INVERSION MONOTONICITY IN SUBCLASSES OF THE 1324-AVOIDERS

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Abstract. A collection B of patterns is called inversion monotone if $\mathrm{av}_n^k(B)$, the number of B-avoiding permutations of length n with k inversions, is weakly increasing in n for any fixed k. In 2012, Claesson, Jelínek and Steingrímsson posed the *inversion monotonicity conjecture*, which states that the pattern 1324 is inversion monotone and implies a new upper bound for its Stanley–Wilf limit.

We prove that the collections $\{1324, 231\}$ and $\{1324, 2314, 3214, 4213\}$ are inversion monotone via explicit injections. The latter follows from a general procedure for constructing inversion-monotone sets. Our results constitute the first known nontrivial examples of inversion-monotone sets.

A key feature of the inversion monotonicity conjecture is that 1324 has a limit sequence: $\mathrm{av}_n^k(1324)$ is constant in n when n is large. We characterize the sets of patterns that have limit sequences, and determine the limit sequences of all pairs $\{1324, p\}$, where p is a pattern of length four. Connections to various families of integer partitions arise.

Finally, we expand on work by Linusson and Verkama (2025) on almost decomposable permutations to determine a broad family of sets containing 1324 that are inversion monotone under the assumption $n \geqslant \frac{k+7}{2}$. The method yields an enumeration of $\mathrm{av}_n^k(1324, 1342)$ when $n \geqslant \frac{k+7}{2}$.

Contents

1. Introduction

The distribution of combinatorial statistics over restricted sets of permutations is a popular topic in enumerative combinatorics. One such statistic is the number of inversions, and a common restriction is pattern avoidance. Let $\mathrm{Av}_n^k(p)$ denote the set of length-n permutations with exactly k inversions that avoid a pattern p, and let $\mathrm{av}_n^k(p) = |\mathrm{Av}_n^k(p)|$. This distribution is poorly understood. For example, if $p = 132$, then $\mathrm{av}_n^k(p)$ is the coefficient of x^k in Carlitz's q-Catalan numbers

$$C_n(x) = \sum_{k=0}^{n-1} C_k(x) C_{n-1-k}(x) x^{k(n-k)}, \quad C_0(x) = 1.$$

These polynomials were conjectured to be unimodal by Stanton in 1990 [Sta90], and the conjecture remains open. In this paper, we study a certain feature of the distribution of inversions over the notorious class $\mathrm{Av}(1324)$.

Inversion-monotone patterns. In 2012, Claesson, Jelínek and Steingrímsson [CJS12] introduced *inversion monotonicity* – a pattern p is called inversion monotone if $\mathrm{av}_n^k(p) \leqslant \mathrm{av}_{n+1}^k(p)$ for all n and k. Inversion monotonicity has special implications for $\mathrm{Av}(1324)$, whose exact enumeration and asymptotic growth rate (*Stanley–Wilf limit*, see [MT04]) are long-standing open problems: if 1324 is inversion monotone, then its Stanley–Wilf limit is less than 13.002. Currently, the best known upper and lower bounds for the Stanley–Wilf limit of 1324 are 13.5 and 10.27, respectively [Bev+20], whereas the actual value is estimated to be 11.600 ± 0.003 [CGZ18].

Conjecture 1.1 (Conjecture 20 in [CJS12])**.** *All patterns, except for the identity patterns, are inversion monotone.*

The identity pattern $\mathrm{id}_m \in S_m$ is not inversion monotone, since $\mathrm{av}_n^k(\mathrm{id}_m) = 0$ for all $n \geqslant k + m$ by (1) (Section 2). Furthermore, if p is a pattern that does not start with 1 or does not end with its largest entry, then p is trivially inversion monotone; in the latter case, setting $n + 1$ as the last entry defines an injection from $\mathrm{Av}_n^k(p)$ to $\mathrm{Av}_{n+1}^k(p)$, and the former case is symmetric. However, Conjecture 1.1 is open for *all* nontrivial patterns. Linusson and Verkama recently made progress on the important pattern 1324 by proving combinatorially that $\mathrm{av}_n^k(1324) \leqslant \mathrm{av}_{n+1}^k(1324)$ for all k and $n \geqslant \frac{k+7}{2}$ [LV25].

In this work, we examine the inversion monotonicity of 1324 by imposing additional pattern-avoidance conditions. We are guided by the following idea: if B is an inversion-monotone collection of patterns such that $1324 \in B$, $|B|$ is small and all patterns in $B \setminus \{1324\}$ are long, then 1324 is close to being inversion monotone. To this end, our results are as follows (Theorem 3.2 and Propositions 4.1, 4.3).

Theorem. *The collections* $\{1324, 231\}$ *and* $\{1324, 2314, 3214, 4213\}$ *are inversion monotone.*

The inversion monotonicity of the latter collection follows as an immediate consequence of a procedure that, given an inversion-monotone collection B, constructs a nontrivially inversion-monotone collection B' such that

$$\min\{|p| : p \in B'\} = \min\{|p| : p \in B\} + 1.$$

Section 4 describes this construction. On the other hand, our proof of the inversion monotonicity of $\{1324, 231\}$ relies on an intricate injection. These are the first proofs of inversion monotonicity for any nontrivial collection of patterns.

Limit sequences. We mentioned above that if 1324 is inversion monotone, then its Stanley–Wilf limit is less than 13.002. The estimate is based on the fact that if $n \geqslant k + 2$, then

$$\mathrm{av}_n^k(1324) = \sum_{i=0}^{k} p(i)p(k-i) =: a(k),$$

where $p(k)$ is the number of integer partitions of k (see Section 2). In particular, $a(k)$ is independent of n. If 1324 is inversion monotone, the number of 1324-avoiders of length n is

$$\sum_{k=0}^{\binom{n}{2}} \mathrm{av}_n^k(1324) \leqslant \sum_{k=0}^{\binom{n}{2}} a(k) \leqslant \left(\binom{n}{2} + 1\right) a\left(\binom{n}{2}\right),$$

on which classical estimates may be applied. It is clear that the *limit sequence* $a(0), a(1), a(2), \ldots$ of 1324 is important.

After proving that $\{1324, 231\}$ is inversion monotone, it is natural to examine the pairs $\{1324, p\}$, where p is a pattern of length four. We could not prove that any of these pairs are inversion monotone, but we are able to determine all of their limit sequences (Section 5, in particular Table 3). In most cases, the problem reduces to counting *indecomposable* permutations avoiding 132 and a pattern of length four by the number of inversions, expanding on work by Franklín [Fra25] (Section 7). The limit sequences are combinatorially rich, with connections to several interesting classes of integer partitions, such as the sand pile model and penny arrangements. Furthermore, we obtain the following characterization of sets that have limit sequences (Proposition 5.1).

Proposition. *A collection B has a limit sequence if and only if B contains a pattern p such that* $\mathrm{inv}(p) \leqslant 1$.

Half-monotone collections. In [LV25], Linusson and Verkama defined a certain injective mapping on the *almost decomposable* 1324-avoiding permutations and proved that if $k \leqslant 2n - 7$, every permutation in $\mathrm{Av}_n^k(1324)$ is decomposable or almost decomposable. As a consequence, $\mathrm{av}_n^k(1324) \leqslant \mathrm{av}_{n+1}^k(1324)$ for every $n \geqslant \frac{k+7}{2}$; we say that 1324 is *half-monotone*. In this work, we expand on the result by determining necessary (Theorem 6.8) and sufficient (Theorem 6.6) conditions for a pattern p to have the following property: if $\pi \in \mathrm{Av}(1324, p)$ is almost decomposable, then the image of π under the Linusson–Verkama injection avoids $\{1324, p\}$. If p is such a pattern – we say that p is *compatible* – then $\{1324, p\}$ is half-monotone. Some special cases of the results in Section 6 are summarized in the following theorem.

Theorem. *The compatible patterns of length four are*

$$1432,\ 4231,\ 4321,$$

and the compatible patterns of length five are

$$\begin{gathered} 14523,\ 14532,\ 15342,\ 15423,\ 15432,\ 34125, \\ 52341,\ 52431,\ 53241,\ 53421,\ 54231,\ 54321. \end{gathered}$$

Furthermore, if $p \in S_m$ is any pattern such that $p_1 = m$ and $p_m = 1$, then p is compatible.

Lastly, we study the pair $\{1324, 1342\}$. The pattern 1342 is not fully compatible, but any almost decomposable permutation $\pi \in \mathrm{Av}_n(1324, 1342)$ such that its image under the Linusson–Verkama injection contains 1342 must satisfy $\mathrm{inv}(\pi) \geqslant 2n-4$ (Theorem 6.9). Half-monotonicity of $\{1324, 1342\}$ follows. Furthermore, we determine the differences $\mathrm{av}_{n+1}^k(1324, 1342) - \mathrm{av}_n^k(1324, 1342)$ when $k \leqslant 2n-7$ (similar to the case 1324 in [LV25]), leading to the following enumeration result (Theorem 6.12).

Theorem. *For every $n \geqslant \frac{k+7}{2}$, the difference $\mathrm{av}_{n+1}^k(1324, 1342) - \mathrm{av}_n^k(1324, 1342)$ is nonnegative, and equals the coefficient of x^k in the generating function*

$$x^{n-1}(2+2x) \cdot \prod_{i \geqslant 1} \frac{1+x^i}{1-x^i}.$$

In particular, when $n \geqslant \frac{k+7}{2}$,

$$\mathrm{av}_n^k(1324, 1342) = [x^k]\left(\frac{1-x-x^{n-1}(2+2x)}{1-x} \cdot \prod_{i \geqslant 1} \frac{1+x^i}{1-x^i}\right).$$

Outline. Section 2 recalls the necessary preliminaries. In Section 3 we prove that $\{1324, 231\}$ is inversion monotone, whereas Section 4 describes the procedure for building new, larger inversion monotone collections. Our analysis of the limit sequences of pairs $\{1324, p\}$ with $p \in S_4$, together with some other observations on pairs of length-four patterns, are in Section 5. Section 6 discusses almost decomposable permutations and compatible patterns. The enumeration results for indecomposable permutations avoiding $\{132, p\}$, needed for the limit sequences, are in Section 7. Finally, Section 8 contains some concluding remarks and open problems, and Appendix A shows data for the values of $\mathrm{av}_n^k(1324, p)$ and $\mathrm{av}_{n+1}^k(1324, p) - \mathrm{av}_n^k(1324, p)$ for $p \in S_4$.

2. Preliminaries

A permutation $\pi \in S_n$ *contains* a pattern $p \in S_m$ if π has a subsequence that is order-isomorphic to p. If π does not contain p, then π *avoids* p. If B is a set of patterns, we say that π avoids B if π avoids every pattern contained in B. We write $\mathrm{Av}(B)$ for the set of all permutations that avoid B, $\mathrm{Av}_n(B) = \mathrm{Av}(B) \cap S_n$ and $\mathrm{av}_n(B) = |\mathrm{Av}_n(B)|$. Furthermore, if $\pi \in S_n$, we write $|\pi| = n$.

It is convenient for us to have notation for the pattern obtained by deleting certain entries. If $\pi \in S_n$ and $A \subseteq [n]$, let $\pi \setminus A$ denote the permutation that is order-isomorphic to the subsequence obtained by removing all entries with values in A from the one-line notation of π. For singletons, we write $\pi \setminus e = \pi \setminus \{e\}$.

Inversions and decomposability. An *inversion* in a permutation π is a pair (i, j) of indices such that $i < j$ and $\pi_i > \pi_j$. Let $\mathrm{inv}(\pi)$ denote the number of inversions in π. We write $\mathrm{Av}_n^k(B)$ for the set of B-avoiding permutations of length n with exactly k inversions, and $\mathrm{av}_n^k(B) = |\mathrm{Av}_n^k(B)|$. There are three (nontrivial) symmetries preserving the number of inversions of a permutation $\pi \in S_n$: the inverse π^{-1}, the reverse-complement π^{rc}, which is given by $\pi_i^{\mathrm{rc}} = n + 1 - \pi_{n+1-i}$, as well as their composition $(\pi^{-1})^{\mathrm{rc}}$. The reverse-complement is the composition of two symmetries that sends inversions to noninversions, namely the reverse $\pi_i^{\mathrm{rev}} = \pi_{n+1-i}$ and the complement $\pi_i^{\mathrm{c}} = n + 1 - \pi_i$.

We define the *direct sum* of $\sigma \in S_n$ and $\tau \in S_m$ as the permutation $\sigma \oplus \tau \in S_{n+m}$ given by

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i) & \text{if } i \leqslant n, \\ n + \tau(i - n) & \text{if } i > n. \end{cases}$$

Similarly, the *skew sum* $\sigma \ominus \tau$ is defined by

$$(\sigma \ominus \tau)(i) = \begin{cases} m + \sigma(i) & \text{if } i \leqslant n, \\ \tau(i - n) & \text{if } i > n. \end{cases}$$

There is a nice visual interpretation. We will illustrate permutations using their plots in cartesian coordinates: $\pi \in S_n$ becomes $\{i, \pi_i : i \in [n]\}$. See Figure 1 for an example: if $\sigma = 21$ and $\tau = 231$, then $\sigma \oplus \tau = 21453$ and $\sigma \ominus \tau = 54231$.

A permutation is called *indecomposable* if it cannot be written as the direct sum of two nonempty permutations. Every permutation π can be written uniquely as a direct sum

$$\pi = \pi^{(1)} \oplus \pi^{(2)} \oplus \ldots \oplus \pi^{(r)}$$

of indecomposable permutations $\pi^{(i)}$, called the *components* of π. Let $\mathrm{comp}(\pi) = r$ denote the number of components of π. The identity permutation of length n is denoted by id_n. If the length is irrelevant and can be left unspecified, we simply write id. Note that

$$\mathrm{id}_n = \underbrace{1 \oplus 1 \oplus \ldots \oplus 1}_{n \text{ times}}.$$

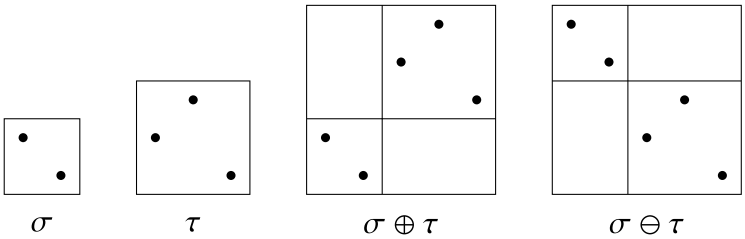

Figure 1. The direct sum and skew sum of $\sigma = 21$ and $\tau = 231$.

The number of inversions and components of a permutation $\pi \in S_n$ are related by the Erdős–Szekeres-type inequality

$$\mathrm{inv}(\pi) + \mathrm{comp}(\pi) \geqslant n, \tag{1}$$

see [CJS12, Lemma 8]. In particular, if $n \geqslant \mathrm{inv}(\pi) + 2$, then π must be decomposable.

An important observation is that a decomposable permutation π avoids 1324 if and only if it is of the form $\pi = \sigma \oplus \mathrm{id} \oplus \tau$, where σ is an indecomposable 132-avoider and τ is an indecomposable 213-avoider [CJS12].

Definition 2.1. A collection B of patterns is called *inversion monotone* if for all k and n,

$$\mathrm{av}_n^k(B) \leqslant \mathrm{av}_{n+1}^k(B).$$

If $B = \{p\}$, we say that p itself is inversion monotone.

Limit sequences. We often find that the sequences $(\mathrm{av}_n^k(B))_n$ converge to constant values c_k. In these cases, analyzing the limiting sequence $(c_k)_k$ can be interesting.

