

We now define the maps that relate different Catalan structures via the unique decomposition of each. Let $(A, e_A, \lambda_A, \oplus_A)$ and $(B, e_B, \lambda_B, \oplus_B)$ be two Catalan structures. A *canonical bijection* $\varphi : A \rightarrow B$ is obtained by

$$\begin{aligned} \varphi(e_A) &= e_B \\ \varphi(\lambda_A(t)) &= \lambda_B(\varphi(t)) \\ \varphi(u \oplus_A v) &= \varphi(u) \oplus_B \varphi(v). \end{aligned}$$

Definition 2.6. The *map* h is defined by

$$\begin{aligned} h(e) &= e \\ h(\lambda(t)) &= \gamma(h(t)) \\ h(u \oplus v) &= h(v) \ominus h(u). \end{aligned}$$

Equivalently, using the first-return decomposition:

$$\begin{aligned} h(e) &= e \\ h(\lambda(u) \oplus v) &= h(v) \ominus \gamma(h(u)). \end{aligned}$$

Lemma 2.7 (Last-return form). *The map h is also characterized by*

$$\begin{aligned} h(e) &= e \\ h(u \oplus \lambda(v)) &= h(v) \oplus \lambda(h(u)) \end{aligned}$$

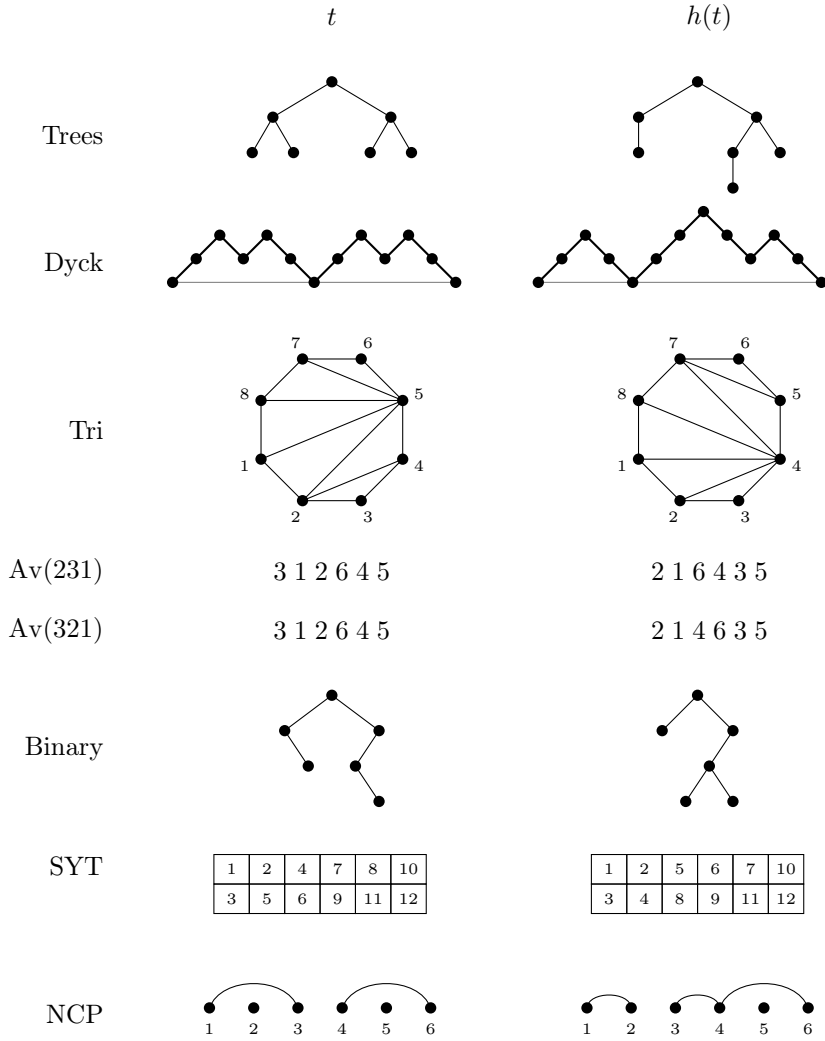


Figure 2: The involution h on a size-6 Catalan object, shown simultaneously across all eight families. Corresponding objects are related by canonical bijections.

Proof. By Lemma 2.3,

$$h(u \oplus \lambda(v)) = h(\lambda(v)) \ominus h(u) = \gamma(h(v)) \ominus h(u) = h(v) \oplus \lambda(h(u)). \quad \square$$

For example, on plane trees, $h([\]) = [\]$, $h([e]) = \gamma(e) = [e]$, and $h([e, e, e]) = \lambda(\lambda(\lambda(e))) = [[[e]]]$: the tree with three children of the root maps to the path of depth three. Figure 2 shows h acting simultaneously on all eight families for a size-6 example.

Lemma 2.8. $h(\gamma(t)) = \lambda(h(t))$ for all t .

Proof. By Lemma 2.3, $h(\gamma(t)) = h(t \oplus \lambda(e)) = h(e) \oplus \lambda(h(t)) = \lambda(h(t))$. \square

Theorem 2.9. h is an involution.

Proof. By induction on size. The base case $h(h(e)) = h(e) = e$ is immediate. For $t = u \oplus \lambda(v)$:

$$h^2(u \oplus \lambda(v)) = h(h(v) \oplus \lambda(h(u))) = h(h(u)) \oplus \lambda(h(h(v))) = u \oplus \lambda(v),$$

using Lemma 2.7 twice and the induction hypothesis.¹ □

Corollary 2.10. $h(u \ominus v) = h(v) \oplus h(u)$ for all u, v .

Proof. By the definition of h and Theorem 2.9,

$$h(h(v) \oplus h(u)) = h(h(u)) \ominus h(h(v)) = u \ominus v.$$

Applying h to both sides and using Theorem 2.9 again, $h(u \ominus v) = h(v) \oplus h(u)$. □

Proposition 2.11. *The fixed points of h are e and the elements*

$$u \oplus \lambda(h(u)) \quad \text{for } u \in C.$$

In particular, there are $|C_k|$ fixed points of size $2k + 1$ (the k -th Catalan number) and none of positive even size.

Proof. The empty element e is fixed. For nonempty t , write $t = u \oplus \lambda(v)$ in its last-return decomposition. By Lemma 2.7, $h(t) = h(v) \oplus \lambda(h(u))$, which is also in last-return form. By uniqueness of the last-return decomposition, $h(t) = t$ is equivalent to the pair of equations $h(v) = u$ and $h(u) = v$ and, since h is an involution (Theorem 2.9), either one implies the other, so both are equivalent to the single condition $h(u) = v$. The fixed points of size n are thus parametrized by $u \in C_k$ via $u \mapsto u \oplus \lambda(h(u))$ when $n = 2k + 1$, giving $|C_k|$ fixed points, and there are none when n is even. □

On the more general $\beta(1, 0)$ -trees (rooted plane trees with certain positive integer labels on the nodes), the fixed-point structure is richer. Kitaev and de Mier [10] show that, in addition to the analogue of the $u \oplus \lambda(h(u))$ construction above, a second recursive family of fixed points arises from the interaction of labels with the rightmost path. They enumerate all fixed points and obtain the sequence A006013 in the OEIS [16].

Proposition 2.12 (Transport). *Let $P(t_1, \dots, t_k) = Q(t_1, \dots, t_k)$ be an equation between expressions built from $e, \lambda, \oplus, \gamma,$ and \ominus . If the equation holds for all t_1, \dots, t_k in the free Catalan structure, then it holds in every Catalan structure.*

Proof. For any Catalan structure (C, e, λ, \oplus) , structural recursion on the three-form decomposition of the free structure defines a unique map φ satisfying $\varphi(e) = e$, $\varphi(\lambda(t)) = \lambda(\varphi(t))$, and $\varphi(u \oplus v) = \varphi(u) \oplus \varphi(v)$. The same recursion applied to the three-form decomposition of C defines an inverse φ^{-1} from C to the free structure, and $\varphi \circ \varphi^{-1} = \varphi^{-1} \circ \varphi = \text{id}$ by induction on the respective decompositions. Since γ and \ominus are defined in terms of λ and \oplus , φ preserves them as well. Any equation between such expressions that holds in the free structure therefore holds in C . □

¹The last identity in the proof of [5, Theorem 5] should read $\dots = h^2(u) \oplus h^2(v) = u \oplus v$.

The involution h is defined by recursion on λ and \oplus , so the same canonical bijection φ preserves it. Consequently, every identity and corollary involving h proved in the free Catalan structure, including the involution property, the fixed-point characterization, and the equidistribution theorem below, holds uniformly across all families.

3 Concrete Catalan families

We now instantiate the abstract framework on eight concrete families. For each, we define e , λ , and \oplus , and describe the resulting γ and \ominus in combinatorial terms. We also give concrete descriptions of h where available.

3.1 Rooted plane trees

A rooted plane tree (henceforth *plane tree*) is represented as a list of the children of the root. The single-node tree (a leaf) has an empty child list. The indecomposable trees are those with exactly one child of the root.

- $e = []$, the single-node tree (empty child list).
- $\lambda(T) = [T]$: a new root with T as its sole child.
- $T_1 \oplus T_2$ concatenates the child lists of the two roots under a common root.
- $\gamma(T)$ appends a leaf as the rightmost child of the root.
- $T_1 \ominus T_2$ replaces the leaf at the end of the rightmost-child chain of T_1 by T_2 .

The involution h has the following description on plane trees. If $T = [c_1, \dots, c_k]$, then the root of $h(T)$ has as children those of $h(c_k)$, followed by one new child. That child's children are those of $h(c_{k-1})$, followed by another new child, and so on until the deepest new node, whose children are those of $h(c_1)$ followed by a leaf.

3.2 Dyck paths

Dyck paths of semilength n are words over $\{u, d\}$ of length $2n$ such that each prefix has at least as many u 's as d 's and equally many in total. The indecomposable Dyck paths are those of the form uwd (touching the x -axis only at the endpoints).

- $e = \epsilon$, the empty path.
- $\lambda(w) = uwd$: enclose w in an up-step and a down-step (lift w one unit).
- $w_1 \oplus w_2 = w_1w_2$: concatenation.
- $\gamma(w) = wud$: append a peak.
- $w_1 \ominus w_2$: replace the final peak ud of w_1 with uw_2d . Every nonempty Dyck path has at least one peak, so this is well-defined, the base case being $\epsilon \ominus w_2 = w_2$.

Writing $D = D_1 \cdots D_k$ with $D_i = uw_i d$, the involution acts by

$$h(D) = h(w_k) u h(w_{k-1}) u \cdots u h(w_1) u d^k.$$

In particular, $h(\epsilon) = \epsilon$ and $h(uwd) = h(w)ud$.

3.3 Triangulations

A triangulation of the convex $(n+2)$ -gon is a maximal set of non-crossing diagonals, with vertices labeled $1, 2, \dots, n+2$ clockwise. It has $n-1$ diagonals, $n+2$ boundary edges, and n triangles, and we say it has *size* n . An indecomposable triangulation is one in which vertices 2 and $n+2$ belong to a common triangle (equivalently, vertex 1 has no incident diagonals).

- e is the single edge $(1, 2)$.
- $\lambda(T)$: relabel each vertex $i \rightarrow i+1$ in T , then add a new vertex 1 adjacent to 2 and to $n+3$.
- $T_1 \oplus T_2$: identify the edge $(1, n_1+2)$ of T_1 with the edge $(1, 2)$ of T_2 , shifting T_2 's labels accordingly.
- $\gamma(T) = T \oplus \lambda(e)$: append a triangle sharing the edge $(1, n+2)$.
- $T_1 \ominus T_2$: if T_1 is decomposable, let V be its last indecomposable summand and write $T_1 = U \oplus V$; then $T_1 \ominus T_2 = U \oplus (V \ominus T_2)$. If T_1 is indecomposable, write $T_1 = \lambda(T'_1)$ (remove vertex 1 and its unique incident triangle, relabel $i \mapsto i-1$) and set $T_1 \ominus T_2 = \lambda(T'_1 \ominus T_2)$.

Figure 3 illustrates these four operations on small triangulations.

Proposition 3.1 (Vertex complement). *On triangulations of the $(n+2)$ -gon, h acts by relabeling each vertex i as $n+3-i$. Equivalently, edge (i, j) maps to $(n+3-i, n+3-j)$.*

Proof. Write \bar{T} for the vertex-complement of a triangulation T , that is, the triangulation obtained by relabeling each vertex i as $n+3-i$. We show $\bar{T} = h(T)$ by induction on size, using Lemma 2.7: $h(e) = e$ and $h(u \oplus \lambda(v)) = h(v) \oplus \lambda(h(u))$.

Base case. The empty triangulation e is the edge $(1, 2)$. The complement sends $1 \mapsto 2, 2 \mapsto 1$, preserving e . So $\bar{e} = e$.

Inductive step. Let $T = T_u \oplus \lambda(T_v)$ be a triangulation of size $n = p+q+1$ on the $(n+2)$ -gon, with $|T_u| = p$ and $|T_v| = q$. The vertex structure in the combined polygon is T_u on $\{1, \dots, p+2\}$; T_v 's vertex i at combined vertex $p+i+1$ (via the λ -shift $i \mapsto i+1$ and the \oplus -shift $j \mapsto p+j$ for $j \geq 2$); and the triangle $(1, p+2, n+2)$ from λ . The complement $i \mapsto n+3-i$ acts as follows:

- (a) T_v 's vertex i (at combined $p+i+1$) maps to $(n+3) - (p+i+1) = q+3-i$, which is the complement on the $(q+2)$ -gon. So the T_v sub-triangulation becomes \bar{T}_v on vertices $\{1, \dots, q+2\}$.

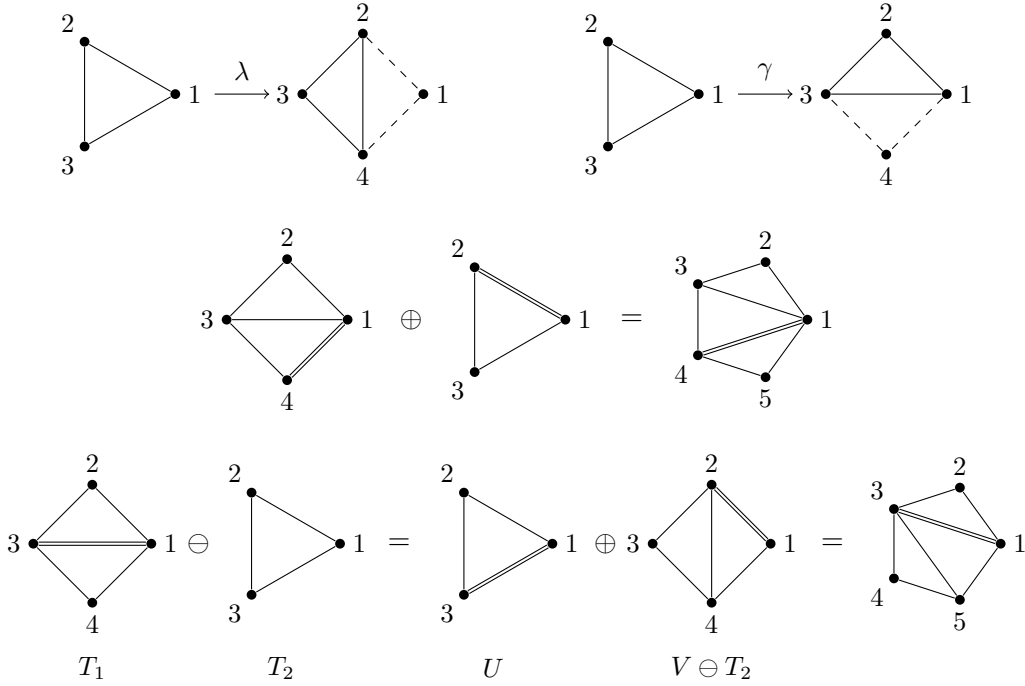


Figure 3: The four triangulation operations on small instances. Dashed edges are those added by λ and γ ; double edges those identified by \oplus and \ominus .