Definition 2.2. If B is a set of patterns such that for every k there exist constants m_k and c_k for which

$$\mathrm{av}_n^k(B) = c_k \quad \text{for all } n \geqslant m_k,$$

we say that $(c_k)_{k \geqslant 0}$ is the *limit sequence* of B, and denote $c_k(B) = c_k$. We call

$$C_B(x) = \sum_{k \geqslant 0} c_k(B) x^k$$

the *limit generating function* of B.

Few limit sequences have been explicitly determined. A simple argument shows that $C_{132}(x) = P(x)$ (using the inversion sequence; see (2) below), from which it follows that $C_{1324}(x) = P(x)^2$, where $P(x)$ is the generating function for the partition numbers. However, for example $C_{1243}(x)$ is unknown. Existence is better understood. Claesson, Jelínek and Steingrímsson proved that a pattern p has a limit sequence if and only if $\mathrm{inv}(p) \leqslant 1$ [CJS12, Proposition 21]. We provide a simple generalization in Proposition 5.1.

Definition 2.3. Let B and B' be collections of patterns.

- B and B' are *inv-Wilf-equivalent* if

$$\mathrm{av}_n^k(B) = \mathrm{av}_n^k(B') \quad \text{for all } n, k.$$

- B and B' are *limit equivalent* if their limit sequences exist and are equal.

Remark 2.4. A collection B of patterns is trivially inv-Wilf-equivalent to

$$\{p^{-1} : p \in B\} \quad \text{and} \quad \{p^{\mathrm{rc}} : p \in B\}.$$

If B and B' are inv-Wilf-equivalent and have a limit sequence, then obviously they are also limit equivalent. Inv-Wilf-equivalence was introduced by Dokos, Dwyer, Johnson, Sagan and Selsor [Dok+12]. Chan proved that nontrivial inv-Wilf-equivalences exist, even for principal classes [Cha15].

132-avoiders and partitions. An essential tool in our analysis of limit sequences is the bijection between indecomposable 132-avoiders and integer partitions, encoding the inversions of the permutation. The *inversion table* or *Lehmer code* $L(\pi) = (b_1, \ldots, b_n)$ of a permutation $\pi \in S_n$ is defined by

$$b_i = |\{j \in \{i+1, \ldots, n\} : \pi_i > \pi_j\}|$$

for all $i \in [n]$. This mapping gives a bijection

$$L : S_n \longrightarrow \{0, \ldots, n-1\} \times \{0, \ldots, n-2\} \times \ldots \times \{0\}.$$

It is easy to show that the inversion table of a permutation π is weakly decreasing if and only if π avoids 132 [Sta12, Section 1.5]. Therefore L induces a mapping Λ from the 132-avoiders to the integer partitions, where $\Lambda(\pi)$ is the partition obtained from $L(\pi)$ by deleting the trailing zeroes. The restriction

$$\Lambda : \{\pi \in \mathrm{Av}(132) : \mathrm{comp}(\pi) = 1\} \longrightarrow \{\text{integer partitions}\}$$

is a bijection. If we fix k and $n \geqslant k+1$, we therefore also have that

$$\Lambda : \mathrm{Av}_n^k(132) \longrightarrow \{\text{integer partitions of } k\} \tag{2}$$

is a bijection. If λ is a partition, we will write $\Lambda^{-1}(\lambda)$ for the unique indecomposable 132-avoider π such that $\Lambda(\pi) = \lambda$. The following elementary result collects some facts about the connection between π and $\lambda = \Lambda(\pi)$.

Lemma 2.5. *Let π be an indecomposable 132-avoiding permutation and $\lambda = \Lambda(\pi)$.*

(a) For every i, π_i is the smallest positive integer ℓ such that $\ell > \lambda_i$ and $\ell \neq \pi_j$ for all $j < i$.

(b) $\lambda_i = \lambda_{i+1}$ *if and only if* $\pi_i < \pi_{i+1}$.

(c) $\lambda_i > \lambda_{i+1}$ *if and only if* $\pi_{i+1} = \lambda_{i+1} + 1$.

Lastly, we show how to determine the limit sequence of 1324. Let $n \geqslant k + 2$. By (1), every permutation $\pi \in \mathrm{Av}_n^k(1324)$ is decomposable, and may be written as $\pi = \sigma \oplus \mathrm{id} \oplus \tau$, where σ is an indecomposable 132-avoider and τ is an indecomposable 213-avoider. Note that $\mathrm{inv}(\pi) = \mathrm{inv}(\sigma) + \mathrm{inv}(\tau)$. The mapping $\pi \mapsto (\Lambda(\sigma), \Lambda(\tau^{\mathrm{rc}}))$ is a bijection between $\mathrm{Av}_n^k(1324)$ and pairs (λ, μ) of integer partitions such that $|\lambda| + |\mu| = k$. Therefore,

$$\mathrm{av}_n^k(1324) = \sum_{i=0}^{k} p(i)p(k-i)$$

for all $n \geqslant k + 2$, and hence $C_{1324}(x) = P(x)^2$.

3. {1324, 231} is inversion monotone

The pattern 1324 contains three distinct patterns of length three: 123, 132 and 213. Furthermore, {1324, 321} is *not* inversion monotone, since

$$\mathrm{av}_{10}^{15}(1324, 321) = 60 > 52 = \mathrm{av}_{11}^{15}(1324, 321).$$

This is interesting in itself, since the decreasing patterns intuitively seem highly inversion monotone. Nevertheless, the only remaining pattern is 231 (and the symmetric 312). We shall prove that the collection {1324, 231} is inversion monotone, but first we establish a lemma.

Lemma 3.1. *An indecomposable permutation* $\pi \in \mathrm{Av}_n^k(213, 231)$ *starts with n, and:*

(a) *For each* $r \in \{0, \dots, n\}$*, there exists a unique* $\sigma \in \mathrm{Av}_{n+1}^{k+r}(213, 231)$ *such that* $\sigma \setminus i = \pi$ *for some i.*

(b) *For each* $r \in \{1, \dots, n-1\}$*, there exists a unique* $\tau \in \mathrm{Av}_{n-1}^{k-r}(213, 231)$ *such that* $\pi \setminus i = \tau$ *for some i.*

Proof. The inverse permutation π^{-1} is {213, 312}-avoiding, thus unimodal. Therefore π^{-1} (and π) is indecomposable if and only if $\pi_n^{-1} = 1$, that is, $\pi_1 = n$. Hence π is of the form indicated in Figure 2 (left): the subsequence of entries π_i with $\pi_i \leqslant \pi_n$ is increasing (the 'lower arm'), and the subsequence of entries π_i with $\pi_i > \pi_n$ is decreasing (the 'upper arm').

Denote the number of points in the upper (resp. lower) arm by a (resp. b), so that $a + b = n$. The rightmost point is considered to be part of the lower arm. There can be several ways to add a point to the lower arm in a given interval between two points in the upper arm, but each insertion yields the same permutation. A point

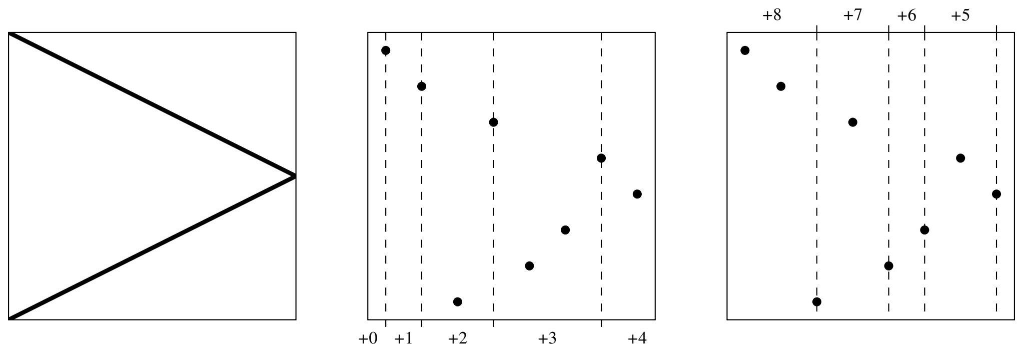

Figure 2. The structure of a $\{213, 231\}$-avoider (left). The different ways to insert a point in the lower arm (middle) and in the upper arm (right), with the increase in inversions indicated.

added to the r-th such interval adds exactly r inversions, for $r \in \{0, \ldots, a\}$; see Figure 2 (middle).

Similarly, adding a point to the upper arm in the r-th interval (from the right) between two points in the lower arm adds exactly $a+r$ inversions, for $r \in \{1, \ldots, b\}$; see Figure 2 (right). This proves part (a). Part (b) is similar: we delete a point instead of adding a point. □

Theorem 3.2. *The collection* $\{1324, 231\}$ *is inversion monotone.*

Proof. We start by analyzing the structure of an arbitrary permutation $\pi \in \mathrm{Av}_n(1324, 231)$. Write $\pi = LnR$, where L (resp. R) is the subsequence of entries to the left (resp. right) of n. Every entry of L must be smaller than every entry of R, since π avoids 231. Furthermore, $L \in \mathrm{Av}(132, 231)$, and if L is nonempty then $R \in \mathrm{Av}(213, 231)$. If L is empty (i.e. n is the first entry of π), then R is an arbitrary $\{1324, 231\}$-avoider. We get the following recursive characterization: $\pi \in S_n$ avoids $\{1324, 231\}$ if and only if $n = 0$, or $\pi_1 = n$ and $\pi \setminus n \in \mathrm{Av}(1324, 231)$, or $\pi = \pi^{(1)} \oplus \pi^{(2)}$ where $\pi^{(1)} \in \mathrm{Av}(132, 231)$ and $\pi^{(2)} \in \mathrm{Av}(213, 231)$. If $\pi^{(2)}$ is indecomposable then it begins with its largest entry, which is the entry n of π.

Hence, if $\pi \in \mathrm{Av}_n(1324, 231)$ is indecomposable and not equal to $\mathrm{id}_n^{\mathrm{rev}}$, then there is some positive integer $\ell < n$ such that $n, n-1, \ldots, n-\ell+1$ appear consecutively in decreasing order as the first ℓ elements of π, and the remaining $n-\ell$ elements form an arbitrary decomposable $\{1324, 231\}$-avoider. We will refer to the last component of this decomposable permutation as the last component of π. Observe that the component avoids $\{213, 231\}$.

With these preliminaries in mind, we can define a mapping from $\mathrm{Av}_n^k(1324, 231)$ to $\mathrm{Av}_{n+1}^k(1324, 231)$ according to Figure 3, which we shall prove is injective. For branches (1) and (2), the injection is straightforward. In branch (3), the mapping is defined as follows for a permutation $\pi \in \mathrm{Av}(1324, 231)$.

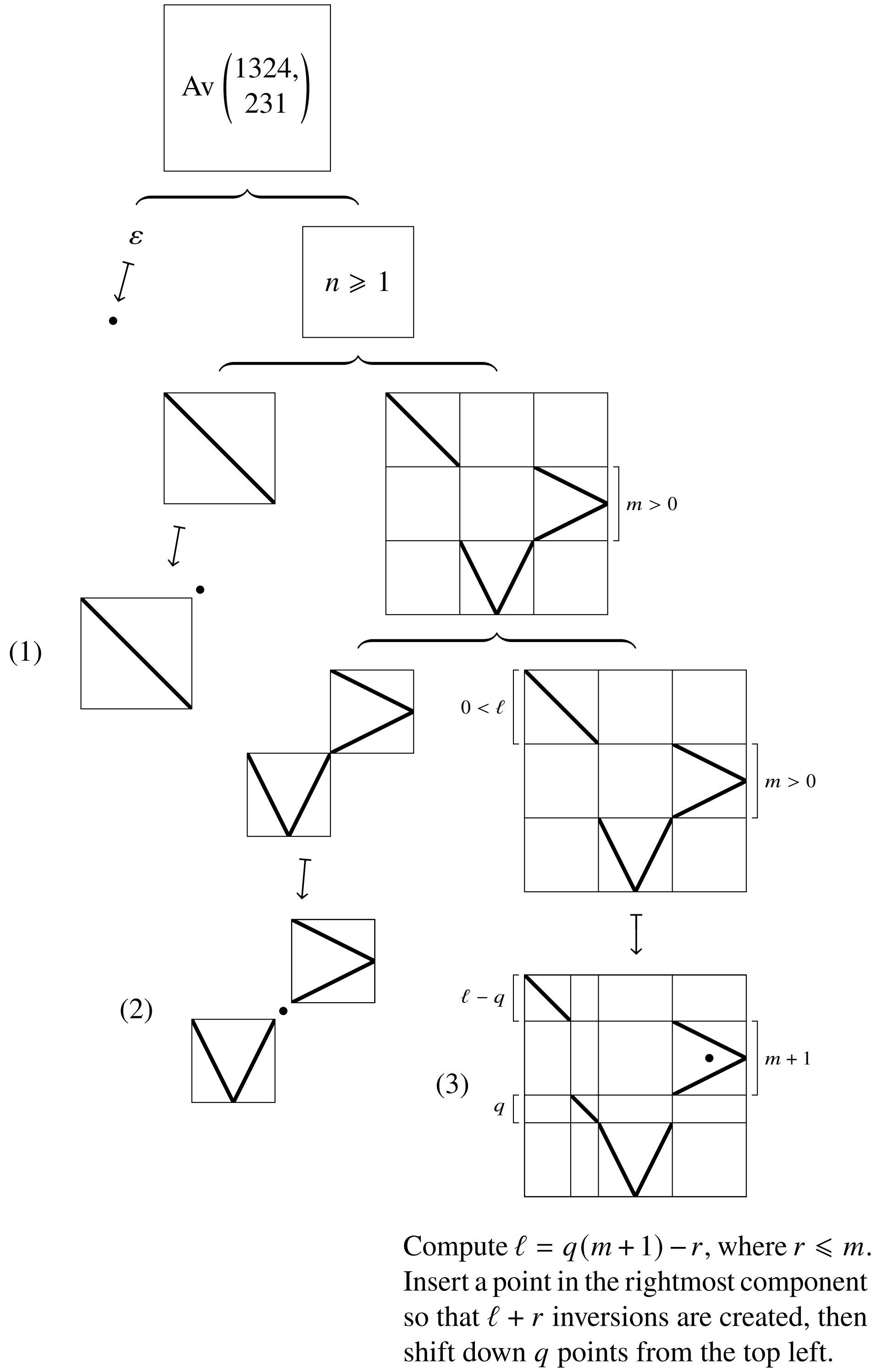

Figure 3. A schematic of the injection $\mathrm{Av}_n^k(1324, 231) \to \mathrm{Av}_{n+1}^k(1324, 231)$.

- There are unique integers $q > 0$ and $0 \leqslant r \leqslant m$ such that $\ell = q(m+1) - r$, where $\ell > 0$ is as before and m is the number of points in the last component of π.
- Insert a point into the last component of π so that its $\{213, 231\}$-avoidance is maintained and r inversions are created within the component, as in Lemma 3.1 (a). The new point also creates inversions with the ℓ first points of the permutation, so in total, $\ell + r$ new inversions are created.
- Shift the last q of the ℓ first points down by $m + 1$ steps each. Each such shift removes $m+1$ inversions, so a total of $q(m+1)$ inversions are removed. Since $\ell + r = q(m+1)$, the resulting permutation has equally many inversions as π.

Here, *shifting* an entry e of a permutation π down (resp. up) by one step means mapping $\pi \mapsto (e\ e-1)\pi$ (resp. $\pi \mapsto (e\ e+1)\pi$), where $(i\ j)$ is the transposition swapping i and j. Shifting e down by m steps is the composition

$$\pi \longmapsto (e-m+1\ e-m)\cdots(e-1\ e-2)(e\ e-1)\pi.$$

When shifting down several entries, start from the smallest and proceed in increasing order. When shifting up, start from the largest.

Let f denote our mapping $\mathrm{Av}_n^k(1324, 231) \longrightarrow \mathrm{Av}_{n+1}^k(1324, 231)$. The restriction of f to branches (1) and (2), i.e. to the set

$$\left\{\pi \in \mathrm{Av}_n^k(1324, 231) : \mathrm{comp}(\pi) \geqslant 2\right\} \cup \left\{\mathrm{id}_n^{\mathrm{rev}}\right\}$$

is clearly injective, and its image is

$$\left\{\pi \in \mathrm{Av}_{n+1}^k(1324, 231) : \mathrm{comp}(\pi) \geqslant 3\right\} \cup \left\{\mathrm{id}_n^{\mathrm{rev}} \oplus 1\right\}.$$

It suffices to show that f restricted to branch (3) is injective, and that its image is disjoint from the above. Denote

$$\mathcal{I}_n^k = \left\{\pi \in \mathrm{Av}_n^k(1324, 231) : \mathrm{comp}(\pi) = 1\right\} \setminus \left\{\mathrm{id}_n^{\mathrm{rev}}\right\},$$

and let $\sigma \in f(\mathcal{I}_n^k)$. We want to construct the inverse of $f|_{\mathcal{I}_n^k}$. To do this, we need to recover the numbers m, q, and r used to construct σ. Firstly:

- $\ell - q$ is the smallest nonnegative integer such that $\sigma_{\ell-q+1}\sigma_{\ell-q+2}\ldots\sigma_{n+1}$ is decomposable. For now, denote $\ell' = \ell - q$.
- $m + 1 = n + 1 - \sigma_{\ell'+1} - \ell'$. This is the number of entries larger than $\sigma_{\ell'+1}$, excluding $\sigma_1, \ldots, \sigma_{\ell'}$.

It remains to determine the value of q, the number of points that were shifted down as the last step in the construction of σ. Observe that there exists some largest positive integer q' such that the entries $\sigma_{\ell'+1}, \sigma_{\ell'+2}, \ldots, \sigma_{\ell'+q'}$ are decreasing and have consecutive values, since we know that $q > 0$. The first of these entries, $\sigma_{\ell'+1}$, is equal to $\sigma_{\ell'} - m - 1$ (if $\ell' = 0$, set $\sigma_{\ell'} = n + 2$). Call these q' entries *eligible*; we must have $q \leqslant q'$.

Since σ was constructed using f, there must exist a positive integer q such that after shifting the first q eligible points up by $m + 1$ steps, the resulting permutation σ' satisfies

$$k + \ell' + q \leqslant \mathrm{inv}(\sigma') \leqslant k + \ell' + q + m.$$

On the other hand, since shifting the first eligible point up by $m + 1$ steps creates $m + 1$ new inversions each time, q is unique. This is what we needed to show. Finally, r is given by

$$\mathrm{inv}(\sigma') = k + \ell' + q + r.$$

By Lemma 3.1 (b) there exists a unique permutation $\pi \in \mathcal{I}_n^k$ such that $\sigma' \setminus e = \pi$ for some e among the last $m + 1$ entries of σ'. Clearly π is the unique preimage of σ under $f|_{\mathcal{I}_n^k}$.

It remains to show that f as a whole is injective.

- Clearly $\mathrm{id}_n^{\mathrm{rev}} \oplus 1 \notin f(\mathcal{I}_n^k)$.
- Each permutation $\sigma \in f(\mathcal{I}_n^k)$ has at most two components. This is clear whenever the inserted point in branch (3) adds at least one inversion to the last component. If it adds zero inversions to the last component, that is, $r = 0$, then $\ell = q(m + 1) \geqslant 2q$, so $\sigma_1 = n + 1$ and $\mathrm{comp}(\sigma) = 1$. □

Example 3.3. Let π be as in Figure 4 (left). We have $n = 12$ and $k = 52$. The last component of π is 21, so $m = 2$. The number of consecutive decreasing entries at the start of π is $\ell = 5$. Hence, we get $\ell = q(m + 1) - r$ with $q = 2$ and $r = 1$.

As per Figure 3, we insert a point into the last component of π so that it creates $r = 1$ inversion; the only way to do this results in the permutation 312. This new point also creates $\ell = 5$ inversions with the first five entries of π, so in total, 6 new inversions are created. Finally, we shift down the last $q = 2$ of the first $\ell = 5$ entries of π by $m + 1 = 3$ steps each. The resulting permutation is $f(\pi)$, shown in Figure 4 (right). It has exactly $k = 52$ inversions. The three circled points in the figure are the two points that were shifted down, and the new point that was inserted.

4. Building inversion-monotone sets

We say that a collection B is *trivially* inversion monotone if

(a) $p_1 \neq 1$ for all $p \in B$, or

(b) $p_m \neq m$ for all $p \in B$, where $m = |p|$.

In this section, we will demonstrate how to build a nontrivially inversion monotone set B' from an inversion monotone set B, such that

$$\min\{|p| : p \in B'\} = \min\{|p| : p \in B\} + 1.$$

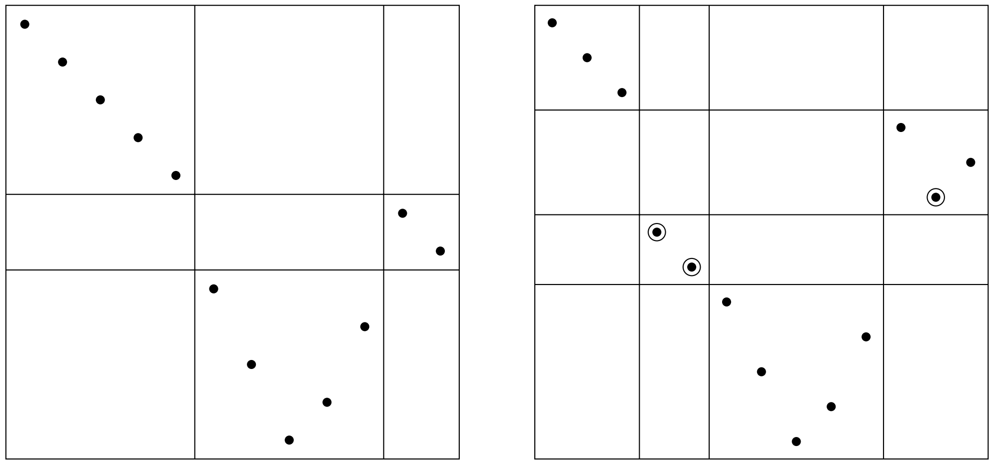

Figure 4. A $\{1324, 231\}$-avoiding permutation π (left) and its image $f(\pi)$ (right).

For a collection B of patterns, we define B^ℓ, B^r, B^u and B^d as follows:

$$\begin{aligned} B^\ell &= \{p : p \setminus p_1 \in B\}, \\ B^r &= \left\{p : p \setminus p_{|p|} \in B\right\}, \\ B^u &= \{p : p \setminus |p| \in B\}, \\ B^d &= \{p : p \setminus 1 \in B\}. \end{aligned}$$

In other words, B^ℓ consists of the patterns obtained by inserting a new first entry to all patterns in B in all possible ways. The other three collections are similar.

Proposition 4.1. *If B is inversion monotone, then so are B^ℓ, B^r, B^u and B^d.*

Proof. It suffices to prove that B^ℓ is inversion monotone, the other cases are symmetrical. Since B is inversion monotone, there is an injection $f : \mathrm{Av}(B) \to \mathrm{Av}(B)$ such that $|f(\pi)| = |\pi| + 1$ and $\mathrm{inv}(f(\pi)) = \mathrm{inv}(\pi)$ for all $\pi \in \mathrm{Av}(B)$. We will first show that if $\pi \in \mathrm{Av}(B^\ell)$, then $\pi \setminus \pi_1 \in \mathrm{Av}(B)$. Indeed, if $\pi \setminus \pi_1$ contains a pattern $p \in B$, then obviously this occurrence of p in π together with π_1 forms a pattern q such that $q \setminus q_1 = p$. So $q \in B^\ell$ by construction, meaning that π does not avoid B^ℓ.

Hence, we can define a map $g : \mathrm{Av}(B^\ell) \to \mathrm{Av}(B^\ell)$ by setting $g(\pi)_1 = \pi_1$ and $g(\pi) \setminus \pi_1 = f(\pi \setminus \pi_1)$. Observe that since $f(\pi \setminus \pi_1) \in \mathrm{Av}(B)$, we must have $g(\pi) \in \mathrm{Av}(B^\ell)$ by the same argument as above. Clearly g is injective, $|g(\pi)| = |\pi| + 1$ and $\mathrm{inv}(g(\pi)) = \mathrm{inv}(\pi)$, so B^ℓ is inversion monotone. □

The result is somewhat surprising, since B^ℓ is much larger than B. If we let $B_0 = B \subseteq S_m$ (for example with $B = \{213\} \subseteq S_3$) and iteratively $B_{i+1} = B_i^\ell$, then

$$|B_i| = (m+i) \cdot |B_{i-1}| = \frac{(m+i)!}{m!} |B|.$$

Example 4.2. Since 213 is inversion monotone, so are

$$\begin{aligned}\{213\}^{\ell} &= \{1324, 2314, 3214, 4213\},\\ \{213\}^{r} &= \{3241, 3142, 2143, 2134\},\\ \{213\}^{u} &= \{4213, 2413, 2143, 2134\},\\ \{213\}^{d} &= \{1324, 3124, 3214, 3241\}.\end{aligned}$$

Note in particular the collections B^{ℓ} and B^{d}, which contain 1324. We record this as a proposition.

Proposition 4.3. *The collection* $\{1324, 2314, 3214, 4213\}$ *is inversion monotone.*

Inspired by the approach in Proposition 4.1 and the discussion right after, one is tempted to try to fix a pattern in the basis and increase the length of the rest. In other words, we would fix some pattern p and collection B_0, and iteratively set B_{i+1} to be either B_i^{ℓ}, B_i^{r}, B_i^{u} or B_i^{d}. Clearly, if $\{p\} \cup B_i$ is inversion monotone for all i, then p is inversion monotone: we have

$$\mathrm{av}_n^k(p) = \mathrm{av}_n^k(\{p\} \cup B_{n+2}) \leqslant \mathrm{av}_{n+1}^k(\{p\} \cup B_{n+2}) = \mathrm{av}_{n+1}^k(p)$$

for all n and k. This is a potential avenue towards the inversion monotonicity conjecture for 1324. We have computational support for the following claim.

Conjecture 4.4. *Let* $B_0 = \{213\}$ *and* $B_{i+1} = B_i^{\ell}$ *for all* $i \geqslant 0$*. The collection* $\{1324\} \cup B_i$ *is inversion monotone for all* i*. In particular,* 1324 *is inversion monotone.*

5. Limit sequences of sets containing 1324

In this section we turn our attention to the pairs $\{1324, p\}$, where p is a pattern of length four. We are not able to show that any such pair is inversion monotone, but we determine all of their limit sequences, and in some cases prove that inversion monotonicity holds under the additional assumption $n \geqslant \frac{k+7}{2}$ (as in [LV25]).

Table 1 shows all pairs of length-four patterns and indicates which of them are not inversion monotone (F), which are trivially inversion monotone (T), which are inversion monotone when $n \geqslant \frac{k+7}{2}$ (T$\frac{1}{2}$), and which are unknown (T?). All cases are checked up to $n = 15$. A pair $\{p, q\}$ is trivially inversion monotone if $p_1 \neq 1$ and $q_1 \neq 1$, or neither p nor q ends with their largest entry.

This section is organized as follows. In Section 5.1 we make some preliminary observations regarding Table 1, characterize the collections that have limit sequences, and identify a class of pattern pairs whose limit sequences are $1, 0, 0, \ldots$. In Section 5.2 we embark on our quest to determine the limit sequences of all pairs $\{1324, p\}$ with $p \in S_4$. Pairs are categorized into three subsections: non-inversion-monotone pairs (Section 5.3), pairs that are inversion monotone for $n \geqslant \frac{k+7}{2}$ (Section 5.4), and the rest (Section 5.5). Proofs of inversion monotonicity

Table 1. Which pairs of length-four patterns are inversion monotone?

	1243	1324	1342	1423	1432	2134	2143	2314	2341	2413	2431	3124	3142	3214	3241	3412	3421	4123	4132	4213	4231	4312	4321
1234	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**
1243		**F**	T	T	T	**F**	T	**F**	T	T	T	**F**	T	**F**	T	T	T	T	T	T	T	T	T
1324			T$\frac{1}{2}$	T$\frac{1}{2}$	T$\frac{1}{2}$	**F**	**F**	T$\frac{1}{2}$	T?	T?	T?	T$\frac{1}{2}$	T?	T$\frac{1}{2}$	T?	T?	T?	T?	T?	T?	T$\frac{1}{2}$	T?	T$\frac{1}{2}$
1342				T	T	**F**	T	T?	T	T	T	T?	T	T?	T	T	T	T	T	T	T	T	T
1423					T	**F**	T	T?	T	T	T	T?	T	T?	T	T	T	T	T	T	T	T	T
1432						**F**	T	T?	T	T	T	T?	T	T?	T	T	T	T	T	T	T	T	T
2134							T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
2143								T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
2314									T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
2341										T	T	T	T	T	T	T	T	T	T	T	T	T	T
2413											T	T	T	T	T	T	T	T	T	T	T	T	T
2431												T	T	T	T	T	T	T	T	T	T	T	T
3124													T	T	T	T	T	T	T	T	T	T	T
3142														T	T	T	T	T	T	T	T	T	T
3214															T	T	T	T	T	T	T	T	T
3241																T	T	T	T	T	T	T	T
3412																	T	T	T	T	T	T	T
3421																		T	T	T	T	T	T
4123																			T	T	T	T	T
4132																				T	T	T	T
4213																					T	T	T
4231																						T	T
4312																							T

holding when $n \geqslant \frac{k+7}{2}$ are in Section 6. By the reasoning in Section 2, determining the limit sequences usually reduces to enumerating indecomposable permutations avoiding $\{132, p\}$ for some pattern p. These results are found in Section 7. The relevant data is in Appendix A.

5.1 On limit sequences

This section contains a few observations about limit sequences. First, note that the first row of Table 1 is all ‘F’s due to the inclusion of the identity pattern 1234: since the limit sequence of 1234 is 0, all pairs $\{1234, p\}$ also have limit sequence 0, so they cannot be inversion monotone. It is curious that all other known ‘F’s (except one) come from pairs containing 1243 or the symmetric 2134. Proposition 5.2 below hints at a reason for this.

Another interesting feature of the table is that apart from the pairs with 1324, the ‘T?’ pairs are exactly

$$\{1342, 1423, 1432\} \times \{2314, 3124, 3214\}. \tag{3}$$

None of these pairs have limit sequences, so we do not study them in this paper. The following result shows why these pairs fail to have limit sequences, and explains why all the pairs $\{1324, p\}$ do. This is related to [CJS12, Proposition 21].

Proposition 5.1. *A collection B of patterns has a limit sequence if and only if B contains a pattern p such that* $\mathrm{inv}(p) \leqslant 1$.

Proof. Suppose first that all patterns in B have at least two inversions. Then clearly

$$\mathrm{Av}_n^1(B) = \{\pi \in S_n : \mathrm{inv}(\pi) = 1\},$$

so $\mathrm{av}_n^1(B) = n - 1$ and B does not have a limit sequence.