(b) T_u 's vertex i (at combined i) maps to $n+3-i = (p+3-i) + (q+1)$, which is the complement on the $(p+2)$ -gon, shifted by $q+1$. So the T_u sub-triangulation becomes \overline{T}_u placed on vertices $\{q+2, \dots, n+2\}$.

(c) The triangle $(1, p+2, n+2)$ maps to $(n+2, q+2, 1) = (1, q+2, n+2)$.

Now compare with $\overline{T}_v \oplus \lambda(\overline{T}_u)$. Here $\lambda(\overline{T}_u)$ is on the $(p+3)$ -gon with triangle $(1, 2, p+3)$, and \overline{T}_u on vertices $\{2, \dots, p+3\}$. After the \oplus -shift (vertex $j \mapsto q+j$ for $j \geq 2$), \overline{T}_u sits on $\{q+2, \dots, n+2\}$ and the triangle becomes $(1, q+2, n+2)$, matching (b) and (c). By the inductive hypothesis, $\overline{T}_u = h(T_u)$ and $\overline{T}_v = h(T_v)$, so $\overline{T} = h(T_v) \oplus \lambda(h(T_u)) = h(T)$. \square

Corollary 3.2. *Since h is vertex complement, its fixed points are the symmetric triangulations. There are none for even polygons, and $|C_k|$ for the $(2k+3)$ -gon. The axis of symmetry runs through vertex $k+2$ and the midpoint of the opposite edge $\{1, 2k+3\}$, so one side determines the other.*

3.4 231-avoiding permutations

A permutation π of $[n] = \{1, \dots, n\}$ is 231-avoiding if no subsequence of length 3 has elements in the relative order of size 2, 3, 1. We write $\text{Av}(231)$ for the set of 231-avoiding permutations. A position i is a *left-to-right maximum* of π if $\pi(j) < \pi(i)$ for all $j < i$, and a *right-to-left minimum* of π if $\pi(i) < \pi(j)$ for all $j > i$. The indecomposable 231-avoiding permutations are those that begin with their maximum, that is, $\pi(1) = n$. The *direct sum* of two permutations π_1 and π_2 is the concatenation $\pi_1 \oplus \pi_2 = \pi_1 \pi_2'$ where $\pi_2'(i) = \pi_2(i) + |\pi_1|$. Closure under λ and \oplus is immediate;

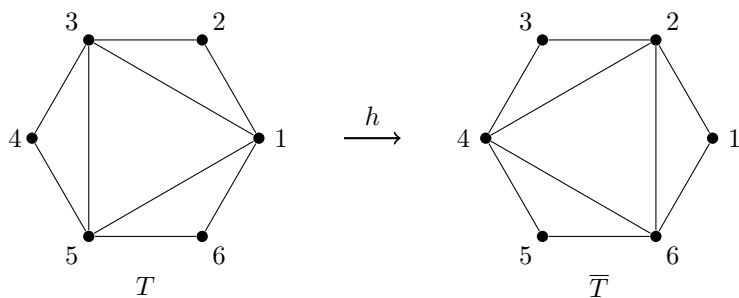


Figure 4: Vertex complement $i \mapsto 7 - i$ on a hexagon triangulation ($n = 4$). The diagonals $\{(1, 3), (3, 5), (1, 5)\}$ of T , forming the central triangle $(1, 3, 5)$, map to $\{(6, 4), (4, 2), (6, 2)\}$, the central triangle to $(2, 4, 6)$.

prepending a new maximum cannot create a 231 pattern, and direct sum preserves avoidance.

- $e = \epsilon$, the empty permutation.
- $\lambda(\pi) = (n+1)\pi$: prepend a new maximum.
- $\pi_1 \oplus \pi_2$: direct sum.²
- $\gamma(\pi) = \pi(n+1)$: append a new fixed point.
- $\pi_1 \ominus \pi_2$: let $x = \pi_1(n)$ be the last entry of π_1 . Shift all values $\geq x$ in π_1 up by $|\pi_2|$ to open a gap, then append $\pi_2 + (x-1)$, where $\pi_2 + (x-1)$ denotes the word obtained from π_2 by adding $x - 1$ to each entry.

That $|\text{Av}(231) \cap S_n| = |C_n|$ is classical [11, §2.2.1, Ex. 4–5]. No closed form for h on $\text{Av}(231)$ is known; h is computed via the generic last-return recursion of Lemma 2.7.

3.5 321-avoiding permutations

A permutation π of $[n]$ is 321-avoiding if no subsequence of length 3 is decreasing. We write $\text{Av}(321)$ for the set of 321-avoiding permutations. The indecomposable 321-avoiding permutations are those whose only direct-sum component is the whole permutation. Unlike the other families, the λ operation on $\text{Av}(321)$ is not immediate from the definition. The construction below is the map β of [3]; we prove closure and bijectivity.

- $e = \epsilon$, the empty permutation.
- $\lambda(\pi)$: insert $n+1$ at the position of 1, then cyclically shift a distinguished subset of positions (described below).
- $\pi_1 \oplus \pi_2$: direct sum.

²The notation \oplus is used both for the abstract Catalan operation and for the direct sum of permutations; on $\text{Av}(231)$ and $\text{Av}(321)$ the two coincide.

- $\gamma(\pi) = \pi(n+1)$: append a new fixed point.
- $\pi_1 \ominus \pi_2$: write $\pi_1 = \alpha_1 \oplus \cdots \oplus \alpha_k$ in its components and replace the last, α_k , by $\alpha_k \ominus \pi_2$. Since α_k is indecomposable, $\alpha_k = \lambda(u)$ and $\alpha_k \ominus \pi_2 = \lambda(u \ominus \pi_2)$, with λ as defined below.

More precisely, $\lambda(\pi)$ works as follows.

Pseudocode.

1. Let $n = |\pi|$. If $n = 0$, return 1.
2. Let i be the position of 1 in π .
3. Identify the “boxed” positions in π : these are the left-to-right maxima of π to the right of position i that are not also right-to-left minima of π .
4. Insert $n+1$ at position i , shifting subsequent entries right, and add this new position to the boxed set.
5. Cyclically shift the values at the boxed positions one step to the left: the value at each boxed position moves to the previous boxed position, and the first boxed value wraps to the last boxed position.
6. Return the result.

An example is shown in Figure 5: $\pi = 241357698$ has $i = 3$ and boxed positions $\{6, 8\}$ (the left-to-right maximum values 7 and 9 to the right of 1 are not right-to-left minima). Inserting 10 at position 3 and boxing it gives boxed set $\{3, 7, 9\}$; the cyclic left-shift of 10, 7, 9 produces 7, 9, 10.

$$\begin{array}{cccccccc}
 \pi & = & 2 & 4 & 1 & 3 & 5 & \boxed{7} & 6 & \boxed{9} & 8 \\
 & & 2 & 4 & \boxed{10} & 1 & 3 & 5 & \boxed{7} & 6 & \boxed{9} & 8 \\
 \lambda(\pi) & = & 2 & 4 & \boxed{7} & 1 & 3 & 5 & \boxed{9} & 6 & \boxed{10} & 8
 \end{array}$$

Figure 5: The map λ on $\text{Av}(321)$. First row: box the left-to-right maxima to the right of 1 that are not right-to-left minima. Second row: insert $n+1$ at the position of 1 and box it. Third row: cyclically shift the boxed values one step to the left.

Lemma 3.3. *A permutation τ avoids 321 if and only if the subsequence of values at positions that are not left-to-right maxima is increasing.*

Proof. (\Rightarrow) Suppose $b < c$ are positions, neither a left-to-right maximum, with $\tau(b) > \tau(c)$. Since b is not a left-to-right maximum, there exists $a < b$ with $\tau(a) > \tau(b) > \tau(c)$, a 321 pattern.

(\Leftarrow) If τ contains a 321 pattern $\tau(a) > \tau(b) > \tau(c)$ with $a < b < c$, then neither b nor c is a left-to-right maximum, so the subsequence of values at non-left-to-right-maximum positions has a descent. □

Proposition 3.4. *If π avoids 321, then $\lambda(\pi)$ avoids 321.*

Proof. For $n = 0$ the result is trivial ($\lambda(e) = 1$). Assume $n \geq 1$ and let p be the position of 1 in π . Write $\sigma = \lambda(\pi)$. We show that the values of σ at positions that are not left-to-right maxima form an increasing subsequence; by Lemma 3.3, this implies $\sigma \in \text{Av}(321)$.

Left-to-right maxima of σ . Let $j_1 < \dots < j_m$ be the boxed positions in π and write $b_0 = p$, $b_k = j_k + 1$ for $1 \leq k \leq m$. The left-to-right maxima of σ are exactly the positions $j < p$ together with b_0, \dots, b_m . Indeed, every $j < p$ is a left-to-right maximum of π (otherwise some $k < j$ with $\pi(k) > \pi(j) > 1 = \pi(p)$ gives a 321 pattern); these positions retain their values in σ and remain left-to-right maxima. The shifted values $\sigma(b_0) = \pi(j_1)$, $\sigma(b_1) = \pi(j_2)$, \dots , $\sigma(b_{m-1}) = \pi(j_m)$, $\sigma(b_m) = n+1$ are strictly increasing (since $j_1 < \dots < j_m$ are successive left-to-right maxima of π), so each b_k is a left-to-right maximum of σ . Conversely, a non-boxed position $j > p$ has $\sigma(j) = \pi(j-1)$, and the nearest preceding boxed position b_k satisfies $\sigma(b_k) > \sigma(j)$ (since $\sigma(b_k)$ is either $\pi(j_{k+1})$ with $j_{k+1} > j-1$ and j_{k+1} a left-to-right maximum, or $n+1$), so j is not a left-to-right maximum.

Values at non-left-to-right-maximum positions are increasing. These positions in σ are the non-boxed positions $> p$, with values $\pi(p), \pi(q_1), \dots, \pi(q_r)$ where $p < q_1 < \dots < q_r$ are the non-boxed positions $> p$ in π . Since $\pi(p) = 1$ is minimal, it suffices to show $\pi(q_i) < \pi(q_j)$ for $i < j$. If q_i is not a left-to-right maximum of π , some $k < q_i$ has $\pi(k) > \pi(q_i)$, so $\pi(q_i) \geq \pi(q_j)$ would yield a 321 pattern at $k < q_i < q_j$. If q_i is a left-to-right maximum of π (hence also a right-to-left minimum, since non-boxed left-to-right maxima to the right of 1 are right-to-left minima), then $\pi(q_i) < \pi(\ell)$ for all $\ell > q_i$. \square

Proposition 3.5. *The map λ is a bijection from $\text{Av}(321)$ of size n to the set of indecomposable 321-avoiding permutations of size $n+1$.*

Proof. Image. Let p be the position of 1 in π . By Proposition 3.4, $\sigma = \lambda(\pi)$ avoids 321, and $|\sigma| = n+1$, so it remains to verify that σ is indecomposable. A component boundary at a position k with $1 \leq k < n+1$ would require $\{1, \dots, k\}$ to be exactly the first k values of σ . For $k \leq p$, the value 1 lies at position $p+1 > k$ (it is shifted one place right by the insertion of $n+1$ at position p , and position $p+1$ is not boxed), so 1 is not among the first k values, a contradiction. For $k > p$, the left-to-right-maxima claim in the closure proof gives a boxed position $b \leq k$ with $\sigma(b) > k$ (the largest boxed position not exceeding k), so the first k values contain $\sigma(b) \notin \{1, \dots, k\}$, again a contradiction. Hence σ is indecomposable.

Injectivity. Given $\sigma = \lambda(\pi)$, we recover π uniquely. As noted, the value 1 appears in σ at position $p+1$, so p is determined by σ . The boxed positions used in the cyclic shift are exactly the left-to-right maxima of σ at positions $\geq p$ (by the left-to-right-maxima claim in the closure proof), identifiable from σ alone. Reversing the cyclic shift at these positions and deleting $n+1$ at position p recovers π uniquely.

Cardinality. It is well known that $|\text{Av}(321) \cap S_n| = |C_n|$. The generating function for indecomposable elements of any Catalan-counted family is $x C(x)$ (since $C(x) = 1/(1 - x C(x))$), so the number of indecomposable 321-avoiders of size $n+1$ is also $|C_n|$. An injective map between two sets of equal finite cardinality is a bijection. \square

No closed form for h on $\text{Av}(321)$ is known; h is computed via the generic last-return recursion of Lemma 2.7.

3.6 Binary trees

A binary tree is either the empty tree \emptyset (no nodes) or a node with a left and a right subtree, that is, a tuple (L, R) , each itself a binary tree. An indecomposable binary tree is one of the form (T, \emptyset) , that is, a root whose right subtree is empty. The *right spine* is the maximal sequence of right edges from the root.

- $e = \emptyset$, the empty tree.
- $\lambda(T) = (T, \emptyset)$: make T the left child of a new root with no right child.
- $T_1 \oplus T_2$ attaches T_2 at the bottom of the right spine of T_1 . Explicitly, $\emptyset \oplus B = B$ and $(T, B_1) \oplus B_2 = (T, B_1 \oplus B_2)$.
- $\gamma(B) = B \oplus (\emptyset, \emptyset)$: adjoin a leaf at the end of the right spine.
- $B \ominus C$: descend from the root, at each node taking the right subtree if it is nonempty and the left subtree otherwise, until reaching a leaf; replace that leaf by $\lambda(C) = (C, \emptyset)$. If $B = \emptyset$, the result is C .

No closed form for h on binary trees is known; h is computed via the generic last-return recursion of Lemma 2.7.

3.7 $2 \times n$ standard Young tableaux

A $2 \times n$ standard Young tableau (SYT) is a filling of a $2 \times n$ grid with $1, 2, \dots, 2n$ such that rows and columns increase. An indecomposable SYT is one that cannot be split as $T_1 \oplus T_2$ with both parts nonempty, that is, there is no k with $0 < k < n$ such that the first k columns contain exactly $\{1, \dots, 2k\}$.