Conversely, suppose that B contains a pattern p with at most one inversion, and write $p = \mathrm{id}_a \oplus 21 \oplus \mathrm{id}_b$, where $a, b \geqslant 0$. We will show that $\mathrm{av}_n^k(B) = \mathrm{av}_{n+1}^k(B)$ for all

$$n \geqslant k + a + b + \max\{|q| : q \in B\}. \tag{4}$$

Indeed, any permutation $\pi \in S_n$ with $\mathrm{inv}(\pi) = k$ satisfying (4) must have

$$\mathrm{comp}(\pi) \geqslant a + b + \max\{|q| : q \in B\}.$$

by (1). Write $\pi = \pi^{(1)} \oplus \pi^{(2)} \oplus \ldots \oplus \pi^{(r)}$, where each $\pi^{(i)}$ is indecomposable. If π avoids B (in particular, π avoids p), then we must have $\pi^{(i)} = 1$ for all i such that $a + 1 \leqslant i \leqslant r - b$. In other words,

$$\pi = \pi^{(1)} \oplus \ldots \oplus \pi^{(a)} \oplus \mathrm{id}_m \oplus \pi^{(r-b+1)} \oplus \ldots \oplus \pi^{(r)},$$

where

$$m \geqslant r - a - b \geqslant \max\{|q| : q \in B\}.$$

Hence, we can define a bijection $\mathrm{Av}_n^k(B) \to \mathrm{Av}_{n+1}^k(B)$ by

$$\pi \longmapsto \pi^{(1)} \oplus \ldots \oplus \pi^{(a)} \oplus \mathrm{id}_{m+1} \oplus \pi^{(r-b+1)} \oplus \ldots \oplus \pi^{(r)} =: \sigma.$$

Observe that σ avoids B, since an occurrence of a pattern $q \in B$ in σ would have to use all of the entries of id_{m+1}, contradicting the fact that $m + 1 > |q|$. This concludes the proof. □

Proposition 5.2. *We have that*

$$\mathrm{av}_n^k\left(\mathrm{id}_a \oplus 21, 21 \oplus \mathrm{id}_b\right) = 0$$

for all $k \geqslant 1$ and $n \geqslant k + a + b$.

Remark 5.3. An example of this is the failure of inversion monotonicity for the pair $\{1243, 2134\}$: we have $\mathrm{av}_n^k(1243, 2134) = 0$ for all $k \geqslant 1$ and $n \geqslant k + 4$.

Proof of Proposition 5.2. Suppose that $k \geqslant 1$ and $n \geqslant k+a+b$. Every permutation $\pi \in \mathrm{Av}_n^k\left(\mathrm{id}_a \oplus 21, 21 \oplus \mathrm{id}_b\right)$ satisfies $\mathrm{comp}(\pi) \geqslant n-k \geqslant a+b$ by (1). Decompose

$$\pi = \pi^{(1)} \oplus \pi^{(2)} \oplus \ldots \oplus \pi^{(c)},$$

and suppose that a component $\pi^{(i)}$ contains an inversion. We must have $i \leqslant a$, or

$$\pi^{(1)} \oplus \pi^{(2)} \oplus \ldots \oplus \pi^{(a)} \oplus \pi^{(i)}$$

contains $\mathrm{id}_a \oplus 21$. Similarly, we must have $i \geqslant c - b + 1$, or

$$\pi^{(i)} \oplus \pi^{(c-b+1)} \oplus \pi^{(c-b+2)} \oplus \ldots \oplus \pi^{(c)}$$

contains $21 \oplus \mathrm{id}_b$. But then $c \leqslant i + b - 1 \leqslant a + b - 1$, a contradiction. □

Table 2. Inversion-preserving symmetries for pairs containing 1324.

Pair	Symmetry	Image
1324, 1423	$\pi \mapsto \pi^{-1}$	1324, 1342
1324, 2134	$\pi \mapsto \pi^{\mathrm{rc}}$	1324, 1243
1324, 2314	$\pi \mapsto \pi^{\mathrm{rc}}$	1324, 1423
1324, 3124	$\pi \mapsto \pi^{\mathrm{rc}}$	1324, 1342
1324, 3142	$\pi \mapsto \pi^{-1}$	1324, 2413
1324, 3214	$\pi \mapsto \pi^{\mathrm{rc}}$	1324, 1432
1324, 3241	$\pi \mapsto (\pi^{-1})^{\mathrm{rc}}$	1324, 2431
1324, 4123	$\pi \mapsto \pi^{-1}$	1324, 2341
1324, 4132	$\pi \mapsto \pi^{-1}$	1324, 2431
1324, 4213	$\pi \mapsto \pi^{\mathrm{rc}}$	1324, 2431
1324, 4312	$\pi \mapsto \pi^{-1}$	1324, 3421

5.2 Representatives and overview

Many entries in Table 1 are redundant due to inversion-preserving symmetries. We will begin by choosing a set of representative pairs $\{1324, p\}$ with $p \in S_4$.

Proposition 5.4. *The pairs of length-four patterns containing* 1324 *are represented up to inv-Wilf-equivalence by the following twelve patterns:*

$$1234, 1243, 1342, 1432, 2143, 2341, 2413, 2431, 3412, 3421, 4231, 4321. \quad (5)$$

All of the pairs have distinct limit sequences.

Proof. The permutations not listed in (5) are

$$1423, 2134, 2314, 3124, 3142, 3214, 3241, 4123, 4132, 4213, 4312.$$

Table 2 shows the inversion-preserving symmetries mapping these patterns to some listed in (5). That the pairs listed in the claim have distinct limit sequences can be checked computationally, see Tables 5–25 in Appendix A. □

We omit the pair $\{1324, 1234\}$ from our analysis, since its limit sequence is 0. Our findings for the remaining eleven pairs are summarized in Table 3. We give combinatorial interpretations for all of their limit sequences, in most cases obtain limit generating functions, and in many cases prove that they are inversion monotone when $n \geqslant \frac{k+7}{2}$ (marked 'half' in the table). The limit sequences exhibit a wide variety of combinatorial structures, including restricted partitions, the sand pile model, and penny arrangements (Section 7). For more information on these permutation classes, PermPAL is a useful resource [Alb+25].

A common theme is the appearance of a *secondary* limit sequence in the row differences $\mathrm{av}_{n+1}^{k}(1324, p) - \mathrm{av}_{n}^{k}(1324, p)$. Specifically, when n is large enough and k is fixed, the values

$$\mathrm{av}_{n+1+i}^{k+i}(1324, p) - \mathrm{av}_{n+i}^{k+i}(1324, p)$$

are constant in i. Linusson and Verkama proved that $\mathrm{Av}(1324)$ has a secondary limit sequence, and used this fact to enumerate $\mathrm{Av}_{n}^{k}(1324)$ when $n \geqslant \frac{k+7}{2}$ [LV25]. We observe the same phenomenon for all pairs $\{1324, p\}$ with p in

$$1243,\ 1342,\ 1432,\ 2341,\ 3421,\ 4321,$$

but prove it only for $p = 1342$ (Section 6). The nonzero terms of the secondary limit sequence are sometimes given by the limit generating function multiplied by a low-degree polynomial; for 1324 and $\{1324, 1342\}$ the polynomials are $2 + 4x$ and $2 + 2x$, respectively.

If $\{1324, p\}$ does not have a secondary limit sequence, we can, for some reason, find a *tertiary* limit sequence in the *second* differences. Specifically, let

$$b(n, k) = (a(n + 2, k + 1) - a(n + 1, k + 1)) + (a(n + 1, k) - a(n, k)),$$

where $a(n, k) = \mathrm{av}_{n}^{k}(1324, p)$. When n is large enough and p is any of the patterns with no secondary limit sequence, i.e.

$$2143,\ 2413,\ 2431,\ 3412,\ 4231,$$

the sequence $b(n + i, k + i)$ is constant in i. This sequence is often related to the limit sequence, and we have no explanation for why it appears.

5.3 Non-inversion-monotone pairs

1324, 1243

The class $\mathrm{Av}(1324, 1243)$ is enumerated by the large Schröder numbers, entry A006318 in the OEIS [OEI26]. Table 5 shows the values of $\mathrm{av}_{n}^{k}(1324, 1243)$. The limit sequence $\mathrm{av}_{n}^{k}(1324, 1243)$, $n \geqslant k + 3$, is

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, \ldots,$$

the partition numbers.

Proposition 5.5. *We have* $\mathrm{av}_{n}^{k}(1324, 1243) = p(k)$ *for all* $n \geqslant k + 3$.

Proof. A permutation π with $\mathrm{comp}(\pi) \geqslant 3$ avoids $\{1324, 1243\}$ if and only if it avoids 132. Therefore

$$\mathrm{av}_{n}^{k}(1324, 1243) = \mathrm{av}_{n}^{k}(132) = p(k)$$

for all $n \geqslant k + 3$. □

Table 3. Limit sequences of pairs containing 1324.

Pair	Inv-monotone	Limit sequence	Reference
1243	No	Partition numbers	Prop. 5.5
2143	No	$2 \cdot$ partition numbers	Prop. 5.6
1342	Half	Overpartitions	Prop. 5.7
1432	Half	Partitions with a divider	Prop. 5.8
4231	Half	(g.f. of convex partitions)^2	Prop. 5.9
4321	Half	(g.f. of partitions with 2 distinct parts)^2	Prop. 5.10
2341	Conjectured	(g.f. of sand pile model)^2	Prop. 5.11
2413	Conjectured	(g.f. of partitions)^2	Prop. 5.12
2431	Conjectured	$P(x) \cdot$ (g.f. of steep partitions)	Prop. 5.13
3412	Conjectured	(g.f. of convex penny arrangements$^{\mathrm{c}}$)^2	Prop. 5.14
3421	Conjectured	(g.f. of A115029)^2	Prop. 5.15

Table 6 shows the row differences $\mathrm{av}_{n+1}^k(1324, 1243) - \mathrm{av}_n^k(1324, 1243)$. We have no explanation for the (negative) secondary limit sequence

$$-4, -2, -2, -6, -6, -8, \ldots .$$

1324, 2143

Av(1324, 2143) is the class of *smooth* permutations – a permutation is smooth if the Schubert cell indexed by it is smooth [LS90]. See the OEIS entry A032351 for more information. Table 7 shows that the limit sequence is

$$1, 2, 4, 6, 10, 14, 22, 30, 44, 60, \ldots,$$

which, for $k \geqslant 1$, is given by $2p(k)$.

Proposition 5.6. *For every $k \geqslant 1$ and $n \geqslant k+2$ we have $\mathrm{av}_n^k(1324, 2143) = 2p(k)$.*

Proof. A decomposable permutation π avoids $\{1324, 2143\}$ if and only if it is of the form $\pi = \sigma \oplus \mathrm{id} \oplus \tau$, where either $\sigma \in \mathrm{Av}(132)$ and $\tau = 1$, or $\sigma = 1$ and $\tau \in \mathrm{Av}(213)$. □

The row differences in Table 8 are fascinating. This is the only case we know of with nontrivial zeros. We have no explanation for the tertiary limit sequence

$$-2, 0, 0, 0, 0, \ldots .$$

5.4 Half-monotone pairs

1324, 1342

Av(1324, 1342) is enumerated by the large Schröder numbers. From Table 9, the limit sequence is

$$1, 2, 4, 8, 14, 24, 40, 64, 100, 154, \ldots,$$

the number of *overpartitions* of k; A015128 in the OEIS. An overpartition of k is a partition of k in which the first occurrence of each part may either be overlined or not.

Proposition 5.7. *For each $n \geqslant k + 2$, $\mathrm{av}_n^k(1324, 1342)$ equals the number of overpartitions of k. In particular,*

$$C_{1324,1342}(x) = \prod_{k \geqslant 1} \frac{1 + x^k}{1 - x^k}.$$

Proof. A decomposable permutation π avoids $\{1324, 1342\}$ if and only if it is of the form $\sigma \oplus \mathrm{id} \oplus \tau$, where $\sigma \in \mathrm{Av}(132)$ and $\tau \in \mathrm{Av}(213, 231)$. The indecomposable $\{213, 231\}$ avoiders with ℓ inversions are in bijection with partitions of ℓ into distinct parts [Fra25], so $\mathrm{Av}_n^k(1324, 1342)$ is in bijection with pairs (λ, μ) of partitions such that $|\lambda| + |\mu| = k$ and all parts of μ are distinct (when $n \geqslant k + 2$). This collection is, in turn, in bijection with the overpartitions of k as follows: overline all parts of μ, then merge λ and μ so that the parts are weakly decreasing, and every overlined part comes before the equal non-overlined parts. The generating function is easy to see. □

Table 10 shows the row differences $\mathrm{av}_{n+1}^k(1324, 1342) - \mathrm{av}_n^k(1324, 1342)$. There is a secondary limit sequence

$$2, 6, 12, 24, 44, 76, 128, \ldots,$$

whose generating function is $(2 + 2x)C_{1324,1342}(x)$. We prove this, as well as inversion monotonicity for $n \geqslant \frac{k+7}{2}$, in Section 6.

1324, 1432

The class Av(1324, 1432) has been enumerated by PermPAL, but there is no known closed form expression for its generating function; see also the OEIS entry A165542. Table 11 shows that the limit sequence is

$$1, 2, 5, 9, 17, 27, 46, 69, 108, 158, \ldots,$$

which is the number of partitions of k into blue and red parts, such that all blue parts are greater than or equal to all red parts; this is sequence A093694 in the OEIS.

Proposition 5.8. *For every $n \geqslant k+2$, $\mathrm{av}_n^k(1324, 1432)$ equals the number of partitions of k into blue and red parts, such that all blue parts are greater than or equal to all red parts. In particular,*

$$C_{1324,1432}(x) = \sum_{k \geqslant 0} (k+1)x^k \cdot \prod_{i=1}^{k} \frac{1}{1-x^i}.$$

Proof. A decomposable permutation π avoids $\{1324, 1432\}$ if and only if it is of the form $\pi = \sigma \oplus \mathrm{id} \oplus \tau$, where $\sigma \in \mathrm{Av}(132)$ and $\tau \in \mathrm{Av}(213, 321)$. The indecomposable $\{213, 321\}$-avoiders with ℓ inversions are in bijection with partitions of ℓ into equal parts [Fra25], so $\mathrm{Av}_n^k(1324, 1432)$ is in bijection with pairs (λ, μ) of partitions such that $|\lambda| + |\mu| = k$ and all parts of μ are equal (when $n \geqslant k+2$). These pairs are, in turn, in bijection with the desired partitions of k as follows: let i denote the size of the parts of μ. Color all parts of λ greater than or equal to i blue, and color the rest of the parts of λ along with all parts of μ red. Then merge λ and μ into one partition, so that all blue parts precede all red parts. See the OEIS entry for the generating function. □

Table 12 shows the row differences $\mathrm{av}_{n+1}^k(1324, 1432) - \mathrm{av}_n^k(1324, 1432)$. We have no explanation for the secondary limit sequence

$$4, 6, 12, 22, 38, 62, \ldots.$$

In Section 6, we prove that $\{1324, 1432\}$ is inversion monotone when $n \geqslant \frac{k+7}{2}$.

1324, 4231

The class $\mathrm{Av}(1324, 4231)$ was first enumerated by Albert, Atkinson and Vatter [AAV09]. Table 13 shows the values $\mathrm{av}_n^k(1324, 4231)$, and the limit sequence

$$1, 2, 5, 10, 20, 34, 59, 96, 151, 230, \ldots$$

does not appear in the OEIS. Since a decomposable permutation π avoids $\{1324, 4231\}$ if and only if it is of the form $\pi = \sigma \oplus \mathrm{id} \oplus \tau$, where $\sigma \in \mathrm{Av}(132, 4231)$ and $\tau \in \mathrm{Av}(213, 4231)$, we have that

$$C_{1324,4231}(x) = C_{132,4231}(x) \cdot C_{213,4231}(x) = C_{132,4231}(x)^2,$$

and the limit sequence of $\{132, 4231\}$ is

$$1, 1, 2, 3, 5, 6, 9, 12, 15, 19, 25, \ldots.$$

This sequence is not in the OEIS either, but we enumerate it in Section 7.5.

Proposition 5.9. *We have*

$$C_{1324,4231}(x) = \left(\prod_{i \geqslant 1} (1+x^i) + \sum_{a,b \geqslant 0} x^{(a+2)(b+1)} \cdot \prod_{i=1}^{a} (1+x^i) \cdot \prod_{i=1}^{b} (1+x^i) \right)^2.$$

Table 14 shows the row differences. There is no secondary limit sequence, but there appears to be a tertiary limit sequence starting with

$$0, 0, 2, 4, 6, 12, 18, 28, \ldots .$$

We have no explanation for this sequence. In Section 6, we prove that $\{1324, 4231\}$ is inversion monotone when $n \geqslant \frac{k+7}{2}$.

1324, 4321

The class $\mathrm{Av}(1324, 4321)$ was enumerated by Vatter [Vat12]. Table 15 shows the values $\mathrm{av}_n^k(1324, 4321)$, and the limit sequence is

$$1, 2, 5, 10, 20, 36, 63, 104, 167, 256, \ldots ,$$

which does not appear in the OEIS. Again, we have that

$$C_{1324,4321}(x) = C_{132,4321}(x) \cdot C_{213,4321}(x) = C_{132,4321}(x)^2,$$

and the limit sequence of $\{132, 4321\}$ is

$$1, 1, 2, 3, 5, 7, 10, 13, 17, 20, \ldots .$$

This is entry A265250 in the OEIS: the number of partitions of k having parts of at most two sizes. We prove that this is the case in Section 7.6.

Proposition 5.10. *We have*

$$C_{1324,4321}(x) = \left(1 + \sum_{k\geqslant 1} \frac{x^k}{1-x^k} + \sum_{k\geqslant 1} \sum_{i\geqslant k+1} \frac{x^{k+i}}{(1-x^k)(1-x^i)}\right)^2 .$$

Table 16 shows that there is a secondary limit sequence

$$4, 10, 24, 46, 88, 144, \ldots ,$$

which we do not understand. In Section 6, we prove that $\{1324, 4321\}$ is inversion monotone when $n \geqslant \frac{k+7}{2}$.

5.5 The remaining pairs

1324, 2341

The class $\mathrm{Av}(1324, 2341)$ was first enumerated by Miner [Min16]. The limit sequence in Table 17 starts with

$$1, 2, 5, 8, 16, 26, 42, 66, 104, 156, \ldots ,$$

which is not in the OEIS. Again, we have that

$$C_{1324,2341}(x) = C_{132,2341}(x) \cdot C_{213,2341}(x) = C_{132,2341}(x)^2,$$

and the limit sequence $c_k(132, 2341)$ starts with

$$1, 1, 2, 2, 4, 5, 6, 9, 13, 15, \dots.$$

Excluding the first term, this shows up as sequence A056219 in the OEIS: the number of partitions in the *sand pile model* $\mathrm{SPM}(k)$ – a discrete dynamical system originating in physics. See Section 7.1 for more details. The set $\mathrm{SPM}(k)$ is defined recursively as follows: $(k) \in \mathrm{SPM}(k)$, and if $\lambda \in \mathrm{SPM}(k)$ then every partition that can be obtained from λ by subtracting one from a part and adding one to the next part is also in $\mathrm{SPM}(k)$. For example,

$$\mathrm{SPM}(5) = \{(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1)\}.$$

The next result follows from Proposition 7.2, and [CG02] for the generating function.

Proposition 5.11. *Indecomposable* $\{132, 2341\}$*-avoiders with* k *inversions are in bijection with* $\mathrm{SPM}(k)$*, and thus*

$$C_{1324,2341}(x) = \left(1 + \sum_{k\geqslant 1} x^{\frac{k(k+1)}{2}} \cdot \prod_{i=1}^{k} \left(x + \frac{1}{1 - x^i}\right)\right)^2.$$

The differences $\mathrm{av}^k_{n+1}(1324, 2341) - \mathrm{av}^k_n(1324, 2341)$ in Table 18 exhibit a secondary limit sequence

$$3, 7, 17, 31, 60, 104, 170, \dots,$$

which we have no explanation for.

1324, 2413

The class $\mathrm{Av}(1324, 2413)$ is equinumerous to the smooth permutations (OEIS A032351) as shown by Bóna [Bón98]. According to Table 19 the limit sequence is

$$1, 2, 5, 10, 20, 36, 65, 110, 185, 300, \dots,$$

which is given by $P(x)^2$, where $P(x)$ is the generating function for the partition numbers. This is the same as the limit sequence of the 1324-avoiders.

Proposition 5.12. *For all* k *and* $n \geqslant k + 2$*,*

$$\mathrm{av}^k_n(1324, 2413) = \mathrm{av}^k_n(1324) = \sum_{i=0}^{k} p(i)p(k - i).$$

In particular, $C_{1324,2413}(x) = P(x)^2$.

Proof. Observe that 2413 contains both 132 and 213 as patterns. Therefore a decomposable permutation avoids $\{1324, 2413\}$ if and only if it avoids 1324. $\square$

Table 20 shows that the differences $\mathrm{av}^k_{n+1}(1324, 2413) - \mathrm{av}^k_n(1324, 2413)$ do not exhibit a secondary limit sequence. However, there is a tertiary limit sequence

$$3, 6, 15, 30, 60, 108, 195, 330, 555, \dots,$$

which agrees with the limit sequence multiplied by three. We have not been able to prove this.

1324, 2431

The class $\mathrm{Av}(1324, 2431)$ was enumerated by Albert, Atkinson and Vatter [AAV14]. Table 21 shows that the limit sequence is

$$1, 2, 5, 10, 19, 34, 59, 97, 158, 250, \dots,$$

which does not appear in the OEIS. Since 2431 contains 132 as a pattern, we have

$$C_{1324,2431}(x) = C_{132}(x) \cdot C_{213,2431}(x) = P(x) \cdot C_{213,2431}(x).$$

The limit sequence of $\{213, 2431\}$ is

$$1, 1, 2, 3, 4, 6, 8, 10, 14, 19, \dots,$$

which does not appear in the OEIS either. In Section 7.2 we are able to interpret this sequence as the number of partitions of k such that the difference between any two consecutive distinct parts is at least the multiplicity of the smaller part. We call these partitions *steep*. We have not been able to find an expression for the generating function of $\mathrm{steep}(k)$, the number of steep partitions of k.

Proposition 5.13. *We have*

$$C_{1324,2431}(x) = P(x) \cdot \sum_{k \geqslant 0} \mathrm{steep}(k) x^k,$$

where $\mathrm{steep}(k)$ *is the number of steep partitions of* k.

Proof. In Proposition 7.3, we show that the indecomposable $\{132, 3241\}$-avoiders with k inversions are in bijection with the steep partitions of k. The mapping $\pi \mapsto (\pi^{-1})^{\mathrm{rc}}$ sends $\{213, 2431\}$ to $\{132, 3241\}$, so the claim follows. $\square$

Table 22 shows that there is no secondary limit sequence, but there is a tertiary limit sequence

$$0, 1, 2, 5, 10, \dots.$$

Excluding the zero, this seems to be equal to the limit sequence itself.

1324, 3412

The class Av(1324, 3412) was enumerated by Albert, Atkinson and Brignall [AAB11]. Table 23 shows the values $\mathrm{av}_n^k(1324, 3412)$, and the limit sequence is

$$1, 2, 5, 10, 18, 34, 57, 96, 154, 246, \ldots .$$

Since 3412 contains neither 132 nor 213 and $3412^{\mathrm{rc}} = 3412$, we have

$$C_{1324,3412}(x) = C_{132,3412}(x) \cdot C_{213,3412}(x) = C_{132,3412}(x)^2,$$

and the limit sequence $c_k(132, 3412)$ is

$$1, 1, 2, 3, 4, 7, 9, 13, 17, 25, \ldots .$$

This sequence appears as entry A005576 in the OEIS, which arises from certain *penny arrangements* – see Section 7.3 for details.

Proposition 5.14. *We have*

$$C_{1324,3412}(x) = \left(\sum_{k \geqslant 0} \mathrm{A005576}(k) x^k \right)^2 .$$

By Table 23 there is no secondary limit sequence, but there is a tertiary limit sequence

$$0, 2, 4, 10, 20, 36, 68, 114, \ldots .$$

Excluding the zero, this seems to be exactly the limit sequence multiplied by two.

1324, 3421

The class Av(1324, 3421) was enumerated by Albert, Atkinson and Brignall [AAB12]. According to Table 25, its limit sequence is

$$1, 2, 5, 10, 20, 34, 61, 98, 159, 246, \ldots ,$$

which does not appear in the OEIS. As in some of the previous cases, we have that

$$C_{1324,3421}(x) = C_{132,3421}(x) \cdot C_{213,3421}(x) = C_{132,3421}(x)^2,$$

and the limit sequence $c_k(132, 3421)$ is

$$1, 1, 2, 3, 5, 6, 10, 12, 17, 22, \ldots .$$

This sequence is entry A115029 in the OEIS: the number of partitions of k such that all parts, except possibly the smallest, are distinct. We prove this in Section 7.4.

Proposition 5.15. *We have*

$$C_{1324,3421}(x) = \left(1 + \sum_{k\geqslant 1} \frac{x^k}{1-x^k} \cdot \prod_{i\geqslant k+1} (1+x^i)\right)^2.$$

See Table 26 for the differences $\mathrm{av}_{n+1}^k(1324, 3421) - \mathrm{av}_n^k(1324, 3421)$. There is a secondary limit sequence

$$4, 10, 24, 52, 103, 185, \ldots,$$

that we do not understand.

6. Almost decomposability

In this section, we provide a wide class of patterns p such that

$$\mathrm{av}_n^k(1324, p) \leqslant \mathrm{av}_{n+1}^k(1324, p) \quad \text{for all } k \text{ and } n \geqslant \frac{k+7}{2}.$$

In particular, this holds for $p \in \{1342, 1432, 4231, 4321\}$, which covers all the half-monotone pairs studied in Section 5.4. In the case of $p = 1342$, we also obtain an enumeration of $\mathrm{av}_n^k(1324, p)$ for $n \geqslant \frac{k+7}{2}$. Our methods are based on the notion of *almost decomposability* introduced in [LV25], as well as the accompanying injective mapping f defined on almost decomposable 1324-avoiders, with the following properties: $f(\pi)$ avoids 1324, $\mathrm{inv}(f(\pi)) = \mathrm{inv}(\pi)$ and $|f(\pi)| = |\pi| + 1$.

Let us recall the preliminaries. We say that $\pi \in S_n$ is *almost decomposable* if it is indecomposable, but there exists an entry $e \in \{1, \pi_1, n, \pi_n\}$ such that $\pi \setminus e$ is decomposable.

Theorem 6.1 (Theorem 8 in [LV25])**.** *If π avoids 1324 and $\mathrm{inv}(\pi) \leqslant 2|\pi| - 7$, then π is decomposable or almost decomposable.*

A decomposable 1324-avoider π is of the form $\pi = \sigma \oplus \mathrm{id}_m \oplus \tau$, where σ is an indecomposable 132-avoider and τ is an indecomposable 213-avoider. This allows us to define

$$\widetilde{f}(\pi) = \sigma \oplus \mathrm{id}_{m+1} \oplus \tau,$$

which clearly preserves 1324-avoidance and the number of inversions. If $\pi \in \mathrm{Av}_n(1324)$ is almost decomposable, we define $f(\pi)$ as follows.

F1 If $\pi \setminus \pi_1$ is decomposable, let $f(\pi)_1 = \pi_1$ and $f(\pi) \setminus \pi_1 = \widetilde{f}(\pi \setminus \pi_1)$.

F2 If $\pi \setminus 1$ is decomposable, let $f(\pi) = f(\pi^{-1})^{-1}$.

F3 Otherwise, let $f(\pi) = f(\pi^{\mathrm{rc}})^{\mathrm{rc}}$.

In other words, we remove a point from the boundary of the plot of π to make it decomposable, use $\widetilde{f}$ on the resulting permutation, and then insert the point back in its original position.

The mapping is presented as above in [LV25], but the second and third cases can be written out more explicitly.

F2′ If $\pi \setminus 1$ is decomposable, let $f(\pi)^{-1}_1 = \pi^{-1}_1$ and $f(\pi) \setminus 1 = \widetilde{f}(\pi \setminus 1)$.

F3′ If $\pi \setminus \pi_n$ is decomposable, let $f(\pi)_{n+1} = \pi_n + 1$ and $f(\pi) \setminus f(\pi)_{n+1} = \widetilde{f}(\pi \setminus \pi_n)$.

F4′ If $\pi \setminus n$ is decomposable, let $f(\pi)^{-1}_{n+1} = \pi^{-1}_n + 1$ and $f(\pi) \setminus \{n+1\} = \widetilde{f}(\pi \setminus n)$.

For notational convenience, we set $f(\pi) := \widetilde{f}(\pi)$ when π is decomposable.

Remark 6.2. It is important to note that a certain priority order is chosen in the definition of f. If $\pi \setminus \pi_1$ and, say, $\pi \setminus \pi_n$ are *both* decomposable, we define $f(\pi)$ according to case F1. We only use case F3 if $\pi \setminus \pi_1$ and $\pi \setminus 1$ are both indecomposable. It is, however, impossible for both $\pi \setminus \pi_1$ and $\pi \setminus 1$ to be decomposable simultaneously, so there is no priority issue between cases F1 and F2, or between F3′ and F4′ [LV25, Proposition 7].

The choice of priority is arbitrary, and the mapping could just as well have been defined by prioritizing F3 over F1 and F2. Some results regarding f in this section are, for this reason, not closed under taking reverse complements of the patterns appearing in them, but similar results valid for those patterns can be obtained by changing priority.

Theorem 6.3 (Theorem 19 in [LV25]). *The mapping f is injective,* $\mathrm{inv}(f(\pi)) = \mathrm{inv}(\pi)$*, and* $f(\pi) \in \mathrm{Av}(1324)$ *for any decomposable or almost decomposable* 1324*-avoider π. In particular, if $n \geqslant \frac{k+7}{2}$, then*

$$\mathrm{av}^k_n(1324) \leqslant \mathrm{av}^k_{n+1}(1324).$$

This section is structured as follows. In Section 6.1, we study the notion of *f-compatibility*: a pattern p is f-compatible if f preserves p-avoidance. If p is such a pattern, then Theorems 6.1 and 6.3 immediately imply that

$$\mathrm{av}^k_n(1324, p) \leqslant \mathrm{av}^k_{n+1}(1324, p)$$

for all $n \geqslant \frac{k+7}{2}$. We give necessary and sufficient conditions for f-compatibility, but fall short of a full characterization. In Section 6.2, we show that the pattern 1342 is not f-compatible, but that all counterexamples have $n < \frac{k+7}{2}$. An analysis of the properties of f restricted to $\mathrm{Av}(1324, 1342)$ yields an exact enumeration of $\mathrm{av}^k_n(1324, 1342)$ for $n \geqslant \frac{k+7}{2}$.

6.1 Compatible patterns

This subsection examines patterns p for which f always preserves p-avoidance.