- $e = ([], [])$, the empty tableau.
- $\lambda(T)$: increment all entries by 1, prepend 1 to the top row, and append $2n+2$ to the bottom row.
- $T_1 \oplus T_2$: shift all entries of T_2 up by $2|T_1|$ and adjoin the columns to the right.
- $\gamma(T)$: append the column $(2n+1, 2n+2)$.
- $T_1 \ominus T_2$: if T_1 is empty, $T_1 \ominus T_2 = T_2$. If T_1 is decomposable, let V be its last indecomposable summand and U the rest, so $T_1 = U \oplus V$; then $T_1 \ominus T_2 = U \oplus (V \ominus T_2)$. If T_1 is indecomposable, write $T_1 = \lambda(T'_1)$ (remove 1 from the top row and $2n$ from the bottom row, subtract 1 from all remaining entries) and set $T_1 \ominus T_2 = \lambda(T'_1 \ominus T_2)$.

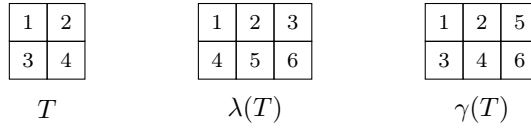


Figure 6: Actions of λ and γ on a 2×2 SYT T . The operation λ increments all entries by 1, prepends 1 to the top row, and appends $2n+2 = 6$ to the bottom row, whereas γ appends the column $(2n+1, 2n+2) = (5, 6)$.

See Figure 6.

No closed form for h on $2 \times n$ SYT is known; h is computed via the generic last-return recursion of Lemma 2.7. For example,

$$h \left[\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \right] = (\gamma \circ h) \left[\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right] = \gamma \left[\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right] = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array}.$$

3.8 Non-crossing partitions

A non-crossing partition [13] of $[n]$ is a set partition whose blocks do not cross, that is, there are no $a < b < c < d$ with a, c in one block and b, d in another. An indecomposable non-crossing partition is one in which 1 and n belong to the same block. The decomposition below, in which λ adds a new maximum to the block of 1, appears in [15, Proposition 9.4].

- e is the empty partition.
- $\lambda(P)$: add $n+1$ to the block containing 1. In particular, $\lambda(e) = \{\{1\}\}$.
- $P_1 \oplus P_2 = P_1 \cup (P_2 + |P_1|)$: shift the elements of P_2 by $|P_1|$ and take the union.
- $\gamma(P)$: adjoin the singleton block $\{n+1\}$.
- $P_1 \ominus P_2$: if P_1 is empty, $P_1 \ominus P_2 = P_2$. If P_1 is decomposable, let V be its last indecomposable summand and U the rest, so $P_1 = U \oplus V$; then $P_1 \ominus P_2 = U \oplus (V \ominus P_2)$. If P_1 is indecomposable, write $P_1 = \lambda(P'_1)$ (remove n from the block of 1, leaving a partition of $[n-1]$) and set $P_1 \ominus P_2 = \lambda(P'_1 \ominus P_2)$, that is, add the new maximum to the block of 1.

See Figure 7.

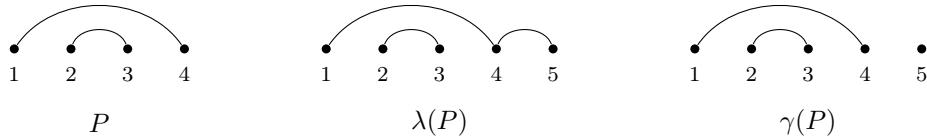


Figure 7: The non-crossing partition $P = \{\{1, 4\}, \{2, 3\}\}$ with its blocks drawn as arcs, and its images under λ and γ : $\lambda(P) = \{\{1, 4, 5\}, \{2, 3\}\}$ adds the new maximum 5 to the block containing 1, while $\gamma(P) = \{\{1, 4\}, \{2, 3\}, \{5\}\}$ adjoins $\{5\}$ as a new singleton block.

No closed form for h on non-crossing partitions is known; h is computed via the generic last-return recursion of Lemma 2.7.

3.9 Concrete descriptions of the canonical bijections

There are $\binom{8}{2} = 28$ canonical bijections $\varphi_{A \rightarrow B}$ between our eight families, each defined recursively via the abstract framework, so their effect on a given object is not immediately clear. For a spanning set of pairs we give concrete, usually non-recursive descriptions. The seven pairs below connect all eight families (they form a spanning tree of K_8), so any $\varphi_{A \rightarrow B}$ is a composition of bijections from this set, and these seven descriptions determine all 28. To verify that a concrete description matches the canonical bijection, it suffices to check by structural induction the following three identities.

Proposition 3.6. *A bijection $f : A \rightarrow B$ between two Catalan structures is the canonical bijection if and only if:*

- $f(e_A) = e_B$,
- $f(\lambda_A(t)) = \lambda_B(f(t))$, and
- $f(u \oplus_A v) = f(u) \oplus_B f(v)$.

Since decomposition into e , λ and \oplus is unique, these identities force $f = \varphi_{A \rightarrow B}$. The verifications are carried out in Appendix A.

Triangulations \rightarrow plane trees. Vertex 1 of the $(n+2)$ -gon becomes the root. The vertices adjacent to 1 (other than $n+2$) become the children of the root; vertex $n+2$ is a dummy that disappears. The construction recurses into each subpolygon between consecutive neighbours of 1. To reverse: label the non-root vertices of the tree in preorder and recover the triangulation. See Figure 8.

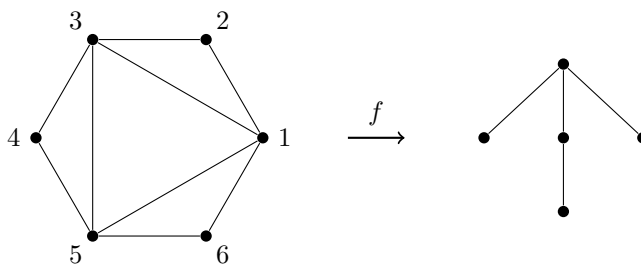


Figure 8: The triangulations-to-trees bijection. Vertex 1 becomes the root, and its neighbours 2, 3, 5 (excluding the dummy vertex $n+2 = 6$) become its three children. The three subpolygons recurse into the three subtrees: each of the edges $(2, 3)$ and $(5, 6)$ yields a leaf, while the triangle $(3, 4, 5)$ yields a subtree with a single edge.

Plane trees \rightarrow Dyck paths. Traverse the tree by depth-first search with children visited left to right; each step down an edge produces a u -step, each step back up produces a d -step.

Dyck paths $\rightarrow 2 \times n$ SYT. The i -th step of the Dyck path is u if i lies in the top row of the tableau, and d if i lies in the bottom row.

Plane trees \rightarrow binary trees. This is the left-child/right-sibling encoding. If T is a single node, $f(T) = \emptyset$. Otherwise, let η be the leftmost edge of T , let T_1 be the subtree below η , and let T_2 be the plane tree obtained from T by deleting T_1 (so T_2 shares its root with T). Then $f(T)$ is the binary tree with root corresponding to η , left subtree $f(T_1)$, and right subtree $f(T_2)$. See Figure 9.

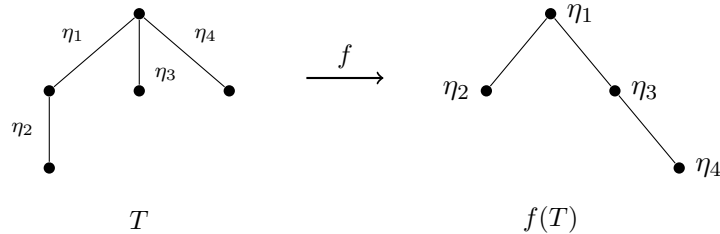


Figure 9: The left-child/right-sibling encoding. Each edge η_i of the plane tree T becomes a node of the binary tree $f(T)$: its left child in $f(T)$ is the leftmost child edge of η_i 's lower endpoint (if any), and its right child in $f(T)$ is the next sibling edge of η_i (if any). Here T_1 is the subtree below η_1 (contributing η_2) and T_2 is the rest of T sharing its root (contributing η_3, η_4).

Plane trees \rightarrow non-crossing partitions. Label the edges of T in postorder with labels $1, 2, \dots, n$ (equivalently, label them $n, n-1, \dots, 1$ in the first-descent order of a depth-first search with children visited right to left; see Figure 10). The labels along the leftmost path form the first block of the partition. Remove these edges and recurse on the resulting forest.

Plane trees \rightarrow 231-avoiding permutations. Label the edges of T in postorder. Then read the labels in preorder. The resulting sequence is a 231-avoiding permutation.

Plane trees \rightarrow 321-avoiding permutations. Label the edges of T in postorder. Traverse T by depth-first search (children left to right). Going down an edge not incident to a leaf: reserve an empty slot. Going down an edge incident to a leaf: place its label immediately. Going up a non-leaf edge: place its label in the leftmost available slot. The result is a 321-avoiding permutation.

Remark 3.7. Most of these concrete descriptions are classical bijections: The DFS encoding of trees as Dyck paths, the step/row correspondence between Dyck paths and SYT, and the left-child/right-sibling encoding of plane trees as binary trees are all standard. Non-crossing partitions and 231-avoiders appear as items (pp) and (ff) (via reverse-complement) of Stanley's list [20, Exercise 6.19]. For non-crossing partitions, a tree bijection via preorder vertex labeling is given in the solution to item 159 of [21]; this is the Dershowitz–Zaks construction [6], and, while similar, it differs from the postorder edge labeling used above. The slot-filling procedure for 321-avoiders appears to be less standard.

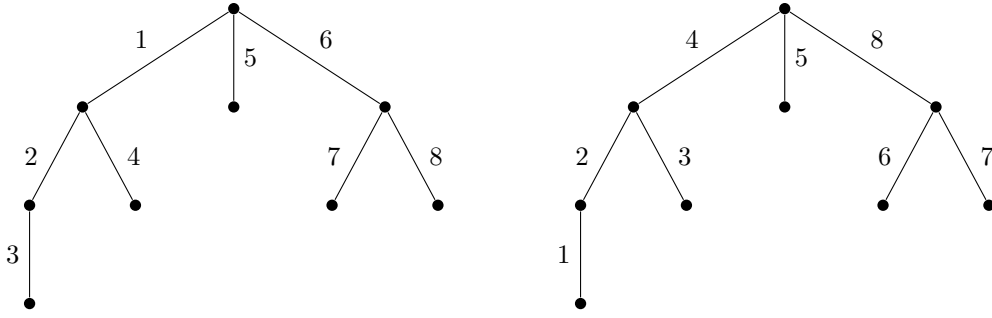


Figure 10: Edge labelings of a plane tree with eight edges. Left: Preorder (labels assigned in the first-descent order of a depth-first search with children visited left to right). Right: Postorder (equivalently, labels assigned in descending order $8, 7, \dots, 1$ in the first-descent order of a depth-first search with children visited right to left). The constructions for non-crossing partitions, $\text{Av}(231)$, and $\text{Av}(321)$ in Section 3.9 use the postorder labeling.

4 Statistics and transport

With the concrete families in hand, we turn to the equidistribution theorem and the dictionary that translates it into per-family statements.

4.1 Canonical statistics and equidistribution

The four *canonical statistics* `leaves`, `sub`, `internal` and `rpath` are defined by the following recursive equations on the free Catalan structure (C, e, λ, \oplus) , where each $u \oplus v$ clause has both summands nonempty:

$$\begin{array}{ll}
 \text{leaves}(e) = 1 & \text{sub}(e) = 0 \\
 \text{leaves}(\lambda(t)) = \text{leaves}(t) & \text{sub}(\lambda(t)) = 1 \\
 \text{leaves}(u \oplus v) = \text{leaves}(u) + \text{leaves}(v) & \text{sub}(u \oplus v) = \text{sub}(u) + \text{sub}(v) \\
 \text{internal}(e) = 0 & \text{rpath}(e) = 0 \\
 \text{internal}(\lambda(t)) = 1 + \text{internal}(t) & \text{rpath}(\lambda(t)) = 1 + \text{rpath}(t) \\
 \text{internal}(u \oplus v) = \text{internal}(u) + \text{internal}(v) - 1 & \text{rpath}(u \oplus v) = \text{rpath}(v).
 \end{array}$$

On plane trees, these have concrete descriptions: `leaves` is the number of leaves, `internal` the number of non-leaf vertices ($= |t| + 1 - \text{leaves}(t)$ for $|t| \geq 1$), `sub` the number of subtrees of the root (that is, the number of children), and `rpath` the length of the rightmost path.

Theorem 4.1 (Equidistribution). *For all t with $|t| \geq 1$,*

$$\text{rpath}(h(t)) = \text{sub}(t) \quad \text{and} \quad \text{leaves}(h(t)) = \text{internal}(t).$$

In particular, for $n \geq 1$, $(\text{sub}, \text{leaves})$ and $(\text{rpath}, \text{internal})$ have the same joint distribution on C_n .

The equidistribution can be seen as a special case of Theorem 6 in [5]. We give a self-contained proof.

Proof. It suffices to show $\text{rpath}(h(t)) = \text{sub}(t)$ for all t , and $\text{leaves}(h(t)) = \text{internal}(t)$ for $|t| \geq 1$. The identities $\text{sub}(h(t)) = \text{rpath}(t)$ and $\text{internal}(h(t)) = \text{leaves}(t)$ then follow from $h^2 = \text{id}$. We use two auxiliary identities, each proved by induction on a using the defining recurrence of \ominus :

$$\text{rpath}(a \ominus b) = \text{rpath}(a) + \text{rpath}(b), \quad \text{leaves}(a \ominus b) + 1 = \text{leaves}(a) + \text{leaves}(b).$$

We proceed by induction on $|t|$. The base cases are immediate, since h fixes both e and $\lambda(e)$: the identity $\text{rpath}(h(t)) = \text{sub}(t)$ holds at $t = e$ and at $t = \lambda(e)$, while $\text{leaves}(h(t)) = \text{internal}(t)$, asserted only for $|t| \geq 1$, holds at $t = \lambda(e)$.

For the inductive step ($|t| \geq 2$), write t in first-return form.

Case 1: $t = \lambda(u)$ with $|u| \geq 1$. Since $h(u) \neq e$, we have $h(t) = \gamma(h(u)) = h(u) \oplus \lambda(e)$, so

$$\begin{aligned} \text{rpath}(h(t)) &= \text{rpath}(\lambda(e)) = 1 = \text{sub}(t), \\ \text{leaves}(h(t)) &= \text{leaves}(h(u)) + 1 = \text{internal}(u) + 1 = \text{internal}(t), \end{aligned}$$

using the induction hypothesis on u in the second line.