Definition 6.4. A collection B of patterns is called *f-compatible* if $f(\pi)$ avoids B for any decomposable or almost decomposable permutation $\pi \in \mathrm{Av}(1324, B)$. If $B = \{p\}$ is f-compatible, we say that p itself is f-compatible. If B is not f-compatible, we say that it is *f-incompatible*.

The following lemma collects some simple properties of f-compatible sets.

Lemma 6.5.

(a) If p is an f-compatible pattern, then p^{-1} is also f-compatible.

(b) If B and B' are f-compatible, then $B \cup B'$ is f-compatible. In particular, a collection of f-compatible patterns is f-compatible.

(c) If p contains 1324, then p is f-compatible.

(d) If B or B^{rc} is f-compatible, then

$$\mathrm{av}_n^k(1324, B) \leqslant \mathrm{av}_{n+1}^k(1324, B)$$

for all k and $n \geqslant \frac{k+7}{2}$.

Proof. Part (a) follows from the symmetry in the definition of f. Parts (b), (c) and (d) are obvious from the definition of f-compatibility and Remark 6.2. □

We are not able to give a concise, complete characterization of the f-compatible patterns, but we get close. Theorems 6.6 and 6.8 below give necessary and sufficient conditions for f-incompatibility, respectively, and thus provide upper and lower bounds on the total number of f-(in)compatible patterns in $\mathrm{Av}_n(1324)$. In Table 4, we compare these bounds with a value obtained computationally as follows: for each $p \in \mathrm{Av}_n(1324)$, we computed $f(\pi)$ for all decomposable and almost decomposable permutations $\pi \in \mathrm{Av}_m(1324, p)$ with $n - 1 \leqslant m \leqslant n + 2$, and checked if $f(\pi)$ avoids p. If any $f(\pi)$ contained p, then p was declared f-incompatible. The total number of such patterns is another lower bound on the number of f-incompatible patterns, likely to be very close to the actual number.

For $n \leqslant 5$ our bounds are tight. The f-compatible patterns of length four are

$$1432,\ 4231,\ 4321,$$

and the f-compatible patterns of length five are

$$14523,\ 14532,\ 15342,\ 15423,\ 15432,\ 34125,$$
$$52341,\ 52431,\ 53241,\ 53421,\ 54231,\ 54321.$$

In Corollary 6.7 below, we collect two simple families of f-compatible patterns. For $n \leqslant 5$, the only f-compatible pattern that is not in one of these families is 34125.

Table 4. Upper and lower bound for the number of f-(in)compatible patterns provided by Theorems 6.6 and 6.8, compared with the also rigorous lower bound (CLB) and upper bound (CUB) obtained computationally.

n	f-incompatible			f-compatible		
	Thm. 6.8	CLB	Thm. 6.6	Thm. 6.6	CUB	Thm. 6.8
3	4	4	4	2	2	2
4	18	20	20	3	3	5
5	87	91	91	12	12	16
6	425	447	451	62	66	88
7	1973	2087	2122	640	675	789

Theorem 6.6. *If $p \in \mathrm{Av}_n(1324)$ is f-incompatible, then at least one of the patterns $q \in \{p, p^{-1}, p^{\mathrm{rc}}, (p^{\mathrm{rc}})^{-1}\}$ satisfies one of the following conditions:*

(a) $\mathrm{comp}(q) \geqslant 3$.

(b) $\mathrm{comp}(q \setminus q_1) > \mathrm{comp}(q)$.

(c) $q_1 > 1$, $\mathrm{comp}(q) = 2$, *and the first component of q begins with its largest entry. Furthermore, if $q \in \{p^{\mathrm{rc}}, (p^{\mathrm{rc}})^{-1}\}$, then $q_n < n$.*

(d) $1 < q_1 < n$ *and* $q \setminus q_1$ *avoids* 213.

Proof. Since p is not f-compatible, there exists a decomposable or almost decomposable permutation $\pi \in \mathrm{Av}(1324, p)$ such that $f(\pi)$ contains p. Recall first that if π is decomposable, then $\pi = \sigma \oplus \mathrm{id}_m \oplus \tau$, where σ and τ are indecomposable, and $f(\pi) = \sigma \oplus \mathrm{id}_{m+1} \oplus \tau$. Since π avoids p, it must be that the occurrence of p in $f(\pi)$ uses every entry of id_{m+1}. This implies that $\mathrm{comp}(p) \geqslant 3$, condition (a).

Assume instead that π is almost decomposable, and that $\pi \setminus \pi_1$ is decomposable (so $f(\pi)$ is defined as in case F1). Write $\pi \setminus \pi_1 = \sigma \oplus \mathrm{id}_m \oplus \tau$, where σ and τ are indecomposable. We analyze how p can occur in $f(\pi)$.

- If p is contained in $f(\pi \setminus \pi_1)$, then $\mathrm{comp}(p) \geqslant 3$ as above. So, assume instead that every occurrence of p in $f(\pi)$ uses the entry $f(\pi)_1 = \pi_1$.
- If an occurrence of p uses only π_1 and points of σ, then π also contains p. So, every occurrence of p in $f(\pi)$ must use at least one point of id_{m+1} or τ.
- If an occurrence of p uses π_1 as well as points from both σ and id_{m+1} (and possibly τ), then $\mathrm{comp}(p \setminus p_1) > \mathrm{comp}(p)$ since $\pi_1 \geqslant |\sigma| + m + 1$. This is condition (b).
- If an occurrence of p uses only π_1 and points from id_{m+1}, then there is another occurrence of p that uses points from σ: replace p_2 with any point from σ. This case was handled above.

- Suppose an occurrence of p uses only π_1 and points from τ. It is clear that if $p_1 = n$ then π also contains p. Furthermore, if $p_1 = 1$ then π contains p by taking $p \setminus p_1$ from τ together with any point from σ. Hence, $1 < p_1 < n$ and condition (d) holds.
- Suppose an occurrence of p uses only π_1 as well as points from both id_{m+1} and τ. Again, $p \setminus p_1$ avoids 213. If $p_1 = n$, then π also contains p: take p_1 as π_1, p_2 arbitrarily from σ, and $p \setminus \{p_1, p_2\}$ from $\mathrm{id}_m \oplus \tau$. Since we clearly have $p_1 > 1$, condition (d) holds.
- Finally, suppose that an occurrence of p uses π_1 as well as points from σ and τ, but not from id_{m+1}. We have $\mathrm{comp}(p \setminus p_1) \geqslant 2$, so if $\mathrm{comp}(p) = 1$ we have condition (b), and if $\mathrm{comp}(p) \geqslant 3$ we have condition (a). Assume that $\mathrm{comp}(p) = 2$. The first component of p is contained in $\pi_1\sigma$ and uses π_1, which is larger than every entry of σ. Hence, the first component of p begins with its largest entry, condition (c). We know that $p_1 > 1$, since p uses entries from σ.

If instead $\pi \setminus 1$ is decomposable, the same arguments apply to π^{-1} and $q = p^{-1}$. If both $\pi \setminus \pi_1$ and $\pi \setminus 1$ are indecomposable, then take π^{rc} and $q = p^{\mathrm{rc}}$ instead. We only need to show that the additional property holds in condition (c). So, suppose $q = p^{\mathrm{rc}}$ is contained in $f(\pi)$, where $\pi \setminus \pi_1 = \sigma \oplus \mathrm{id}_m \oplus \tau$ is decomposable, and the occurrence of q in $f(\pi)$ uses only $f(\pi)_1 = \pi_1$ and points from both σ and τ. Again, we may assume that q has exactly two components. If $q_n = n$, then the second component is simply 1. If our occurrence of q contains any other entries of τ, then $q \setminus q_1$ has at least three components, which gives us condition (b).

Hence, we assume that the occurrence of q uses only π_1 and points from σ, as well as one point from τ. Now, observe that $\pi_1 < n$: otherwise the last entry of π^{rc} is 1, $\pi^{\mathrm{rc}} \setminus 1$ is decomposable, and $f(\pi^{\mathrm{rc}})$ is defined according to case F2 instead of F3. This means that there exists some entry in τ that is larger than π_1, in π. Taking this entry together with our occurrence of $q \setminus q_n$ in $f(\pi)$ creates an occurrence of q in π. This is a contradiction, so we must have $q_n \neq n$, concluding the proof. □

Corollary 6.7. *If $p \in \mathrm{Av}_n(1324)$ satisfies any of the following conditions, then p is f-compatible:*

- $p_1 = n$ *and* $p_n = 1$.
- $p = 1 \oplus \tau$, *where* τ, $\tau \setminus \tau_{n-1}$ *and* $\tau \setminus \{n-1\}$ *are indecomposable* ($n \geqslant 4$).

Proof. It suffices to verify that if p satisfies one of the two conditions, then none of the patterns $q \in \{p, p^{-1}, p^{\mathrm{rc}}, (p^{\mathrm{rc}})^{-1}\}$ satisfy any of the conditions in Theorem 6.6, which is straightforward. □

Theorem 6.8. *Let $p \in \mathrm{Av}_n(1324)$ be a pattern. If at least one of the patterns $q \in \{p, p^{-1}, p^{\mathrm{rc}}, (p^{\mathrm{rc}})^{-1}\}$ satisfies one of the following conditions, then p is f-incompatible:*

(a) $\mathrm{comp}(q) \geqslant 3$.

(b) $q_1 < n$ *and* $\mathrm{comp}(q \setminus q_1) > \mathrm{comp}(q)$.

(c) $q_1 > 1$, $\mathrm{comp}(q) = 2$, *and the first component of* q *begins with its largest entry.*

(d) $1 < q_1 < n$ *and* $q \setminus q_1$ *avoids* 213.

In cases (b), (c) and (d), if $q \in \{p^{\mathrm{rc}}, (p^{\mathrm{rc}})^{-1}\}$, *then require further that* $q_1 < n - 1$.

Proof. In each case, it suffices to find a permutation $\pi \in \mathrm{Av}(1324, p)$ such that $f(\pi)$ contains p. We begin with the case $q = p$. The case $q = p^{-1}$ is analogous – take π^{-1} instead of π. In (a) we have $p = \sigma \oplus \mathrm{id}_m \oplus \tau$ with $m \geqslant 1$, so we can take $\pi = \sigma \oplus \mathrm{id}_{m-1} \oplus \tau$. Then π obviously avoids p, and $f(\pi) = p$. Henceforth, assume that $\mathrm{comp}(p) \leqslant 2$.

Suppose that p satisfies (b). Assume first that $\mathrm{comp}(p) = 2$, say $p = \sigma \oplus \tau$. Here σ is an indecomposable 132-avoider such that $\mathrm{comp}(\sigma \setminus \sigma_1) \geqslant 2$, and it is easy to see that such a permutation must begin with its largest entry – condition (c) holds, and this case will be handled in the next paragraph. Assume instead that p is indecomposable. If $\mathrm{comp}(p \setminus p_1) \geqslant 3$, we can let π be the preimage $f^{-1}(p)$. Otherwise, let π be defined by $\pi \setminus \pi_1 = p \setminus p_1$ and $\pi_1 = p_1 + 1$ (since $p_1 < n$, this is possible). Clearly π avoids p, and $f(\pi)$ contains p.

Now, assume p satisfies (c). As above, define π by $\pi \setminus \pi_1 = p \setminus p_1$, $\pi_1 = p_1 + 1$. Clearly π is indecomposable, π avoids p, and $f(\pi)$ contains p.

Finally, suppose that p satisfies (d). Define π by $\pi \setminus 1 = 1 \oplus p$ and $\pi_1 = p_1 + 2$ (i.e. the first entry of $\pi \setminus 1$ is $p_1 + 1$). It is easy to see that π avoids p, and $f(\pi)$ contains p.

If instead $q \in \{p^{\mathrm{rc}}, (p^{\mathrm{rc}})^{-1}\}$ satisfies one of the conditions, we want to construct π in an analogous manner. However, care must be taken to ensure that $\pi \setminus 1$ and $\pi \setminus \pi_1$ are indecomposable; otherwise $f(\pi)$ is not defined according to case F3. Concretely, this is what we want to show: if $q = p^{\mathrm{rc}}$ satisfies (b), (c) or (d), and $q_1 < n-1$, then there exists a permutation $\pi \in \mathrm{Av}(1324, q)$ such that $f(\pi)$ contains q, and both $\pi \setminus n$ and $\pi \setminus \pi_n$ are indecomposable.

First, suppose $q = p^{\mathrm{rc}}$ satisfies condition (b) or (c): either q is indecomposable and $\mathrm{comp}(q \setminus q_1) > 2$, or q has exactly two components, the first of which starts with its largest entry. Write $q \setminus q_1 = \sigma \oplus \tau$, where $\tau \in \mathrm{Av}_m(213)$ is indecomposable. We define

$$\tau' = \begin{cases} \tau & \text{if } \tau_1 = m, \tau_m = 1, \\ 1 \ominus \tau & \text{if } \tau_1 \neq m, \tau_m = 1, \\ \tau \ominus 1 & \text{if } \tau_1 = m, \tau_m \neq 1, \\ 1 \ominus \tau \ominus 1 & \text{if } \tau_1 \neq m, \tau_m \neq 1, \end{cases} \tag{6}$$

in each case ensuring that τ' begins with its largest entry and ends with 1, so that in particular τ' is not almost decomposable. Now, define π by $\pi \setminus \pi_1 = \sigma \oplus \tau'$ and

$$\pi_1 = \begin{cases} q_1 + 1 & \text{if } \tau_m = 1, \\ q_1 + 2 & \text{if } \tau_m \neq 1. \end{cases}$$

We can check that π avoids q: an occurrence of q in π cannot use the new entries added to τ, so it must consist of π_1 and $\sigma \oplus \tau$ – but π_1 was shifted up one step to prevent this. Furthermore, $f(\pi)$ contains q and both $\pi \setminus n$ and $\pi \setminus \pi_n$ are indecomposable, since τ' is not almost decomposable and critically, π_1 is not the largest entry of π. (This is where we use the assumption that $q_1 < n - 1$.)

Lastly, suppose that $q = p^{\mathrm{rc}}$ satisfies condition (d). Let $q \setminus q_1 = \tau \in \mathrm{Av}_m(213)$ and define τ' as in (6). Define π by $\pi \setminus \pi_1 = 1 \oplus \tau'$ and

$$\pi_1 = \begin{cases} q_1 + 2 & \text{if } \tau_m = 1, \\ q_1 + 3 & \text{if } \tau_m \neq 1. \end{cases}$$

The argument is analogous to the previous case. Since the case $q = (p^{\mathrm{rc}})^{-1}$ is symmetrical, this concludes the proof. □

6.2 1342 is almost compatible

The pattern 1342 is not f-compatible: if $\pi = 34152$ then $f(\pi) = 241563$, whose subsequence 2563 forms a 1342-pattern. However, we can show that every such counterexample has many inversions.

Theorem 6.9. *If $\pi \in \mathrm{Av}_n(1324, 1342)$ is decomposable or almost decomposable and $\mathrm{inv}(\pi) \leqslant 2n - 5$, then $f(\pi)$ avoids 1342. In particular, for all k and $n \geqslant \frac{k+7}{2}$, we have*

$$\mathrm{av}_n^k(1324, 1342) \leqslant \mathrm{av}_{n+1}^k(1324, 1342).$$

Remark 6.10. The pattern 1342 is the only such case among the length four patterns. All others are either f-compatible, or there are counterexamples $\pi \in \mathrm{Av}_n(1324, p)$ with n, $n + 1$ or $n + 2$ inversions, so that $f(\pi)$ contains p.

Proof of Theorem 6.9. First of all, it is clear that if π is decomposable, then $f(\pi) = \widetilde{f}(\pi)$ avoids 1342. So, let us assume that π is an almost decomposable $\{1324, 1342\}$-avoiding permutation of length n. We will examine how the 1342-pattern can appear in $f(\pi)$. The goal is to prove that $f(\pi)$ must have been constructed according to case F3′ in the definition of f.