Case 2: $t = \lambda(u) \oplus v$ with $v \neq e$. By the definition of h ,

$$h(t) = h(v) \ominus \gamma(h(u)) = h(v) \ominus h(\lambda(u)).$$

Applying the auxiliary identities together with the induction hypothesis on v and on $\lambda(u)$ (both of size $< |t|$ and ≥ 1),

$$\begin{aligned} \text{rpath}(h(t)) &= \text{rpath}(h(v)) + \text{rpath}(h(\lambda(u))) \\ &= \text{sub}(v) + \text{sub}(\lambda(u)) = \text{sub}(t), \\ \text{leaves}(h(t)) + 1 &= \text{leaves}(h(v)) + \text{leaves}(h(\lambda(u))) \\ &= \text{internal}(v) + \text{internal}(\lambda(u)) = \text{internal}(t) + 1, \end{aligned}$$

where the last equality uses $\text{internal}(u' \oplus v') = \text{internal}(u') + \text{internal}(v') - 1$ for nonempty u', v' . \square

4.2 Translating canonical statistics to native ones

With the equidistribution established at the abstract level, it remains to identify the canonical statistics with classical, natively defined statistics on each concrete family. The full dictionary is as follows (the native statistics are defined after the table):

Trees	Dyck	Tri	Av(231)	Av(321)	Binary	SYT	NCP
sub	comp	comp	comp	comp	comp	comp	comp
rpath	rcomp	rcomp	rmax	rcomp	rcomp	rcomp	rcomp
leaves	peak	tr	rmin	rmin	redge	noncons	blocks
internal	nonpeak	nontr	nonrmin	nonrmin	ledge	cons	nonmin

Convention: Some statistics are shifted from their naive combinatorial definition. For example, on binary trees **redge** is one plus the number of right edges, where a right edge is an edge from a node to a nonempty right child. The base-case caveat (Remark 4.2) applies to the last two rows of the dictionary. The complement statistic $n + 1 - \text{leaves}$ is not independent but appears on the right-hand side of the equidistribution, so its native descriptions are included for each family.

Dyck paths.

- $\text{comp} =$ number of returns to the x -axis
- $\text{rcomp} =$ length of the terminal d -run
- $\text{peak} =$ number of occurrences of ud
- $\text{nonpeak} = 1 +$ number of occurrences of uu

Triangulations of the $(n+2)$ -gon.

- $\text{comp} =$ number of edges incident to 1, minus 1
- $\text{rcomp} =$ number of edges incident to $n+2$, minus 1
- $\text{tr} =$ number of $i \in \{2, \dots, n+1\}$ such that some $j < i$ is adjacent to both i and $i+1$
- $\text{nontr} = n + 1 - \text{tr}$

Note that tr counts the triangles whose two largest-labeled vertices are joined by a boundary edge.

231-avoiders.

- $\text{comp} =$ number of components (each begins with its max)
- $\text{rmax} =$ number of right-to-left maxima
- $\text{rmin} =$ number of right-to-left minima
- $\text{nonrmin} = 1 +$ number of non-right-to-left-minima

321-avoiders.

- $\text{comp} =$ number of components
- $\text{rcomp} = n - \pi(n) + 1$, where $\pi(n)$ is the last entry of π
- $\text{rmin} =$ number of right-to-left minima
- $\text{nonrmin} = 1 +$ number of non-right-to-left-minima

Binary trees.

- comp = number of nodes on the right spine
- rcomp = number of right-spine peelings (see Section B.5)
- redge = $1 +$ number of right edges
- $\text{ledge} = n + 1 - \text{redge}$

$2 \times n$ SYT.

- comp = number of components
- rcomp = length of maximal consecutive segment at right end of bottom row
- $\text{cons} = 1 +$ number of top-row entries i such that $i+1$ is also in the top row
- $\text{noncons} =$ number of top-row entries i such that $i+1$ is in the bottom row

Non-crossing partitions.

- comp = number of components
- rcomp = number of right-to-left maxima in flattened form
- blocks = number of blocks
- $\text{nonmin} = n + 1 - \text{blocks}$

4.3 Verifying the dictionary

Each entry in the dictionary of Section 4.2 is a claim, namely that the transported canonical statistic coincides with a classical, natively defined statistic on the target family. For instance, the entry $\text{leaves} \mapsto \text{peak}$ for Dyck paths asserts that $\text{leaves}(\varphi^{-1}(w)) = \text{peak}(w)$ for every Dyck path w . These identifications are where the main combinatorial work lies; without them, the per-family equidistribution statements in Section 4.4 are only formal rewritings.

The verification strategy is uniform: show that the native statistic satisfies the same recursion (under e , λ , \oplus) as the canonical statistic. The full verification of every dictionary entry is carried out in Appendix B. As a model, we prove the $\text{leaves} \mapsto \text{peak}$ entry in detail.

The canonical bijection $\varphi : \text{Trees} \rightarrow \text{Dyck}$ is given by

$$\begin{aligned}\varphi(e) &= \epsilon \\ \varphi(\lambda(t)) &= u\varphi(t)d \\ \varphi(s \oplus t) &= \varphi(s)\varphi(t).\end{aligned}$$

We must show that peak satisfies the same recursion as leaves through this bijection, taking the two operations in turn.

Concatenation. If both w_1 and w_2 are nonempty then w_1 ends with d and w_2 starts with u , so pairing the last letter of w_1 with the first letter of w_2 yields du , not a peak; if either word is empty the claim is trivial. Therefore

$$\text{peak}(w_1w_2) = \text{peak}(w_1) + \text{peak}(w_2),$$

matching $\text{leaves}(s \oplus t) = \text{leaves}(s) + \text{leaves}(t)$.

Lifting. When $w \neq \epsilon$, w starts with u and ends with d , so $\text{peak}(uwd) = \text{peak}(w)$, matching $\text{leaves}(\lambda(t)) = \text{leaves}(t)$. When $w = \epsilon$, $\text{peak}(ud) = 1 = \text{leaves}(\lambda(e))$, completing the induction for $n \geq 1$.

Remark 4.2 (Base-case caveat). All dictionary identifications for leaves and internal hold for $n \geq 1$. At $n = 0$ the native statistics may differ from the canonical values $\text{leaves}(e) = 1$ and $\text{internal}(e) = 0$. For instance, $\text{peak}(\epsilon) = 0 \neq 1 = \text{leaves}(e)$. This discrepancy does not affect the equidistribution theorem, which concerns $n \geq 1$.

4.4 Per-family corollaries

Combining the equidistribution theorem (Theorem 4.1) with the dictionary identifications from Section 4.2 gives the following per-family statements, each holding for $n \geq 1$. The involution h sends:

- (comp, peak) to (rcomp, nonpeak) on Dyck paths,
- (comp, tr) to (rcomp, nontr) on triangulations,
- (comp, rmin) to (rmax, nonrmin) on $\text{Av}(231)$,
- (comp, rmin) to (rcomp, nonrmin) on $\text{Av}(321)$,
- (comp, redge) to (rcomp, ledge) on binary trees,
- (comp, noncons) to (rcomp, cons) on $2 \times n$ SYT, and
- (comp, blocks) to (rcomp, nonmin) on non-crossing partitions.

The equidistribution of the individual statistics (for example, that peaks on Dyck paths have the Narayana distribution [7, 14]) is classical. What the present framework provides is the *joint* equidistribution via an explicit involution. In [5] this joint result is established on $\beta(1, 0)$ -trees, but the per-family statements above, which require the dictionary identifications of Section 4.2 to translate the canonical statistics into native ones, are new.

4.5 Generating functions

The four canonical statistics admit a joint generating function, which we now derive. Define

$$F = F(t, u) = \sum_{T \in \mathcal{C}} t^{|T|} u^{\text{leaves}(T)}.$$

The empty tree contributes u . Every nonempty tree is either indecomposable, $T = \lambda(A)$, or decomposable, $T = \lambda(A) \oplus B$ with $B \neq e$. In the indecomposable case $\text{leaves}(T) = \text{leaves}(A)$ and $|T| = 1 + |A|$, so these trees contribute tF . In the decomposable case $\text{leaves}(T) = \text{leaves}(A) + \text{leaves}(B)$ and $|T| = 1 + |A| + |B|$, and the sums over A (all trees) and B (nonempty trees) factor, contributing $tF(F - u)$. Therefore $F - u = tF + tF(F - u)$, that is,

$$tF^2 + (t - tu - 1)F + u = 0.$$

Writing $\Delta = (1 - t + tu)^2 - 4tu$ for the discriminant,

$$F(t, u) = \frac{1 - t + tu - \sqrt{\Delta}}{2t}.$$

For $n \geq 1$ and $1 \leq k \leq n$, the coefficient to $t^n u^k$ in F is the Narayana number $N(n, k)$. Specializing $u = 1$ gives $tF^2 - F + 1 = 0$, and hence

$$F(t, 1) = \frac{1 - \sqrt{1 - 4t}}{2t} = \sum_{n \geq 0} |C_n| t^n,$$

the Catalan generating function.

Introducing additional variables p for sub and q for rpath , define

$$G = G(t, p, q, u) = \sum_{T \in \mathcal{C}} t^{|T|} p^{\text{sub}(T)} q^{\text{rpath}(T)} u^{\text{leaves}(T)}.$$

Theorem 4.3 (Generating function). *With $F = F(t, u)$ as above,*

$$G(t, p, q, u) = u + \frac{tpqu(1 - tF)}{(1 - tpF)(1 - tF - tq)}.$$

Equivalently, with $\Delta = (1 - t + tu)^2 - 4tu$,

$$G(t, p, q, u) = u + \frac{tpqu(1 + t - tu + \sqrt{\Delta})}{2(1 - tq - tpu) + (1 - t + tu - \sqrt{\Delta})(tp(u + q - 1) - 1)}.$$

Proof. The empty tree contributes u to G . For each nonempty tree T , apply its first-return decomposition.

Indecomposable case ($T = \lambda(A)$): $\text{sub}(T) = 1$, $\text{rpath}(T) = 1 + \text{rpath}(A)$, $\text{leaves}(T) = \text{leaves}(A)$, $|T| = 1 + |A|$. Summing over A gives $tpqG(t, 1, q, u)$, since the sum tracks rpath and leaves but not sub .

Decomposable case ($T = \lambda(A) \oplus B$, $B \neq e$): $\text{sub}(T) = 1 + \text{sub}(B)$, $\text{rpath}(T) = \text{rpath}(B)$, $\text{leaves}(T) = \text{leaves}(A) + \text{leaves}(B)$, $|T| = 1 + |A| + |B|$. The sum over A contributes F (only t and u are involved), the sum over nonempty B contributes $G - u$, the extra t from $|T| = 1 + |A| + |B|$ contributes a factor t , and $p^{1 + \text{sub}(B)} = p \cdot p^{\text{sub}(B)}$ introduces a factor p , giving $tpF(G - u)$ in total.

The two cases together yield

$$G - u = tpqG(t, 1, q, u) + tpF(G - u),$$

and hence

$$G = u + \frac{tpq G(t, 1, q, u)}{1 - tpF}.$$

When $p = 1$ this becomes

$$G(t, 1, q, u) = u + \frac{tq G(t, 1, q, u)}{1 - tF},$$

so $G(t, 1, q, u) = u(1 - tF)/(1 - tF - tq)$. Feeding this back gives

$$G = u + \frac{tpq}{1 - tpF} \frac{u(1 - tF)}{1 - tF - tq} = u + \frac{tpqu(1 - tF)}{(1 - tpF)(1 - tF - tq)}.$$

For the explicit form, expand the denominator $D = (1 - tpF)(1 - tF - tq)$:

$$D = 1 - tF - tq - tpF + t^2pF^2 + t^2pqF.$$

The quadratic relation gives $tF^2 = (1 - t + tu)F - u$, so

$$D = (1 - tq - tpu) + tF(tp(u + q - 1) - 1).$$

Substituting $tF = \frac{1}{2}(1 - t + tu - \sqrt{\Delta})$ and $1 - tF = \frac{1}{2}(1 + t - tu + \sqrt{\Delta})$ yields the second form. \square

Since the canonical bijections preserve all four statistics, the generating function G is the same for every concrete Catalan family.

5 Donaghey's map M and iterated secondary structures

The involution h can be factored as a composition of two simpler involutions. This factorization underlies the connection to Donaghey's automorphism on plane trees and the period theorem for iterated secondary structures.

5.1 The involutions rev and corev

On binary trees (see Section 3.6), rev reverses the sequence of left subtrees along the right spine, while corev swaps left and right subtrees at every node; both recurse into every subtree. On the free Catalan structure we define rev and corev by

$$\begin{aligned} \text{rev}(e) &= e \\ \text{rev}(\lambda(u) \oplus v) &= \text{rev}(v) \oplus \lambda(\text{rev}(u)) \end{aligned}$$

and

$$\begin{aligned} \text{corev}(e) &= e \\ \text{corev}(\lambda(u) \oplus v) &= \lambda(\text{corev}(v)) \oplus \text{corev}(u). \end{aligned}$$

Note that setting $v = e$ above shows that rev commutes with λ .

Lemma 5.1. *For all $u, v \in C$,*

$$\text{rev}(u \oplus \lambda(v)) = \lambda(\text{rev}(v)) \oplus \text{rev}(u).$$

Proof. By induction on the size of u . If $u = e$ then both sides equal $\lambda(\text{rev}(v))$. If $u \neq e$, write $u = \lambda(a) \oplus b$. Then

$$\begin{aligned}
\text{rev}(u \oplus \lambda(v)) &= \text{rev}(\lambda(a) \oplus (b \oplus \lambda(v))) \\
&= \text{rev}(b \oplus \lambda(v)) \oplus \lambda(\text{rev}(a)) \\
&= (\lambda(\text{rev}(v)) \oplus \text{rev}(b)) \oplus \lambda(\text{rev}(a)) \\
&= \lambda(\text{rev}(v)) \oplus (\text{rev}(b) \oplus \lambda(\text{rev}(a))) \\
&= \lambda(\text{rev}(v)) \oplus \text{rev}(u)
\end{aligned}$$

where the third step uses the induction hypothesis on b , and the last step folds the definition of rev . \square

Corollary 5.2. *rev and corev are involutions.*

Proof. We treat rev first. The base case $\text{rev}(\text{rev}(e)) = e$ is immediate. For the inductive step, write $t = \lambda(u) \oplus v$ in first-return form, where v may be e . Then $\text{rev}(t) = \text{rev}(v) \oplus \lambda(\text{rev}(u))$, and by Lemma 5.1,

$$\text{rev}(\text{rev}(t)) = \lambda(\text{rev}(\text{rev}(u))) \oplus \text{rev}(\text{rev}(v)) = \lambda(u) \oplus v = t.$$

The argument for corev is similar:

$$\begin{aligned}
\text{corev}(\text{corev}(\lambda(u) \oplus v)) &= \text{corev}(\lambda(\text{corev}(v)) \oplus \text{corev}(u)) \\
&= \lambda(\text{corev}(\text{corev}(u))) \oplus \text{corev}(\text{corev}(v)) \\
&= \lambda(u) \oplus v.
\end{aligned}$$

\square

Theorem 5.3. $h = \text{rev} \circ \text{corev} \circ \text{rev}$.

Proof. Set $F = \text{rev} \circ \text{corev} \circ \text{rev}$. Then $F(e) = e$. For $t = u \oplus \lambda(v)$:

$$\begin{aligned}
F(u \oplus \lambda(v)) &= \text{rev}(\text{corev}(\text{rev}(u \oplus \lambda(v)))) \\
&= \text{rev}(\text{corev}(\lambda(\text{rev}(v)) \oplus \text{rev}(u))) && \text{[Lemma 5.1]} \\
&= \text{rev}(\lambda(\text{corev}(\text{rev}(u))) \oplus \text{corev}(\text{rev}(v))) && \text{[corev def.]} \\
&= \text{rev}(\text{corev}(\text{rev}(v))) \oplus \lambda(\text{rev}(\text{corev}(\text{rev}(u)))) && \text{[rev def.]} \\
&= F(v) \oplus \lambda(F(u)).
\end{aligned}$$

So F satisfies $F(e) = e$ and $F(u \oplus \lambda(v)) = F(v) \oplus \lambda(F(u))$. Since the last-return decomposition is unique, these two equations determine F by structural induction, and h satisfies the same recursion (Lemma 2.7), so $F = h$. \square

Remark 5.4. Computing $h(t)$ from its defining recursion takes time $O(n^2)$ in the size $n = |t|$, since each application of \ominus traverses a rightmost path. The factorization $h = \text{rev} \circ \text{corev} \circ \text{rev}$ does better. Represent the structure as a binary tree, each node holding pointers to its two subtrees. Then corev swaps the two pointers at every node, and rev reverses the right-spine list and recurses into every subtree. Each is a single traversal of the tree, hence $O(n)$, and so is h .

5.2 Donaghey's map M

Since $h = \text{rev} \circ \text{corev} \circ \text{rev}$ (Theorem 5.3), define $M = h \circ \text{rev} = \text{rev} \circ \text{corev}$. This map turns out to *intertwine* the primary and secondary structures, meaning that M conjugates λ into γ and \oplus into \ominus . Since these properties uniquely determine a size-preserving map, M must coincide with any other map that intertwines the two structures, in particular with Donaghey's automorphism [8], as we verify below.

Theorem 5.5.

$$\begin{aligned} M(e) &= e \\ M(\lambda(t)) &= \gamma(M(t)) \\ M(u \oplus v) &= M(u) \ominus M(v) \end{aligned}$$

Proof. The first equation is immediate. For the second, since rev commutes with λ , we have

$$M(\lambda(t)) = h(\text{rev}(\lambda(t))) = h(\lambda(\text{rev}(t))) = \gamma(h(\text{rev}(t))) = \gamma(M(t)).$$

For the third, we first note that for any $u = \lambda(a) \oplus b$,

$$M(u) = M(a) \oplus \lambda(M(b)). \tag{1}$$

Indeed,

$$\begin{aligned} M(\lambda(a) \oplus b) &= h(\text{rev}(b) \oplus \lambda(\text{rev}(a))) && [\text{rev def.}] \\ &= h(\text{rev}(a) \oplus \lambda(h(\text{rev}(b)))) && [h \text{ last-return}] \\ &= M(a) \oplus \lambda(M(b)). \end{aligned}$$

Now we induct on the size of u . The base case $u = e$ is trivial. For the inductive step, write $u = \lambda(a) \oplus b$, so that $u \oplus v = \lambda(a) \oplus (b \oplus v)$. Then

$$\begin{aligned} M(u \oplus v) &= M(a) \oplus \lambda(M(b \oplus v)) && [(1)] \\ &= M(a) \oplus \lambda(M(b) \ominus M(v)) && [\text{ind. hyp.}] \\ &= (M(a) \oplus \lambda(M(b))) \ominus M(v) && [\text{Lemma 2.3}] \\ &= M(u) \ominus M(v). && [(1)] \end{aligned}$$

This concludes the proof. □

Corollary 5.6. $M^{-1} = \text{rev} \circ M \circ \text{rev}$. In particular, M is conjugate to its own inverse by rev .

Proof. Since rev and h are involutions,

$$M^{-1} = (h \circ \text{rev})^{-1} = \text{rev} \circ h = \text{rev} \circ h \circ \text{rev} \circ \text{rev} = \text{rev} \circ M \circ \text{rev}. \quad \square$$

Theorem 5.7. The only fixed points of M are e and $\lambda(e)$.

Proof. Both e and $\lambda(e)$ are fixed by M (since rev and h each fix them). For the converse, since $M = h \circ \text{rev}$, $M(t) = t$ implies $h(t) = \text{rev}(t)$. For each nonempty element $t = \lambda(s_1) \oplus \cdots \oplus \lambda(s_k)$, define $\text{first}(t) = 1 + |s_1|$ and $\text{last}(t) = 1 + |s_k|$ (the sizes of the first and last indecomposable summands).

We claim that $\text{last}(h(t)) = |t| - \text{last}(t) + 1$ for all nonempty t . If $t = \lambda(a)$, then $h(t) = \gamma(h(a))$ and the last summand of $\gamma(h(a))$ is $\lambda(e)$, giving $\text{last}(h(t)) = 1 = |t| - |t| + 1$. If $t = \lambda(a) \oplus v$ with v nonempty, then $h(t) = h(v) \oplus \gamma(h(a))$, which replaces the leaf at the end of the rightmost-child chain in the last summand of $h(v)$ by $\gamma(h(a))$, increasing that summand's size by $|\gamma(h(a))| = |a| + 1$. By induction, $\text{last}(h(t)) = \text{last}(h(v)) + |a| + 1 = (|v| - \text{last}(v) + 1) + |a| + 1 = |t| - \text{last}(t) + 1$.

Similarly, $\text{last}(\text{rev}(t)) = \text{first}(t)$ since rev reverses the sequence of indecomposable summands (recursing into each).

Now if $h(t) = \text{rev}(t)$ for nonempty t , applying last to both sides gives $|t| - \text{last}(t) + 1 = \text{first}(t)$, that is, $|t| + 1 = \text{first}(t) + \text{last}(t)$. We consider three cases. For $k = 1$: $\text{first}(t) = \text{last}(t) = |t|$, giving $|t| + 1 = 2|t|$, so $|t| = 1$. For $k = 2$: $|t| = \text{first}(t) + \text{last}(t)$ (no middle summands), giving $|t| + 1 = |t|$, a contradiction. For $k \geq 3$: the middle summands contribute at least $k - 2 \geq 1$ to $|t|$, so $|t| > \text{first}(t) + \text{last}(t)$, also a contradiction. \square

We now describe M concretely on plane trees. The map rev reverses the child list at every level:

$$\text{rev}([c_1, \dots, c_k]) = [\text{rev}(c_k), \dots, \text{rev}(c_1)].$$

Using (1) from the proof of Theorem 5.5, M admits the recursive description

$$\begin{aligned} M([]) &= [] \\ M([c_1, \dots, c_k]) &= M(c_1) \oplus \lambda(M([c_2, \dots, c_k])) \end{aligned} \quad (2)$$

since $[c_1, \dots, c_k] = \lambda(c_1) \oplus [c_2, \dots, c_k]$ and (1) gives $M(\lambda(c_1) \oplus [c_2, \dots, c_k]) = M(c_1) \oplus \lambda(M([c_2, \dots, c_k]))$. Compare with the description of h on trees in Section 3.1:

$$h([c_1, \dots, c_k]) = h(c_k) \oplus \lambda(h(c_{k-1})) \oplus \lambda(\cdots \oplus \lambda(h(c_1)) \oplus \lambda(e) \cdots).$$

The two have the same form, but M processes children left-to-right and h right-to-left, reflecting $h = M \circ \text{rev}$.

Theorem 5.8 (Donaghey's automorphism). *The recursion (2) coincides with Donaghey's automorphism [8] on plane trees.*

Proof. Donaghey constructs his automorphism as $L^{-1} \circ R$, where R and L are two maps between what he calls "general bracketings" and "plus-binary bracketings" (well-matched parenthesizations, and a variant in which some brackets are replaced by + signs). We describe the construction in our language.

Both types of bracketing are counted by the Catalan numbers and can be represented as plane trees. A general bracketing corresponds to a plane tree in the usual way (nested parentheses encode the parent-child relation). A plus-binary bracketing has a right-nested structure: its tree form is a (possibly empty) right-nested list $[b_1, \dots, b_k]$, where each b_i is itself a plus-binary bracketing.

The two representations correspond to different decompositions in the free Catalan structure. A general bracketing (plane tree) with child list $[c_1, \dots, c_k]$ decodes via the first-return decomposition:

$$\lambda(c_1) \oplus \dots \oplus \lambda(c_k).$$

A plus-binary bracketing $[b_1, \dots, b_k]$ decodes by nesting each successive element inside the last summand:

$$b_1 \oplus \lambda(b_2 \oplus \lambda(\dots \oplus \lambda(b_k \oplus \lambda(e)) \dots)).$$

Figure 11 illustrates the two decodings for $k = 3$.

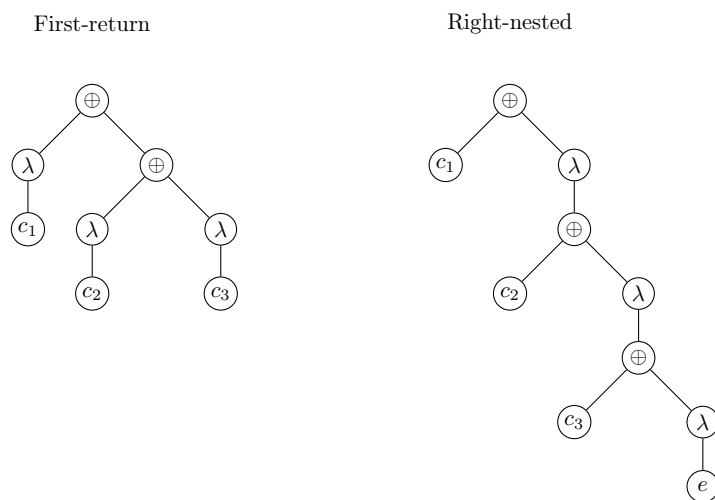


Figure 11: The two decodings of a tree with children c_1, c_2, c_3 . The first-return decomposition (left) produces $\lambda(c_1) \oplus \lambda(c_2) \oplus \lambda(c_3)$. The right-nested decoding (right) produces $c_1 \oplus \lambda(c_2 \oplus \lambda(c_3 \oplus \lambda(e)))$. Donaghey's map M reinterprets the first as the second.

Donaghey's map R (his Property 1) sends the general bracketing $[c_1, \dots, c_k]$ to the plus-binary bracketing $[R(c_1), \dots, R(c_k)]$: it preserves the tree shape but reinterprets the children as a right-nested list. The inverse L^{-1} converts a plus-binary bracketing back to a general bracketing. Thus $L^{-1} \circ R$ takes a plane tree, keeps its shape, but changes the decoding from the first-return pattern to the right-nested pattern, which is exactly the recursion (2). The verification is by induction on size: for $T = []$ both give e ; for $T = [c_1, \dots, c_k]$ with $k \geq 1$, writing $T = \lambda(c_1) \oplus [c_2, \dots, c_k]$ and applying (1) reduces to the induction hypothesis on c_1 and $[c_2, \dots, c_k]$. \square

5.3 The free product structure

Pushkarev and Byzov [17, 18] independently decompose Donaghey's transformation into two involutions, working on *plane planted cubic trees* (PPCTs), that is, rooted plane trees in which every non-leaf has exactly two children (a left son and a right son). PPCTs with n leaves are in bijection with plane trees with $n - 1$ edges (equivalently, with elements of C_{n-1}).

Their two involutions are:

- σ , the *transposition*: swap the left and right son at every non-leaf of a PPCT.
- θ , *Dyck-path reflection*: given a Dyck path $w = w_1 \cdots w_{2n}$, reverse it to $w_{2n} \cdots w_1$ and exchange $u \leftrightarrow d$. This acts on PPCTs via the canonical bijection φ from PPCTs to Dyck paths: $T \mapsto \varphi^{-1}(\theta(\varphi(T)))$.

Donaghey's transformation is then $\varphi^{-1} \circ \theta \circ \varphi \circ \sigma$ [17, Definition 2.3].

Proposition 5.9. *Under the left-child/right-sibling encoding, σ corresponds to corev and $\varphi^{-1} \circ \theta \circ \varphi$ corresponds to rev.*

Proof. In the first-return decomposition $t = \lambda(u) \oplus v$, the left-child/right-sibling encoding maps u to the left subtree and v to the right subtree. Swapping left and right recursively therefore sends $\lambda(u) \oplus v$ to $\lambda(\text{corev}(v)) \oplus \text{corev}(u)$, which is the defining recursion of corev. Thus σ corresponds to corev. For θ : by Theorem 5.3, $M = h \circ \text{rev} = (\text{rev} \circ \text{corev} \circ \text{rev}) \circ \text{rev} = \text{rev} \circ \text{corev}$. Since $\varphi^{-1} \circ \theta \circ \varphi \circ \sigma$ also equals M ([17, Theorem 2.1] and Theorem 5.8), and σ corresponds to corev, we get $\varphi^{-1} \circ \theta \circ \varphi = M \circ \text{corev} = (\text{rev} \circ \text{corev}) \circ \text{corev} = \text{rev}$. \square

Shapiro [19] proved $M^6 = \text{id}$ on compositions and posed the order question in general. Pushkarev and Byzov established the following stronger structural result.

Theorem 5.10 ([17, Theorem 6.4]). *The group generated by σ and $\varphi^{-1} \circ \theta \circ \varphi$ acting on C is the free product $\mathbb{Z}_2 * \mathbb{Z}_2$.*

Corollary 5.11. *$\langle h, \text{rev} \rangle$, the group generated by h and rev , equals $\mathbb{Z}_2 * \mathbb{Z}_2$. In particular, $M = h \circ \text{rev}$ has infinite order and $\text{ord}(M|_{C_n})$ is unbounded.*

Proof. By Proposition 5.9, $\langle \sigma, \varphi^{-1} \circ \theta \circ \varphi \rangle = \langle \text{corev}, \text{rev} \rangle$. Since $h = \text{rev} \circ \text{corev} \circ \text{rev}$ (Theorem 5.3), $\langle \text{corev}, \text{rev} \rangle = \langle h, \text{rev} \rangle$. By the theorem above this group is the free product $\mathbb{Z}_2 * \mathbb{Z}_2$, whose only relations are $h^2 = \text{rev}^2 = \text{id}$. Hence $M = h \circ \text{rev}$ has infinite order, and no single k satisfies $M^k = \text{id}$ on all of C . As each $M|_{C_n}$ permutes the finite set C_n , its order is finite; were these orders bounded, their least common multiple would be such a global k . Hence $\text{ord}(M|_{C_n})$ is unbounded. \square

Remark 5.12. The group $\langle \text{rev}, \text{corev} \rangle$ is the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$, whose reflections are the odd-length words in rev and corev . Besides the generators, the two shortest are $h = \text{rev} \circ \text{corev} \circ \text{rev}$ and $k := \text{corev} \circ \text{rev} \circ \text{corev}$. Since corev is an involution, $k = \text{corev} \circ \text{rev} \circ \text{corev}^{-1}$ is conjugate to rev ; dually h is conjugate to corev . The reflections of this group form exactly two conjugacy classes, represented by rev and corev ; hence k and h lie in different classes and are not conjugate to each other. Since conjugate involutions have the same cycle type, k and rev have equally many fixed points on each C_n . In the Dyck model rev is reversing the path, so its fixed points are the paths symmetric about their vertical midline, and

$$|\{t \in C_n : k(t) = t\}| = |\{t \in C_n : \text{rev}(t) = t\}| = \binom{n}{\lfloor n/2 \rfloor},$$

the central binomial coefficient (OEIS A001405). By contrast, h has $|C_m|$ fixed points of odd size $n = 2m + 1$ and none of positive even size (Proposition 2.11).