- Suppose $\pi \setminus \pi_1$ is decomposable. Since $\widetilde{f}(\pi \setminus \pi_1)$ avoids 1342, we know that if $f(\pi)$ contains 1342, then $f(\pi)_1$ must be the 1 in the pattern. However, $\pi_1 > \pi_2$ and hence $f(\pi)_1 > f(\pi)_2$, so $\widetilde{f}(\pi \setminus \pi_1)$ must also contain 1342. This is a contradiction, so $\pi \setminus \pi_1$ must be indecomposable.
- By an identical argument, $\pi \setminus 1$ must be indecomposable.

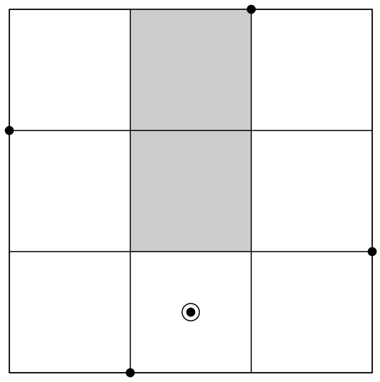

Figure 5. The structure of an almost decomposable permutation in $\mathrm{Av}(1324, 1342)$ whose image under f contains 1342.

- Suppose that $\pi \setminus \pi_1$ and $\pi \setminus 1$ are indecomposable, and $\pi \setminus n$ is decomposable. Again, $\widetilde{f}(\pi \setminus n)$ avoids 1342, so the entry $n+1$ of $f(\pi)$ must be the 4 in any occurrence of 1342. Since $\pi_n^{-1} < \pi_{n-1}^{-1}$ we have that $f(\pi)_{n+1}^{-1} < f(\pi)_n^{-1}$. Now, fix an occurrence $abcd$ of 1342 in $f(\pi)$, where $c = n+1$. Note that d must come after n in $f(\pi)$, as otherwise $abdn$ forms a 1324-pattern. But this means that $abnd$ is an occurrence of 1342 in $f(\pi)$ that does not use the entry $n+1$, which is a contradiction. Hence, $\pi \setminus n$ must be indecomposable.

In conclusion, we have showed that if $f(\pi)$ contains 1342, then $\pi \setminus \pi_1$, $\pi \setminus 1$ and $\pi \setminus n$ are all indecomposable. So, $\pi \setminus \pi_n$ must be decomposable. We further claim that $\pi_1 > \pi_n$ must hold. Indeed, since π is indecomposable, there would otherwise be some $i < \pi_n^{-1}$ such that $\pi_i > \pi_n$, and then $\pi_1 \pi_i n \pi_n$ forms a 1342-pattern. Furthermore, $\pi_1^{-1} < \pi_n^{-1}$ since $\pi \setminus \pi_n$ is decomposable.

See Figure 5 for an illustration of the plot of such a permutation π. Points can be contained in any of the white regions, but the shaded region must be empty, since a point there would create a 1342-pattern together with 1, n and π_n. Note that the four 'boundary' points $\pi_1, 1, n, \pi_n$ create $2 \cdot 4 - 5 = 3$ inversions by themselves. Furthermore, any point in one of the empty white regions must create at least two inversions with these boundary points. Lastly, any point π_i in the bottom middle region (containing a circled point in the figure) creates one inversion with π_1, and we claim that it must also create an inversion with a non-boundary ('interior') point. Indeed, otherwise all points (except π_1) before π_i are smaller than it, and all points after π_i are larger, so $\pi \setminus \pi_1$ is decomposable. This is a contradiction.

Construct π from the configuration of its four boundary points by adding the interior points one by one, so that the points in the bottom middle region are added last. The argument above shows that the number of inversions increases by at least two at each step, so $\mathrm{inv}(\pi) \geqslant 2n - 5$ by induction. □

To conclude this section, we follow the methods of [LV25] to enumerate the row differences

$$\mathrm{av}_{n+1}^k(1324, 1342) - \mathrm{av}_n^k(1324, 1342)$$

for $n \geqslant \frac{k+7}{2}$. Since we also know the limit sequence of $\{1324, 1342\}$, this yields an enumeration of $\mathrm{Av}_n^k(1324, 1342)$ for all $n \geqslant \frac{k+7}{2}$. Recall first from [LV25, Section 4] that

$$\mathrm{Av}^k_{n+1}(1324) \setminus f(\mathrm{Av}^k_n(1324)) = R_1 \sqcup R_2 \sqcup R_3,$$

where the collections R_i are defined as follows.

- R_1 contains all permutations σ such that $\sigma_1 = n+1$ or $\sigma_{n+1} = 1$.
- R_2 contains all permutations σ such that $\sigma_2 = n+1$ or $\sigma_{n+1} = 2$.
- R_3 contains all permutations σ such that either $\tau = \sigma$ or $\tau = \sigma^{-1}$ has the following form: $\mathrm{comp}(\tau \smallsetminus \tau_1) \geqslant 3$, $\tau_1 = \ell + 1$ and $\tau_{n+1} = \ell + 2$, where ℓ is the length of the first component of $\tau \smallsetminus \{\tau_1, \tau_{n+1}\}$.

We will show that the difference

$$\mathrm{Av}^k_{n+1}(1324, 1342) \setminus f(\mathrm{Av}^k_n(1324, 1342)) \tag{7}$$

consists of exactly the same collections intersected with $\mathrm{Av}(1342)$.

Lemma 6.11. *If $\pi \in \mathrm{Av}_n(1324)$ is decomposable or almost decomposable and contains 1342, then $f(\pi)$ also contains 1342.*

Proof. This is clear when π is decomposable, so suppose that π is almost decomposable. Assume first that $\pi \smallsetminus \pi_1$ is decomposable. The entries 342 of an occurrence of 1342 in π must be contained in the same component of $\pi \smallsetminus \pi_1$, so it is clear that $f(\pi)$ also contains 1342. The case where $\pi \smallsetminus 1$ is decomposable is analogous.

Now, suppose that $\pi \smallsetminus \pi_1$ and $\pi \smallsetminus 1$ are indecomposable, and that $\pi \smallsetminus \pi_n$ is decomposable. In $f(\pi)$ there must be an occurrence of 12 with both entries larger than $f(\pi)_{n+1}$. Together with 1, these entries form a 1342-pattern in $f(\pi)$. Finally, assume that $\pi \smallsetminus n$ is decomposable. The entries 1 and 3 in our 1342 must both come from the first component of $\pi \smallsetminus n$, so the 2 must also come from the first component. But then the first component contains 132, which is impossible. □

Lemma 6.11 implies that the collection 7 consists exactly of R_1, R_2 and R_3 intersected with $\mathrm{Av}(1342)$, like we wanted. It remains to enumerate these sets.

Theorem 6.12. *For every nonnegative integer k and $n \geqslant \frac{k+7}{2}$, the difference $\mathrm{av}^k_{n+1}(1324, 1342) - \mathrm{av}^k_n(1324, 1342)$ is nonnegative, and equals the coefficient of x^k in the generating function*

$$x^{n-1}(2+2x)C_{1324,1342}(x) = x^{n-1}(2+2x) \cdot \prod_{i \geqslant 1} \frac{1+x^i}{1-x^i}.$$

In particular, when $n \geqslant \frac{k+7}{2}$,

$$\mathrm{av}^k_n(1324, 1342) = [x^k] \left(\frac{1 - x - x^{n-1}(2+2x)}{1-x} \cdot \prod_{i \geqslant 1} \frac{1+x^i}{1-x^i} \right).$$

Proof. The collection $R_1 \cap \mathrm{Av}(1342)$ consists of permutations $1 \ominus \pi$ and $\pi \ominus 1$, where $\pi \in \mathrm{Av}_n^{k-n}(1324, 1342)$ is arbitrary. Hence, its generating function in k is $2x^n C_{1324,1342}(x)$. In $R_2 \cap \mathrm{Av}(1342)$, note that the case $\sigma_{n+1} = 2$ is impossible: there exists a 12-pattern in σ with both entries above σ_{n+1}, which together with 1 form a 1342-pattern. Hence, as in the previous case, $R_2 \cap \mathrm{Av}(1342)$ has generating function $x^{n-1}C_{1324,1342}(x)$. The collection $R_3 \cap \mathrm{Av}(1342)$ is similar: the case where $\tau = \sigma$ (see the definition) is impossible, since there exists a 12-pattern with both entries above τ_{n+1}. On the other hand, when $\tau = \sigma^{-1}$, the decomposable permutation $\sigma \setminus \{1, n+1\} \in \mathrm{Av}_{n-1}^{k-n+2}(1324, 1342)$ is arbitrary, so the generating function of this collection is also $x^{n-1}C_{1324,1342}(x)$. Summing up, we get the generating function

$$x^{n-1}(2+2x)C_{1324,1342}(x) = x^{n-1}(2+2x) \cdot \prod_{i \geqslant 1} \frac{1+x^i}{1-x^i}$$

with $C_{1324,1342}(x)$ from Proposition 5.7. □

7. Some indecomposable 132-avoiders

In this section, we enumerate indecomposable permutations avoiding 132 and certain patterns of length four, by the number of inversions. A similar analysis has been carried out before for collections of length-three patterns [Fra+24; Fra25]. For a collection B of patterns, let

$$I_k(B) = \{\pi \in \mathrm{Av}(B) : \mathrm{comp}(\pi) = 1 \text{ and } \mathrm{inv}(\pi) = k\} \quad \text{and} \quad i_k(B) = |I_k(B)|.$$

Observe that $I_k(B)$ can contain permutations of different sizes. The fact that $I_k(B)$ is finite is easily deduced from the inequality $\mathrm{inv}(\pi) + \mathrm{comp}(\pi) \geqslant |\pi|$ due to Claesson, Jelínek and Steingrímsson [CJS12].

Taking the inversion table of a permutation defines a bijection Λ from $I_k(132)$ to the partitions of k, which means that if B contains 132, then $I_k(B)$ can be identified with a certain subclass of partitions. This subclass often has another nice description, and thus a known generating function. The converse is true in some cases, in the sense that the pattern-avoidance perspective illuminates the partition class. Note that since a 132-avoiding permutation is decomposable if and only if it ends with its largest entry, the generating function $C_B(x)$ of the limit sequence of B satisfies

$$C_B(x) = \sum_{k \geqslant 0} i_k(B) x^k$$

for any basis B containing 132. The existence of the limit sequence is guaranteed by Proposition 5.1.

Throughout, we will use the following convention: if $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition, then $\lambda_i = 0$ for all $i > \ell$. This is convenient, since $\Lambda(\pi)$ is always shorter than π itself for an indecomposable 132-avoider π, and we do not necessarily know by how much.

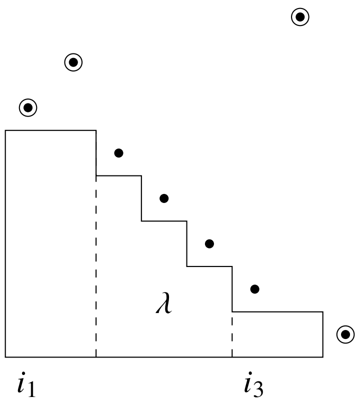

Figure 6. A partition violating condition (b) of Proposition 7.1 corresponds to a 132-avoiding permutation containing the pattern 2341. The picture shows the relevant part of the partition as explained in the proof of Proposition 7.2, as well as the corresponding points of the permutation. The points forming the 2341 pattern are circled.

7.1 2341: the sand pile model

The *sand pile model* is a discrete dynamical system originating in physics [BTW88] that has since been studied by combinatorialists as a chip-firing game, and computer scientists as a cellular automaton. In combinatorial terms, the model consists of a set $\mathrm{SPM}(k)$ of partitions of k defined recursively as follows: $(k) \in \mathrm{SPM}(k)$, and if $\lambda \in \mathrm{SPM}(k)$ then every partition that can be obtained from λ by subtracting one from a part and adding one to the next part is also in $\mathrm{SPM}(k)$. These partitions admit the following characterization.

Proposition 7.1 ([Pha99])**.** *A partition λ of k is contained in* $\mathrm{SPM}(k)$ *if and only if*

(a) λ has no three equal parts, and

(b) if $\lambda_{i_1} = \lambda_{i_1+1}$ and $\lambda_{i_3} = \lambda_{i_3+1}$ for some $i_1 < i_3$, then there exists some i_2 such that $i_1 < i_2 < i_3$ and $\lambda_{i_2} - \lambda_{i_2+1} \geqslant 2$.

The first few terms of the sequence $|\mathrm{SPM}(k)|$ are

$$1, 1, 2, 2, 4, 5, 6, 9, 13, 15, \ldots,$$

see entry A056219 in the OEIS. For example,

$$\mathrm{SPM}(5) = \{(5), (4,1), (3,2), (3,1,1), (2,2,1)\}.$$

Proposition 7.2. *For every k, we have* $\Lambda(I_k(132, 2341)) = \mathrm{SPM}(k)$. *In particular,*

$$C_{132,2341}(x) = 1 + \sum_{k \geqslant 1} x^{\frac{k(k+1)}{2}} \cdot \prod_{i=1}^{k} \left(x + \frac{1}{1 - x^i} \right).$$

Proof. Suppose first that $\lambda \vdash k$ is not contained in $\mathrm{SPM}(k)$, i.e. condition (a) or (b) of Proposition 7.1 is violated. Assume that (a) does not hold, i.e. there is some i such that $\lambda_i = \lambda_{i+1} = \lambda_{i+2} > 0$. By Lemma 2.5 (b) we have $\pi_i < \pi_{i+1} < \pi_{i+2}$, and so $\pi_i \pi_{i+1} \pi_{i+2} 1$ forms a 2341 pattern. Suppose instead that condition (b) does not

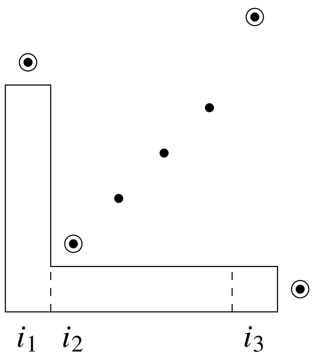

Figure 7. A special occurrence of 3241 in a 132-avoider, and the corresponding non-steep partition. The points forming the 3241 pattern described in the proof of Proposition 7.3 are circled.

hold, i.e. there exist some indices $i_1 < i_3$ such that $\lambda_{i_1} = \lambda_{i_1+1} > \lambda_{i_3} = \lambda_{i_3+1}$, and $\lambda_{i_2} = \lambda_{i_2+1} + 1$ for all $i_1 < i_2 < i_3$. By Lemma 2.5 we get $\pi_{i_1} < \pi_{i_1+1}$, $\pi_{i_2} = \lambda_{i_2} + 1$ for all $i_1 + 1 < i_2 \leqslant i_3$, and therefore finally $\pi_{i_3+1} > \pi_{i_1+1}$. Putting everything together, $\pi_{i_1}\pi_{i_1+1}\pi_{i_3+1}1$ forms a 2341 pattern. See Figure 6 for an illustration.