In [18], Pushkarev and Byzov prove that the number of orbits of M on C_n grows as $\Theta(2^n)$ and construct explicit families of cycles of lengths 6 and 9. The order of M controls the periodicity of iterated secondary structures.

5.4 Iterated secondary structures

Starting from (λ_1, \oplus_1) , the secondary structure construction yields $(\lambda_2, \oplus_2) = (\gamma, \ominus)$. Iterating gives a sequence

$$(\lambda_1, \oplus_1) \rightarrow (\lambda_2, \oplus_2) \rightarrow (\lambda_3, \oplus_3) \rightarrow \cdots .$$

The general step defines λ_{i+1} by

$$\lambda_{i+1}(t) = t \oplus_i \lambda_i(e)$$

and \oplus_{i+1} recursively in the first argument by

$$\begin{aligned} e \oplus_{i+1} v &= v \\ \lambda_i(u) \oplus_{i+1} v &= \lambda_i(u \oplus_{i+1} v) \\ (t \oplus_i u) \oplus_{i+1} v &= t \oplus_i (u \oplus_{i+1} v). \end{aligned}$$

The dual relations are $\lambda_i(t) = \lambda_{i+1}(e) \oplus_{i+1} t$ and

$$\begin{aligned} t \oplus_i e &= t \\ t \oplus_i \lambda_{i+1}(u) &= \lambda_{i+1}(t \oplus_i u) \\ t \oplus_i (u \oplus_{i+1} v) &= (t \oplus_i u) \oplus_{i+1} v \quad (u \neq e). \end{aligned}$$

Since h and rev both preserve size, $M = h \circ \text{rev}$ restricts to a permutation of each finite set C_k .

Theorem 5.13. $(\lambda_{p+1}, \oplus_{p+1}) = (\lambda_1, \oplus_1)$ if and only if $M^p = \text{id}$ on C .

Proof. The intertwining theorem (Theorem 5.5) gives $M \circ \lambda_1 = \lambda_2 \circ M$ and $M(u \oplus_1 v) = M(u) \oplus_2 M(v)$. Induction on i yields $\lambda_{i+1} = M^i \circ \lambda_1 \circ M^{-i}$ and $u \oplus_{i+1} v = M^i(M^{-i}(u) \oplus_1 M^{-i}(v))$. Indeed, the secondary structure construction sends (λ_i, \oplus_i) to $(\lambda_{i+1}, \oplus_{i+1})$ using only λ_i , \oplus_i , and e (see Definition 2.2); conjugating by M preserves these relations, giving $\gamma_i = M^{i-1} \circ \gamma_1 \circ M^{-(i-1)}$ and \ominus_i transforms accordingly.

Now $(\lambda_{p+1}, \oplus_{p+1}) = (\lambda_1, \oplus_1)$ means $M^p \circ \lambda_1 = \lambda_1 \circ M^p$ and $M^p(u \oplus_1 v) = M^p(u) \oplus_1 M^p(v)$. Setting $\phi = M^p$, these two conditions give $\phi(\lambda(t)) = \lambda(\phi(t))$ and $\phi(u \oplus v) = \phi(u) \oplus \phi(v)$. Together with $\phi(e) = e$ (since ϕ preserves size), ϕ preserves the three-form decomposition. Since every element is uniquely built from e , λ , and \oplus , any size-preserving endomorphism that commutes with these operations is the identity. Conversely, $M^p = \text{id}$ trivially implies $(\lambda_{p+1}, \oplus_{p+1}) = (\lambda_1, \oplus_1)$. \square

The smallest p such that M^p is the identity on $\bigsqcup_{k \leq n} C_k$ is the least common multiple $\text{lcm}_{0 \leq k \leq n} \text{ord}(M|_{C_k})$. The first few values are (OEIS A060114):

n	2	3	4	5	6	7	8	9	10
lcm	2	6	6	30	120	720	15120	1164240	15135120

6 Open problems

We collect several questions left open by the present work.

1. *Description of h and the canonical statistics on different Catalan families.* Given the great number of Catalan families [21], it would be interesting to see which ones admit a simple description of h and the canonical statistics.
2. *Further equidistribution results.* The involution h translates more statistics in a meaningful way on some Catalan families. For example, let the *descent word* of a permutation $\pi \in S_n$ be $w(\pi) \in \{A, D\}^{n-1}$, with i th letter D at a descent and A at an ascent, and let rc be the reverse-complement (reverse the word and swap $A \leftrightarrow D$). Using induction, one can show that h reverse-complements the descent word on $\text{Av}(231)$, that is, $w(h(\pi)) = \text{rc}(w(\pi))$. Consequently, $|w(h(\pi))|_x = |w(\pi)|_{\text{rc}(x)}$ for every factor x . Note that $\text{des}(\pi) = |w(\pi)|_D$, $\text{asc}(\pi) = |w(\pi)|_A$, and, similarly, peak , valley , ddes , dasc count the factors AD , DA , DD , AA of the descent word. Since rc swaps $D \leftrightarrow A$ and $DD \leftrightarrow AA$ and fixes AD and DA , it follows that h exchanges descents with ascents and double descents with double ascents while preserving peaks and valleys. Equivalently, $(\text{des}, \text{ddes}, \text{comp}, \text{rmin}, \text{peak}, \text{valley})$ and $(\text{asc}, \text{dasc}, \text{rmax}, \text{nonrmin}, \text{peak}, \text{valley})$ are equidistributed on $\text{Av}(231) \cap S_n$. This is family-specific: on $\text{Av}(321)$, h preserves valleys but $\text{des}(h(\pi)) = \text{asc}(\pi)$ fails already at $n = 3$. It would be interesting to find further equidistributions on the other families.
3. *Cycle structure of Donaghey's map.* By Theorem 5.7, M has no fixed points for $n \geq 2$, and by Section 5.3, $\text{ord}(M|_{C_n})$ is unbounded. Knuth asks [12, Ex. 17, §7.2.1.6] for a characterization of the fixed points of $(T \circ R)^2$, which reduces to characterizing the 2-cycles of M . Beyond the partial results in [17, 18, 19], the cycle structure remains poorly understood.
4. *Order of M on C_n .* The period of the iterated secondary structures equals $\text{lcm}_{0 \leq k \leq n} \text{ord}(M|_{C_k})$ (OEIS A060114; see the table in Section 5.4). No closed form for $\text{ord}(M|_{C_n})$ is known.
5. *A transparent model for both h and rev .* On triangulations, h is vertex complement (Proposition 3.1), a global geometric operation, but rev is recursive. On plane trees, rev reverses child lists at every level (a global operation), but h is recursive. Since $M = h \circ \text{rev}$, a Catalan model on which both h and rev have non-recursive descriptions would give a direct geometric handle on the cycle structure of M . Is there such a model among the over two hundred catalogued in [21]?
6. *Direct description of \oplus_3 .* The tertiary λ_3 has a simple description on each family (on plane trees, for instance, it adds a new leaf as the sole child of the rightmost leaf, so the \oplus_3 -indecomposable trees are those whose rightmost leaf is an only child). However, \oplus_3 involves the full secondary decomposition. Is there a direct, non-recursive description of \oplus_3 on any concrete Catalan family?

7. *Secondary composition on $\text{Av}(321)$.* On $\text{Av}(231)$, the operation \ominus has a direct permutation-level description in terms of the last entry of π_1 (see Section 3.4). On $\text{Av}(321)$, we define \ominus only through the generic recursion on indecomposables (see Section 3.5). Is there a direct description there too?

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A Bijection verification

The verifications in this appendix and Appendix B are included for completeness.

For each of the seven canonical bijections in Section 3.9, we verify the three identities (base, indecomposable, decomposable) that characterize the canonical bijection. Each verification amounts to showing that the concrete bijection f satisfies $f(e_1) = e_2$, $f(\lambda_1(t)) = \lambda_2(f(t))$, and $f(u \oplus_1 v) = f(u) \oplus_2 f(v)$. Notation follows Sections 3.1–3.8.

A.1 Triangulations \rightarrow plane trees

Let f send a triangulation of the $(n+2)$ -gon to a plane tree as described in Section 3.9. The vertex 1 becomes the root, the vertices adjacent to 1 (other than $n+2$) become the children of the root, and the construction recurses into each subpolygon between consecutive neighbours of 1.

Base. The empty triangulation e_{Tri} is the single edge $(1, 2)$, an edge with no diagonals. On the tree side, $e_{\text{Trees}} = []$, the single-node tree. Both are the size-0 object, and $f(e_{\text{Tri}}) = e_{\text{Trees}}$ is immediate (no vertices adjacent to 1 besides $2 = n+2$). The triangle $(1, 2, 3)$ is $\lambda(e)$: vertex 1 is adjacent to 2 and 3, and the sole subpolygon between them contains the edge $(2, 3)$, which is e_{Tri} . The image is $[[[]]] = [e] = \lambda(e)$, as required.

Indecomposable step. We must show $f(\lambda_{\text{Tri}}(T)) = \lambda_{\text{Trees}}(f(T))$. The operation $\lambda_{\text{Tri}}(T)$ relabels $i \rightarrow i+1$ in a triangulation T of the $(n+2)$ -gon and adds vertex 1 adjacent to 2 and $n+3$, creating a boundary triangle $(1, 2, n+3)$. In the resulting $(n+3)$ -gon, vertex 1 has exactly one neighbour besides $n+3$, namely vertex 2. The single subpolygon between them contains the relabeled copy of T . On the tree side, this means the root has exactly one child, whose subtree is $f(T)$. Thus $f(\lambda(T)) = [f(T)] = \lambda(f(T))$.

Decomposable step. We must show $f(T_1 \oplus_{\text{Tri}} T_2) = f(T_1) \oplus_{\text{Trees}} f(T_2)$. The operation $T_1 \oplus T_2$ identifies the edge $(1, n_1+2)$ of T_1 with edge $(1, 2)$ of T_2 (after shifting T_2 's labels). In the combined (n_1+n_2+2) -gon, the neighbours of vertex 1 are those from T_1 (excluding n_1+2) followed by those from the shifted T_2 . Each subpolygon in

T_1 stays intact, and each subpolygon in T_2 stays intact. The subpolygons between consecutive neighbours of 1 correspond to subtrees hanging from the root, so the children of the root in $f(T_1 \oplus T_2)$ are exactly the children of $f(T_1)$ followed by those of $f(T_2)$. This is $f(T_1) \oplus f(T_2)$, which merges child lists.

A.2 Plane trees \rightarrow Dyck paths

Let f send a plane tree to a Dyck path by depth-first search: each step down an edge produces a u -step, each step back up produces a d -step.

Base. $f([\])$ = ϵ : the single-node tree has no edges, so the traversal produces the empty word.

Indecomposable step. We must show $f(\lambda(T)) = \lambda(f(T)) = uf(T)d$. The tree $\lambda(T) = [T]$ has one child of the root, with subtree T hanging below it. The traversal first goes down the root edge (producing u), then traverses the subtree T (producing $f(T)$), then comes back up the root edge (producing d). The result is $uf(T)d = \lambda(f(T))$.

Decomposable step. We must show $f(T_1 \oplus T_2) = f(T_1)f(T_2) = f(T_1) \oplus f(T_2)$. The tree $T_1 \oplus T_2$ merges the child lists of the two roots. The DFS visits the children from left to right, so it first traverses all children of T_1 (producing $f(T_1)$), then all children of T_2 (producing $f(T_2)$). The result is the concatenation $f(T_1)f(T_2)$.

A.3 Dyck paths $\rightarrow 2 \times n$ SYT

Let f send a Dyck path of semilength n to a $2 \times n$ SYT by the following rule. Step i is u if and only if i lies in the top row of the tableau, and step i is d if and only if i lies in the bottom row.

Base. $f(\epsilon) = ([\], [\])$: the empty Dyck path corresponds to the empty tableau.

Indecomposable step. We must show $f(\lambda(w)) = \lambda_{\text{SYT}}(f(w))$. An indecomposable Dyck path $\lambda(w) = uwd$ of semilength n has step 1 as u and step $2n$ as d . So 1 goes in the top row and $2n$ in the bottom row. The inner steps $2, \dots, 2n-1$ encode the Dyck path w (shifted by 1), which by induction gives the SYT $f(w)$ with entries incremented by 1. This is exactly $\lambda_{\text{SYT}}(f(w))$: since $f(w)$ is a $2 \times (n-1)$ SYT, λ_{SYT} prepends 1 to the top row and appends $2(n-1)+2 = 2n$ to the bottom row.

Decomposable step. We must show $f(w_1w_2) = f(w_1) \oplus f(w_2)$. The concatenation w_1w_2 has the first $2n_1$ steps determined by w_1 and the last $2n_2$ steps by w_2 . The first $2n_1$ steps produce $f(w_1)$, and the last $2n_2$ steps produce $f(w_2)$ with entries shifted by $2n_1$. Placing these side by side gives $f(w_1) \oplus f(w_2)$.

A.4 Plane trees \rightarrow binary trees

Let f be the left-child/right-sibling encoding: the leftmost edge of the plane tree becomes the root of the binary tree, its subtree becomes the left child, and the remaining siblings form the right-sibling chain.

Base. $f([\]) = \emptyset$: the single-node tree (no edges) maps to the empty binary tree.

Indecomposable step. We must show $f(\lambda(T)) = \lambda_{\text{Bin}}(f(T)) = (f(T), \emptyset)$. The tree $\lambda(T) = [T]$ has one child. There is a single edge η_0 from root to child, with subtree $T_1 = T$ below it and no siblings (T_2 is a single node). The binary tree has root corresponding to edge η_0 , left subtree $f(T)$, and right subtree \emptyset . This is $(f(T), \emptyset) = \lambda(f(T))$.

Decomposable step. We must show $f(T_1 \oplus T_2) = f(T_1) \oplus_{\text{Bin}} f(T_2)$. The tree $T_1 \oplus T_2$ merges child lists. The leftmost edge of T_1 becomes the root of the binary tree, with its subtree as left child. The remaining children of T_1 followed by all children of T_2 form the right-sibling chain. In the binary tree, this means $f(T_2)$ is attached at the end of the right spine of $f(T_1)$, which is exactly $f(T_1) \oplus_{\text{Bin}} f(T_2)$.

A.5 Plane trees \rightarrow non-crossing partitions

Let f send a plane tree with n edges to a non-crossing partition of $[n]$ as follows: label the edges of T in postorder from 1 to n . The labels along the leftmost path form the first block, and the construction recurses on the remaining subtrees.

Base. $f([\])$ = $[\]$: the single-node tree has no edges, giving the empty partition.

Indecomposable step. We must show $f(\lambda(T)) = \lambda_{\text{NCP}}(f(T))$. The tree $\lambda(T) = [T]$ adds a new root edge. In postorder, the new root edge is the last visited, so it receives label $n+1$. This edge lies on the leftmost path, so $n+1$ is adjoined to the block containing 1. On the NCP side, $\lambda_{\text{NCP}}(P)$ adjoins $n+1$ to the block containing 1. Thus $f(\lambda(T)) = \lambda(f(T))$.

Decomposable step. We must show $f(T_1 \oplus T_2) = f(T_1) \oplus f(T_2)$. The tree $T_1 \oplus T_2$ merges child lists. In postorder, T_1 's subtree is processed first, receiving labels $1, \dots, n_1$, then T_2 's subtree receives labels n_1+1, \dots, n_1+n_2 . The resulting partition is $f(T_1) \cup (f(T_2) + n_1)$, which is $f(T_1) \oplus f(T_2)$: the shifted union.

A.6 Plane trees \rightarrow Av(231)

Let f send a plane tree to a 231-avoiding permutation: label the edges in postorder (as in Section A.5), then read the labels in preorder.

Base. $f([\])$ = ϵ : no edges, empty permutation.

Indecomposable step. We must show $f(\lambda(T)) = \lambda(f(T)) = (n+1)f(T)$. The tree $\lambda(T) = [T]$ has a new root edge. In postorder it is labeled $n+1$ (the largest label). The preorder reading visits the root edge first, so the permutation begins with $n+1$, followed by the preorder reading of T 's edges, which by induction is $f(T)$. Thus $f([\])$ = $(n+1)f(T) = \lambda(f(T))$.

Decomposable step. We must show $f(T_1 \oplus T_2) = f(T_1) \oplus f(T_2)$. The postorder labeling assigns $1, \dots, n_1$ to T_1 's edges and n_1+1, \dots, n_1+n_2 to T_2 's edges. The preorder reading visits T_1 's subtree first (producing $f(T_1)$, a permutation of $\{1, \dots, n_1\}$), then T_2 's subtree (producing $f(T_2)$ shifted by n_1 , a permutation of $\{n_1+1, \dots, n_1+n_2\}$). The result is the direct sum $f(T_1) \oplus f(T_2)$.

A.7 Plane trees $\rightarrow \text{Av}(321)$

Let f send a plane tree to a 321-avoiding permutation: label the edges in postorder, then apply the slot-filling traversal described in Section 3.9.

Lemma A.1. *In the slot-filling traversal of any tree S , the labels written on return (up) steps are placed in strictly increasing order, and each occupies a left-to-right maximum position of $f(S)$.*

Proof. Consider a non-leaf edge η with label ℓ . The edges whose labels are placed before the return of η fall into two groups: (i) edges in η 's subtree, and (ii) edges in left-sibling subtrees at every ancestor level of η . Every such edge η' is strictly smaller than η in postorder. For group (i), η' is a descendant of η and so precedes η in postorder. For group (ii), η' belongs to a left-sibling subtree of some ancestor of η , which is fully processed before η 's ancestor chain. Hence $\ell' < \ell$.

When η returns, its label ℓ is placed in the leftmost available slot. All positions to the left of this slot have already been filled with labels from groups (i) and (ii), each less than ℓ , so ℓ occupies a left-to-right maximum position. Since the set of labels placed before each successive return only grows, the return labels are placed in strictly increasing order. \square

Lemma A.2. *A position j in $\pi = f(T)$ is a return-fill position if and only if it is a left-to-right maximum but not a right-to-left minimum.*

Proof. That a return-fill position is a left-to-right maximum is Lemma A.1. For the “not a right-to-left minimum” part: a non-leaf edge η with label ℓ returning to position j has at least one descendant η' with $\ell' < \ell$ filling some $j' > j$, so j is not a right-to-left minimum.

Conversely, suppose a leaf edge η with label ℓ fills a position j that is a left-to-right maximum. Every edge η' filling a position $j' > j$ is either in a right-sibling subtree of an ancestor of η , or is an ancestor of η returning to a slot $> j$. In both cases η precedes η' in postorder, so $\ell < \ell'$, making j a right-to-left minimum. \square

Verification that f is canonical.

Base. $f([\])$ = ϵ : no edges, empty permutation.

Indecomposable step. We must show $f(\lambda(T)) = \lambda(f(T))$.

Case $T = e$. The tree $\lambda(e) = [e]$ has one leaf edge labeled 1. Slot-filling places it immediately, giving $f([e]) = 1$. On the other side, $\lambda(f(e)) = \lambda(\epsilon) = 1$.

Case $|T| \geq 1$. Write $\pi = f(T)$, $n = |T|$, $i = \pi^{-1}(1)$, and $T' = \lambda(T) = [T]$. Positions in all permutations are numbered $1, 2, \dots$. In the postorder labeling on T' , the new root edge η_* is visited last and therefore receives label $n+1$, while every edge of T keeps its original label.

By Lemma A.2, the return-fill positions of π are the left-to-right maxima that are not right-to-left minima. Since $\pi(i) = 1$ is the global minimum, i is a right-to-left minimum, so i is not a return-fill. The DFS traversal descends the non-leaf edges on the leftmost path (positions $1, 2, \dots, i-1$) before reaching the leftmost leaf (position

i , label 1), so the return-fills $< i$ are exactly $\{1, 2, \dots, i-1\}$. Write $c_1 < \dots < c_q$ for the return-fills $> i$ and set $k = i - 1 + q$.

Run the slot-filling on T' . The initial descent along η_* creates one extra slot at position 1; the remaining events are those of T shifted right by one. The $k+1$ slots in $\pi' = f(T')$ therefore sit at $1, 2, \dots, i, c_1+1, \dots, c_q+1$ and are filled left to right with $\pi(1), \dots, \pi(i-1), \pi(c_1), \dots, \pi(c_q), n+1$. Each non-slot position $l+1$ carries $\pi(l)$.

Now compare with $\lambda(\pi)$. Inserting $n+1$ at position i and shifting positions $\geq i$ right by one gives the same effect on non-return positions: position $j < i$ retains $\pi(j)$, and position $l+1$ for $l \geq i$ retains $\pi(l)$. The boxed positions of λ (see Section 3.5) are $B = \{i, c_1+1, \dots, c_q+1\}$: position i (newly inserted) together with the positions $> i$ that are left-to-right maxima but not right-to-left minima, each shifted by 1. The cyclic left-shift at B sends $n+1$ from position i to c_q+1 , and $\pi(c_j)$ from c_j+1 to $c_{j-1}+1$ (or to i when $j = 1$). This matches the slot-fill values above, so $\pi' = \lambda(\pi) = \lambda(f(T))$.

Decomposable step. We must show $f(T_1 \oplus T_2) = f(T_1) \oplus f(T_2)$. The tree $T_1 \oplus T_2$ concatenates the child lists of T_1 and T_2 under the root. In postorder T_1 's subtree is processed before T_2 's, assigning labels $1, \dots, n_1$ to T_1 's edges and n_1+1, \dots, n_1+n_2 to T_2 's edges; within each subtree the relative labeling is identical to that of T_i alone. The slot-filling procedure processes children left to right, so it fills T_1 's slots first using labels from $\{1, \dots, n_1\}$, producing $f(T_1)$ in positions $1, \dots, n_1$. It then fills T_2 's slots using labels from $\{n_1+1, \dots, n_1+n_2\}$, producing a shifted copy of $f(T_2)$ in positions n_1+1, \dots, n_1+n_2 . No cross-talk occurs because the position ranges and label sets are disjoint. The result is the direct sum $f(T_1) \oplus f(T_2)$.

B Dictionary verification

We verify that each entry in the dictionary table of Section 4.2 satisfies the canonical recursion from Section 4.1. For each family \mathcal{C} and each native statistic s , we check the three cases (base e , indecomposable λ , and decomposable \oplus), showing that s satisfies the same recursion as the corresponding canonical statistic. Once this is verified, the transport proposition (Proposition 2.12) transfers the equidistribution result to \mathcal{C} automatically. The base-case caveat (Remark 4.2) applies to the last two rows of the dictionary.

Lemma B.1. *If the canonical bijection sends leaves to a native statistic L , then it sends internal to $n + 1 - L$ for $n \geq 1$ (with $\text{internal}(e) = 0$ at $n = 0$). This follows from $\text{internal}(t) = |t| + 1 - \text{leaves}(t)$ and size preservation.*

By this lemma, the internal row of the dictionary is determined by the leaves row, so the per-family verifications below omit it. All definitions of e , λ , and \oplus are in Section 3.

B.1 Dyck paths

sub \rightarrow **comp**. The statistic **comp** counts returns to the x -axis. For ϵ , $\text{comp}(\epsilon) = 0$. For uwd : the path stays strictly above the x -axis between the initial u and the final

d , so $\text{comp}(uwd) = 1$. For w_1w_2 : since w_1 ends at height 0 and w_2 begins at height 0, the returns split additively: $\text{comp}(w_1w_2) = \text{comp}(w_1) + \text{comp}(w_2)$.

rpath \rightarrow **rcomp**. The statistic **rcomp** is the length of the maximal run of d -steps at the end. For ϵ , $\text{rcomp}(\epsilon) = 0$. For uwd : the path ends with d ; if $w = \epsilon$ the final run is d of length 1, and if $w \neq \epsilon$ then w ends with d so the final run extends by one step. Thus $\text{rcomp}(uwd) = 1 + \text{rcomp}(w)$. For w_1w_2 : the final run of d -steps is determined entirely by w_2 (since w_2 starts with u , the run cannot extend into w_1), so $\text{rcomp}(w_1w_2) = \text{rcomp}(w_2)$.

leaves \rightarrow **peak**. See the model proof in Section 4.3.

B.2 Triangulations

sub \rightarrow **comp**. The statistic **comp** counts edges incident to vertex 1, minus 1. For $e = (1, 2)$, vertex 1 has one neighbour, so $\text{comp}(e) = 0$. For $\lambda(T)$: vertex 1 is adjacent to exactly 2 and $n+3$, giving $\text{comp} = 1$. For $T_1 \oplus T_2$: the neighbours of 1 in T_1 (other than n_1+2) together with the neighbours of 1 in T_2 (shifted) give $\text{comp}(T_1 \oplus T_2) = \text{comp}(T_1) + \text{comp}(T_2)$.

rpath \rightarrow **rcomp**. The statistic **rcomp** counts edges incident to vertex $n+2$, minus 1. For e , $\text{rcomp}(e) = 0$. For $\lambda(T)$: the last vertex of the $(n+3)$ -gon is $n+3$, which inherits the (shifted) adjacencies of vertex $n+2$ in T and gains vertex 1 as an additional neighbour, so $\text{rcomp}(\lambda(T)) = 1 + \text{rcomp}(T)$. For $T_1 \oplus T_2$: the last vertex n_1+n_2+2 comes from T_2 (shifted), and its adjacencies are entirely determined by T_2 , so $\text{rcomp}(T_1 \oplus T_2) = \text{rcomp}(T_2)$.

leaves \rightarrow **tr**. The statistic **tr** counts indices $i \in \{2, \dots, n+1\}$ such that i and $i+1$ share a common neighbour $j < i$. For e , $\text{tr}(e) = 0$ (Remark 4.2).

For $\lambda(T)$: the shift $i \rightarrow i+1$ sends each qualifying index i in T to $i+1$ in $\lambda(T)$, preserving the triangle structure. The only new candidate is $i = 2$: vertices 2 and 3 share vertex 1 if and only if 1 is adjacent to 3, which holds if and only if $n+3 = 3$, that is, $T = e$. When $T = e$, $\lambda(e)$ is the triangle $(1, 2, 3)$ and $\text{tr} = 1 = \text{leaves}(e)$. When $T \neq e$, vertex 1 is adjacent only to 2 and $n+3 \geq 4$, so $i = 2$ does not qualify, and $\text{tr}(\lambda(T)) = \text{tr}(T)$.

For $T_1 \oplus T_2$: the gluing identifies an edge of T_1 with an edge of T_2 without creating any new triangle. Hence the triangles of the combined polygon are the disjoint union of those of T_1 and the shifted triangles of T_2 , and the qualifying indices split between the two ranges: $\text{tr}(T_1 \oplus T_2) = \text{tr}(T_1) + \text{tr}(T_2)$.

B.3 231-avoiding permutations

sub \rightarrow **comp**. The statistic **comp** counts components, where a 231-avoiding permutation is indecomposable when it starts with its maximum. For e , $\text{comp}(e) = 0$. For $\lambda(\pi) = (n+1)\pi$: the permutation starts with its maximum $n+1$, so it is indecomposable and $\text{comp}(\lambda(\pi)) = 1$. For $\pi_1 \oplus \pi_2$: the prefix π_1 permutes $\{1, \dots, k\}$ and π_2 (shifted) permutes $\{k+1, \dots, k+m\}$, so the components of the direct sum are exactly those of π_1 followed by those of π_2 : $\text{comp}(\pi_1 \oplus \pi_2) = \text{comp}(\pi_1) + \text{comp}(\pi_2)$.

rpath \rightarrow **rmax**. The statistic **rmax** counts right-to-left maxima. For e , $\text{rmax}(\epsilon) = 0$. For $\lambda(\pi) = (n+1)\pi$: since $n+1$ is prepended and all values in π are smaller, the right-to-left maxima of $\lambda(\pi)$ are exactly those of π together with $n+1$ itself. Thus $\text{rmax}(\lambda(\pi)) = 1 + \text{rmax}(\pi)$. For $\pi_1 \oplus \pi_2$: every value of π_2 (shifted) exceeds every value of π_1 , so a right-to-left maximum in π_1 is blocked by the values of π_2 to its right (unless π_2 is empty, but then we are not in the \oplus case). Hence the right-to-left maxima of the direct sum are exactly those of the shifted π_2 : $\text{rmax}(\pi_1 \oplus \pi_2) = \text{rmax}(\pi_2)$.

leaves \rightarrow **rmin**. The statistic **rmin** counts right-to-left minima. For e , $\text{rmin}(\epsilon) = 0$ (Remark 4.2).

For $\lambda(\pi) = (n+1)\pi$: the prepended $n+1$ is the global maximum and is never a right-to-left minimum. The remaining entries are π in the same relative order, so their right-to-left minima are unchanged: $\text{rmin}(\lambda(\pi)) = \text{rmin}(\pi)$.

For $\pi_1 \oplus \pi_2$: the values of π_1 lie in $\{1, \dots, k\}$ and those of π_2 (shifted) lie in $\{k+1, \dots, k+m\}$. Every value in π_1 is smaller than every value to its right coming from π_2 , so right-to-left minima within π_1 remain right-to-left minima of the whole permutation. Right-to-left minima within the shifted π_2 are determined by π_2 alone (no smaller values follow). Thus $\text{rmin}(\pi_1 \oplus \pi_2) = \text{rmin}(\pi_1) + \text{rmin}(\pi_2)$.

B.4 321-avoiding permutations

For $\text{Av}(321)$, $e = \epsilon$, $\lambda(\pi)$ inserts $n+1$ at the position of 1 and cyclically shifts the boxed positions (see Section 3.5 for the full description), and $\pi_1 \oplus \pi_2$ is the direct sum. The closure and bijectivity of λ are proved in Section 3.5. The arguments below use only the following structural property: the right-to-left minima of π are never boxed.

sub \rightarrow **comp**. The statistic **comp** counts the components of the direct-sum decomposition. For e , $\text{comp}(\epsilon) = 0$. For $\lambda(\pi)$: by Proposition 3.5, $\sigma = \lambda(\pi)$ is indecomposable, so $\text{comp}(\lambda(\pi)) = 1$. For the direct sum: $\text{comp}(\pi_1 \oplus \pi_2) = \text{comp}(\pi_1) + \text{comp}(\pi_2)$, by the same argument as for $\text{Av}(231)$.

rpath \rightarrow **rcomp**. The statistic $\text{rcomp}(\pi) = n - \pi(n) + 1$. For e , $\text{rcomp}(\epsilon) = 0$.

For $\lambda(\pi)$: the operation λ inserts $n+1$ at the position of 1 and cyclically shifts the boxed positions. The last entry of $\lambda(\pi)$ satisfies $\lambda(\pi)(n+1) = \pi(n)$: the value $\pi(n)$ is always a right-to-left minimum of π and hence never boxed, so the cyclic shift does not affect the last position. Therefore $\text{rcomp}(\lambda(\pi)) = (n+1) - \pi(n) + 1 = (n - \pi(n) + 1) + 1 = 1 + \text{rcomp}(\pi)$.

For $\pi_1 \oplus \pi_2$ with $|\pi_1| = k$: the last entry is $\pi_2(m) + k$ where $m = |\pi_2|$. Then $\text{rcomp}(\pi_1 \oplus \pi_2) = (k+m) - (\pi_2(m) + k) + 1 = m - \pi_2(m) + 1 = \text{rcomp}(\pi_2)$.

leaves \rightarrow **rmin**. The statistic **rmin** counts right-to-left minima. For e , $\text{rmin}(\epsilon) = 0$ (Remark 4.2).

For $\lambda(\pi)$: the inserted value $n+1$ is the global maximum and cannot be a right-to-left minimum. The cyclic shift permutes the values at the boxed positions b_0, \dots, b_m . Both before and after the shift, these values are left-to-right maxima of σ , each exceeding the suffix minimum at its position (since each boxed position is not a right-to-left minimum, there is a smaller value in its suffix at a non-boxed position). Hence

the shift does not create or destroy any right-to-left minimum, giving $\text{rmin}(\lambda(\pi)) = \text{rmin}(\pi)$.

For $\pi_1 \oplus \pi_2$ with $|\pi_1| = k$: values of π_1 lie in $\{1, \dots, k\}$ and values of the shifted π_2 lie in $\{k+1, \dots, k+m\}$. A position $j \leq k$ is a right-to-left minimum of the whole permutation if and only if it is one in π_1 : the π_2 values to its right are all $> k \geq \pi_1(j)$, so they impose no new constraint, and witnesses within π_1 are preserved. A position $j > k$ is a right-to-left minimum of the whole if and only if it is one in π_2 : all positions to its right also lie in the π_2 block. Hence $\text{rmin}(\pi_1 \oplus \pi_2) = \text{rmin}(\pi_1) + \text{rmin}(\pi_2)$.

B.5 Binary trees

sub \rightarrow **comp**. The statistic **comp** is the number of nodes on the right spine from the root (that is, the number of indecomposable summands). For \emptyset , $\text{comp}(\emptyset) = 0$. For $\lambda(T) = (T, \emptyset)$: the right spine has one node, so $\text{comp} = 1$. For $B_1 \oplus B_2$: the right spine of the combined tree consists of the right spine of B_1 extended by that of B_2 , so $\text{comp}(B_1 \oplus B_2) = \text{comp}(B_1) + \text{comp}(B_2)$.

rpath \rightarrow **rcomp**. The statistic $\text{rcomp}(B)$ is computed by iteratively peeling right spines: go to the last node on the right spine, replace B by that node's left subtree, and repeat; **rcomp** counts the number of iterations before reaching \emptyset . For \emptyset , $\text{rcomp}(\emptyset) = 0$. For $\lambda(T) = (T, \emptyset)$: the right spine consists of the root alone, whose left subtree is T , so one iteration reduces to T : $\text{rcomp}((T, \emptyset)) = 1 + \text{rcomp}(T)$. For $B_1 \oplus B_2$: the last node on the combined right spine is the last node of B_2 's right spine, with the same left subtree, so $\text{rcomp}(B_1 \oplus B_2) = \text{rcomp}(B_2)$.

leaves \rightarrow **redge**. We define $\text{redge}(B) = 1 + (\text{number of right edges of } B)$, where a right edge is an edge from a node to a nonempty right child. Equivalently, **redge** satisfies the three-case recursion

$$\begin{aligned} \text{redge}(\emptyset) &= 1 \\ \text{redge}((L, \emptyset)) &= \text{redge}(L) \\ \text{redge}((L, R)) &= \text{redge}(L) + \text{redge}(R) \quad \text{for } R \neq \emptyset \end{aligned}$$

matching the canonical recursion for **leaves**. Indeed: for $\lambda(T) = (T, \emptyset)$, the root has no right child, so λ introduces no new right edge: $\text{redge}(\lambda(T)) = \text{redge}(T)$. For $B_1 \oplus B_2$: \oplus creates exactly one new right edge (from the last node of B_1 's right spine to the root of B_2), preserving the internal edges. Writing r_i for the number of right edges of B_i : $\text{redge}(B_1 \oplus B_2) = 1 + r_1 + 1 + r_2 = \text{redge}(B_1) + \text{redge}(B_2)$.

B.6 $2 \times n$ SYT

sub \rightarrow **comp**. The statistic **comp** counts components: positions k where the first k columns contain exactly $\{1, \dots, 2k\}$. For e , $\text{comp}(e) = 0$. For $\lambda(T)$: since 1 is in position $(1, 1)$ and $2n+2$ is in position $(2, n+1)$, no proper prefix of columns contains $\{1, \dots, 2j\}$ (the entry 1 prevents early closure and $2n+2$ prevents premature completion), so $\text{comp}(\lambda(T)) = 1$. For $T_1 \oplus T_2$: the blocks are determined independently in the two halves, so $\text{comp}(T_1 \oplus T_2) = \text{comp}(T_1) + \text{comp}(T_2)$.

rpath \rightarrow **rcomp**. The statistic **rcomp** is the length of the maximal consecutive segment at the right end of the bottom row: the largest k such that the last k entries of

the bottom row, read right to left, are $2n, 2n-1, \dots, 2n-k+1$. For e , $\text{rcomp}(e) = 0$. For $\lambda(T)$: the bottom row ends with $2n+2$ (newly appended). The previous bottom entry is the last entry of the shifted inner bottom row, which is $2n'+1$ (where $n' = |T|$) if it ended the inner consecutive segment. Since λ places $2n+2$ at the end and shifts the inner entries by 1, the consecutive segment extends by one: $\text{rcomp}(\lambda(T)) = 1 + \text{rcomp}(T)$. For $T_1 \oplus T_2$: the right end of the bottom row is entirely determined by T_2 (shifted), so $\text{rcomp}(T_1 \oplus T_2) = \text{rcomp}(T_2)$.

leaves \rightarrow **noncons**. The statistic **noncons** counts entries i in the top row such that $i+1$ is not in the top row. For e , $\text{noncons}(e) = 0$ (Remark 4.2).

For $\lambda(T)$: the top row of $\lambda(T)$ is $\{1\} \cup \{t_j + 1 : t_j \in \text{top}(T)\}$. When $T = e$, $\lambda(e) = ([1], [2])$, so entry 1 has 2 in the bottom row and $\text{noncons} = 1 = \text{leaves}(e)$. When $T \neq e$, the top row of T contains 1, so the top row of $\lambda(T)$ contains both 1 and $2 = 1 + 1$; thus entry 1 does not contribute to **noncons**. The shifted entries preserve the **noncons** count: $\text{noncons}(\lambda(T)) = \text{noncons}(T)$.

For $T_1 \oplus T_2$: entries from T_1 and T_2 occupy disjoint value ranges ($\{1, \dots, 2n_1\}$ and $\{2n_1+1, \dots, 2n_1+2n_2\}$). The largest top entry of T_1 is at most $2n_1 - 1$ (since $2n_1$ is always in the bottom row), so for every top entry i of T_1 , the successor $i + 1 \leq 2n_1$ lies within T_1 's range. Likewise, for every top entry i of the shifted T_2 , $i + 1$ lies within T_2 's range. Thus the **noncons** contributions from T_1 and T_2 are independent: $\text{noncons}(T_1 \oplus T_2) = \text{noncons}(T_1) + \text{noncons}(T_2)$.

B.7 Non-crossing partitions

sub \rightarrow **comp**. The statistic **comp** counts components. For e , $\text{comp}(e) = 0$. For $\lambda(P)$: since 1 and $n+1$ belong to the same block, $\lambda(P)$ is indecomposable, so $\text{comp}(\lambda(P)) = 1$. For $P_1 \oplus P_2$: the blocks of P_1 occupy $\{1, \dots, n_1\}$ and those of P_2 (shifted) occupy $\{n_1+1, \dots, n_1+n_2\}$, so the components of the two partitions contribute independently: $\text{comp}(P_1 \oplus P_2) = \text{comp}(P_1) + \text{comp}(P_2)$.

rpath \rightarrow **rcomp**. The statistic **rcomp** counts right-to-left maxima in the flattened form of P , where the flattened form lists the elements of each block in decreasing order, with blocks sorted by their minimal element. For e , $\text{rcomp}(e) = 0$.

For $\lambda(P)$: the operation adjoins $n+1$ to the block of 1, making $n+1$ the first element of that block in the flattened form (since it is the new maximum of the block). The remaining elements appear in the same relative order. Since $n+1$ exceeds all other elements, it is always a right-to-left maximum in the flattened sequence, adding exactly one to the count. Thus $\text{rcomp}(\lambda(P)) = 1 + \text{rcomp}(P)$.

For $P_1 \oplus P_2$: every element of the shifted P_2 exceeds every element of P_1 , so no element of P_1 can be a right-to-left maximum in the combined flattened form. The right-to-left maxima are exactly those within P_2 's portion: $\text{rcomp}(P_1 \oplus P_2) = \text{rcomp}(P_2)$.

leaves \rightarrow **blocks**. The statistic **blocks** counts blocks. For e , $\text{blocks}(e) = 0$ (Remark 4.2).

For $\lambda(P)$: when $P = e$, $\lambda(e) = \{\{1\}\}$ has one block and $\text{leaves}(e) = 1$. When $P \neq e$, λ merges $n+1$ into the existing block of 1 without creating or destroying any block, so $\text{blocks}(\lambda(P)) = \text{blocks}(P)$.

For $P_1 \oplus P_2$: the blocks are the disjoint union of P_1 's blocks and P_2 's (shifted) blocks, so $\text{blocks}(P_1 \oplus P_2) = \text{blocks}(P_1) + \text{blocks}(P_2)$.

References

- [1] R. Brak, A universal bijection for Catalan structures, *arXiv:1808.09078* (2019).
- [2] D. Callan, A bijection on Dyck paths and its cycle structure, *Electron. J. Combin.* **14** (2007), #R28.
- [3] A. Claesson and S. Kitaev, Classification of bijections between 321- and 132-avoiding permutations, *Sém. Lothar. Combin.* **60** (2008), Art. B60d.
- [4] A. Claesson, S. Kitaev, and E. Steingrímsson, Decompositions and statistics for $\beta(1,0)$ -trees and nonseparable permutations, *Adv. Appl. Math.* **42** (2009), 313–328.
- [5] A. Claesson, S. Kitaev, and E. Steingrímsson, An involution on $\beta(1,0)$ -trees, *Adv. Appl. Math.* **51** (2013), 276–284.
- [6] N. Dershowitz and S. Zaks, Ordered trees and non-crossing partitions, *Discrete Math.* **62** (1986), 215–218.
- [7] E. Deutsch, Dyck path enumeration, *Discrete Math.* **204** (1999), 167–202.
- [8] R. Donaghey, Automorphisms on Catalan trees and bracketings, *J. Combin. Theory Ser. B* **29** (1980), 75–90.
- [9] W. Fang, A trinity of duality: non-separable planar maps, $\beta(1,0)$ -trees and synchronized intervals, *Adv. Appl. Math.* **95** (2018), 1–30.
- [10] S. Kitaev and A. de Mier, Enumeration of fixed points of an involution on $\beta(1,0)$ -trees, *Graphs Combin.* **30** (2014), 1207–1221.
- [11] D. E. Knuth, *The Art of Computer Programming, Vol. 1: Fundamental Algorithms*, Addison-Wesley, 1968.
- [12] D. E. Knuth, *The Art of Computer Programming, Vol. 4, Fascicle 4: Generating All Trees*, Addison-Wesley, 2006.
- [13] G. Kreweras, Sur les partitions non croisées d'un cycle, *Discrete Math.* **1** (1972), 333–350.
- [14] T. V. Narayana, Sur les treillis formés par les partitions d'un entier et leurs applications à la théorie des probabilités, *C. R. Acad. Sci. Paris* **240** (1955), 1188–1189.
- [15] A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability*, London Math. Soc. Lecture Note Ser. **335**, Cambridge University Press, 2006.
- [16] The On-Line Encyclopedia of Integer Sequences, <http://oeis.org>,
- [17] I. A. Pushkarev and V. A. Byzov, Donaghey's transformation: an elementary approach, *J. Math. Sci.* **196** (2014), 199–215.

- [18] I. A. Pushkarev and V. A. Byzov, Donaghey's transformation: carousel effects and tame components, *Prikl. Diskr. Mat.* **44** (2019), 12–33.
- [19] L. W. Shapiro, The cycle of six, *Fibonacci Quart.* **17** (1979), 253–259.
- [20] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, 1999.
- [21] R. P. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.