Conversely, suppose that $\pi \in \mathrm{Av}(132)$ contains 2341. A standard argument shows that π contains an occurrence $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_4}$ of 2341 such that $\pi_{i_4} = 1$, $i_2 = i_1 + 1$, and either $i_3 = i_2 + 1$ or $\pi_j = \pi_{j+1} + 1$ for all $i_2 < j < i_3 - 1$. In the first case $\pi_{i_1} < \pi_{i_2} < \pi_{i_3}$, so λ has three equal parts. In the second case, $\lambda_{i_1} = \lambda_{i_1+1}$, $\lambda_{i_3-1} = \lambda_{i_3}$, and there is no $i_1 < j < i_3 - 1$ such that $\lambda_j - \lambda_{j+1} \geqslant 2$. Thus, in either case, λ is not in $\mathrm{SPM}(k)$. The generating function is due to Corteel and Gouyou-Beauchamps [CG02]. □

7.2 3241: steep partitions

We call a partition λ of k *steep* if the difference between any two consecutive distinct parts of λ is greater than or equal to the multiplicity of the smaller part. For example, $(5, 5, 3, 3, 2, 1)$ is steep, but $(3, 2, 2)$ and $(5, 3, 3, 3, 1)$ are not. The enumeration sequence for the steep partitions is

$$1, 1, 2, 3, 4, 6, 8, 10, 14, 19, \dots,$$

and it does not appear in the OEIS. We were not able to find a nice expression for the generating function of this sequence.

Proposition 7.3. *An indecomposable* 132*-avoider* π *avoids* 3241 *if and only if* $\Lambda(\pi)$ *is steep.*

Proof. Suppose first that $\lambda := \Lambda(\pi)$ is not steep. This means that there is some i such that $\lambda_{i+1} = \lambda_{i+2} = \dots = \lambda_{i+m+1}$, where $m := \lambda_i - \lambda_{i+1} \geqslant 1$. Using Lemma 2.5 we see that $\pi_i\pi_{i+1}\pi_{i+m+1}1$ forms a 3241 pattern in π.

Conversely, assume π contains 3241. It is routine to check that π must contain an occurrence $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_4}$ of 3241 such that $i_2 = i_1 + 1$, $\pi_{i_4} = 1$, and

$$\pi_{i_2} = \pi_{i_2+1} - 1 = \pi_{i_2+2} - 2 = \dots = \pi_{i_3-1} - i_3 + i_2 + 1.$$

Lemma 2.5 shows that $\lambda_{i_2} = \lambda_{i_2+1} = \dots = \lambda_{i_3}$. Since the difference $\lambda_{i_1} - \lambda_{i_2}$ equals $i_3 - i_2$ (see Figure 7 for an illustration), λ is not steep. □

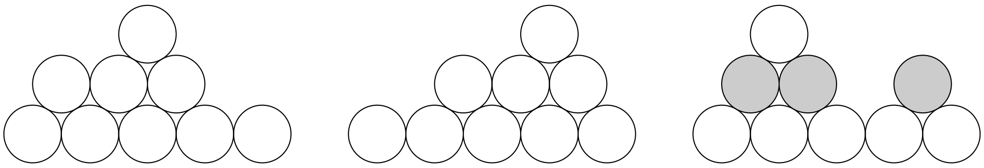

Figure 8. Two convex penny arrangements and a non-convex one for $r = 9$, $s = 5$.

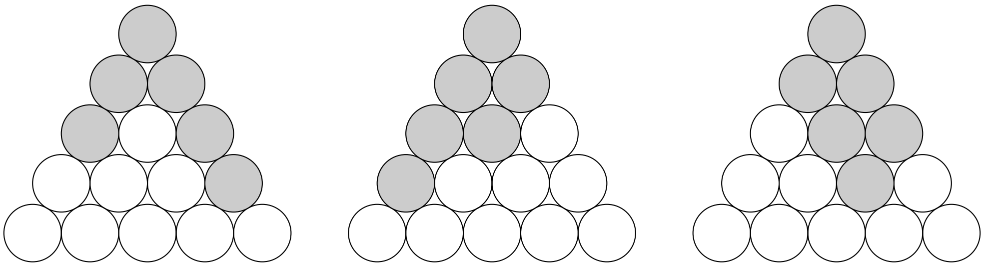

Figure 9. The complements of two convex penny arrangements and a non-convex one. From left to right, the corresponding partitions are $(4, 1, 1)$, $(2, 2, 1, 1)$ and $(3, 3)$.

7.3 3412: convex penny arrangements

The sequence $i_k(132, 3412)$ begins with

$$1, 1, 2, 3, 4, 7, 9, 13, 17, 25, \ldots,$$

sequence A005576 in the OEIS. This sequence is best explained in the following way. A *penny arrangement* is a configuration of pennies in rows in the plane such that the bottom row is contiguous, and each penny in a higher row touches two pennies in the row below it. A penny arrangement is called *convex* if every row is contiguous. Let $P(r, s)$ denote the set of all penny arrangements of r pennies with s pennies in the bottom row, and let $C(r, s) \subseteq P(r, s)$ be the set of convex penny arrangements, $c(r, s) := |C(r, s)|$. This is sequence A259095. Figure 8 illustrates two convex penny arrangements and a non-convex one for $r = 9$, $s = 5$.

The k-th entry $a(k)$ of sequence A005576 equals $c(r(r + 1)/2 - k, r)$ for large r. In other words, it is the limit as r goes to infinity of the number of convex penny arrangements with r pennies in the bottom row, such that the complement of the arrangement – with respect to the full triangular arrangement – contains k pennies. It is easy to see that the limit is attained when $r = k$, so $a(k) = c\left(\binom{k}{2}, k\right)$. If α is a penny arrangement, let α^{c} denote its complement. Figure 9 shows three examples.

Let $\Pi(\alpha)$ be the partition formed by the lengths of the top-to-bottom-right diagonals of α^{c}, going from right to left. In the figure, the partitions are $(4, 1, 1)$, $(2, 2, 1, 1)$ and $(3, 3)$, respectively. Observe that Π gives a bijection from $P\left(\binom{k}{2}, k\right)$ to the partitions of k. Our claim is that when restricted to convex penny arrangements, the image of Π is the set $\Lambda(I_k(132, 3412))$.

Proposition 7.4. *Let π be an indecomposable* 132*-avoider and $\lambda = \Lambda(\pi)$. The following are equivalent:*

(a) *π avoids* 3412*.*

(b) *$\Pi^{-1}(\lambda)$ is a convex penny arrangement.*

(c) *There are no two indices $i < j$ such that $\lambda_i = \lambda_{i+1}$ and $\lambda_j - \lambda_{j+1} \geqslant 2$.*

Proof. The equivalence of (b) and (c) is easy to see from the pictures. It remains to establish the equivalence of (a) and (c), which can be done using standard methods. Suppose first that there are indices $i < j$ such that $\lambda_i = \lambda_{i+1}$ and $\lambda_j - \lambda_{j+1} \geqslant 2$. Suppose further that the distance between the two is minimal, meaning that we have $\lambda_{i+1} = \lambda_{i+2} + 1 = \lambda_{i+3} + 2 = \ldots = \lambda_j + j - i - 1$. Hence, for every $i + 1 < \ell \leqslant j$, π_ℓ is a left-to-right minimum. By Lemma 2.5, we can see that $\pi_i\pi_{i+1}\pi_{j+1}\pi_\ell$ forms a 3412 pattern (for some $\ell > j$).

Conversely, assume that π contains 3412. A standard argument shows that π contains a 3412 pattern $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_4}$ such that $i_2 = i_1 + 1$, and $\pi_{i_2+1}, \pi_{i_2+2}, \ldots, \pi_{i_3-1}$ forms an interval in which every point π_ℓ is a left-to-right minimum, and $\pi_{i_1} > \pi_\ell > \pi_{i_4}$. It follows that $\lambda_{i_1} = \lambda_{i_1+1}$, and $\lambda_{i_3-1} - \lambda_{i_3} \geqslant 2$. □

7.4 3421: distinct parts, except the smallest

The enumeration sequence $i_k(132, 3421)$ begins with

$$1, 1, 2, 3, 5, 6, 10, 12, 17, 22, \ldots,$$

sequence A115029 in the OEIS: the number of partitions of k such that all parts, except possibly the smallest, have multiplicity one.

Proposition 7.5. *An indecomposable* 132*-avoider π avoids* 3421 *if and only if all parts of $\Lambda(\pi)$, except possibly the smallest, have multiplicity one. In particular,*

$$C_{132,3421}(x) = 1 + \sum_{k \geqslant 1} \frac{x^k}{1 - x^k} \cdot \prod_{i \geqslant k+1} \left(1 + x^i\right).$$

Proof. Denote $\lambda = \Lambda(\pi)$. Suppose first that $\lambda_i = \lambda_{i+1}$ for some i, and that λ has a part λ_j smaller than λ_i. By Lemma 2.5, $\pi_i\pi_{i+1}\pi_j1$ forms a 3421 pattern.

Conversely, suppose that π contains 3421. Note first that we can choose the occurrence of 3421 so that the entry 1 of π serves as the 1 in the pattern. Otherwise, the entry 1 of π occurs before the 21 of the pattern, and the three entries together form 132. Furthermore, we can choose the 3 and 4 of the pattern to be adjacent. Here is why: say $\pi_{i_1}\pi_{i_2}\pi_{i_3}1$ is the occurrence of 3421 in π. None of the entries π_i with $i_1 < i < i_2$ can satisfy $\pi_i < \pi_{i_3}$, or $\pi_i\pi_{i_2}\pi_{i_3}$ is a 132 pattern. Hence, we have an occurrence $\pi_i\pi_{i+1}\pi_j1$ of 3421 in π. This implies that $\lambda_i = \lambda_{i+1} > \lambda_j > 0$, proving the claim. The generating function is easy to see. □

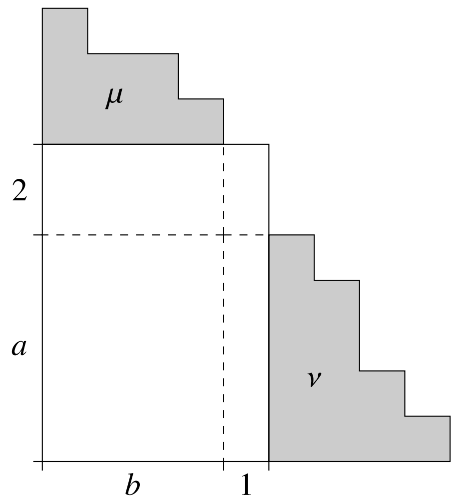

Figure 10. Counting the partitions $\lambda \vdash k$ in bijection with $I_k(132, 4231)$. Either λ has no gap of size greater than one, or it looks like this. The partition μ has no gap of size greater than one and at most b parts, whereas all parts of ν are distinct and the largest part is at most a.

7.5 4231: convex partitions

The enumeration sequence $i_k(132, 4231)$ begins with

$$1, 1, 2, 3, 5, 6, 9, 12, 15, 19, 25, \ldots,$$

which is not in the OEIS. However, we are able to characterize and enumerate the corresponding partitions.

Proposition 7.6. *An indecomposable* 132*-avoider* π *avoids* 4231 *if and only if* $\lambda := \Lambda(\pi)$ *satisfies the following condition: if* $\lambda_i \geqslant \lambda_{i+1} + 2$ *for some* i*, then* $\lambda_{i+1} > \lambda_{i+2} > \ldots$. *In particular,*

$$C_{132,4231}(x) = \prod_{i \geqslant 1}(1 + x^i) + \sum_{a,b \geqslant 0} x^{(a+2)(b+1)} \cdot \prod_{i=1}^{a}(1 + x^i) \cdot \prod_{i=1}^{b}(1 + x^i).$$

Proof. Let $\pi \in I_k(132)$ and $\lambda = \Lambda(\pi)$. We want to show that π avoids 4231 if and only if λ satisfies the following condition:

$$\text{if } \lambda_i \geqslant \lambda_{i+1} + 2 \text{ for some } i, \text{ then } \lambda_{i+1} > \lambda_{i+2} > \ldots. \tag{8}$$

Suppose first that condition (8) does not hold, i.e. that there is some $i_1 < i_2$ with $\lambda_{i_1} \geqslant \lambda_{i_1+1} + 2$ and $\lambda_{i_2} = \lambda_{i_2+1}$. Suppose further that no i with $i_1 < i < i_2$ has either of the two properties. Since $\lambda_{i_1} - \lambda_{i_1+1} \geqslant 2$, there must be some $i_3 > i_2$ such that $\pi_{i_1+1} < \pi_{i_3} < \pi_{i_1}$. If $i_2 + 1$ does not satisfy this property, i.e. $\lambda_{i_2+1} > \lambda_{i_1}$, then $\pi_{i_1}\pi_{i_2+1}\pi_{i_3}$ forms a 132 pattern. Therefore $\lambda_{i_2+1} < \lambda_{i_1}$, and clearly $\pi_{i_2} < \pi_{i_2+1}$. The entry 1 must come after π_{i_1} in π, so $\pi_{i_1}\pi_{i_2}\pi_{i_2+1}1$ forms a 4231 pattern in π.

Conversely, suppose π contains 4231. We can assume that the entry 1 of π serves as the 1 in the pattern. Otherwise the entry 1 of π occurs before the 31 of the pattern, and the three entries together form 132. Furthermore, we can choose the 2 and 3 of the pattern to be adjacent. Here is why: say $\pi_{i_1}\pi_{i_2}\pi_{i_3}1$ is the occurrence of 4231 in π. None of the entries π_i with $i_2 < i < i_3$ can satisfy $\pi_i > \pi_{i_3}$, or $\pi_{i_2}\pi_i\pi_{i_3}$ is a 132 pattern. So, we have an occurrence $\pi_{i_1}\pi_{i_2}\pi_{i_2+1}1$ of 4231 in π. Maximize i_1, so that $\pi_i < \pi_{i_2+1}$ for all $i_1 < i < i_2$. In particular, $\pi_{i_1+1} < \pi_{i_2+1} < \pi_{i_1}$, implying that $\lambda_{i_1} \geqslant \lambda_{i_1+1} + 2$ and $\lambda_{i_2} = \lambda_{i_2+1}$. Thus, λ does not satisfy condition (8).

Figure 10 shows the structural interpretation of the partitions λ satisfying condition (8) leading to the claimed generating function. We have two cases: either λ has no gap of size greater than one, or it has a first gap of size at least two. The first case is enumerated by $\prod_{i\geqslant 1}(1+x^i)$. The second case is given by first choosing $a, b \geqslant 0$ (as in the figure), and then two partitions μ and ν, both with all parts distinct, and largest parts at most b and a, respectively. (Conjugate μ.) We multiply the generating function by $x^{(a+2)(b+1)}$ to account for the rectangle in the figure. □

7.6 132 and the decreasing pattern

The enumeration sequence $i_k(132, 4321)$ begins with

$$1, 1, 2, 3, 5, 7, 10, 13, 17, 20, \ldots,$$

entry A265250 in the OEIS: partitions of k having at most two distinct parts. This generalizes to $i_k(132, \mathrm{id}_m^{\mathrm{rev}})$ for any m.

Proposition 7.7. *An indecomposable* 132*-avoider* π *avoids* $\mathrm{id}_m^{\mathrm{rev}}$ *if and only if* $\Lambda(\pi)$ *has at most* $m-2$ *distinct parts. In particular,*

$$C_{132,4321}(x) = 1 + \sum_{k\geqslant 1} \frac{x^k}{1-x^k} + \sum_{k\geqslant 1}\sum_{i\geqslant k+1} \frac{x^{k+i}}{(1-x^k)(1-x^i)}.$$

Proof. Let $\pi \in I_k(132)$ and $\lambda = \Lambda(\pi)$. We want to show that π avoids $\mathrm{id}_m^{\mathrm{rev}}$ if and only if λ has at most $m-2$ distinct parts. Suppose first that π contains $\mathrm{id}_m^{\mathrm{rev}}$. Since the entries between any two consecutive left-to-right minima of π are increasing, there must be an occurrence of $\mathrm{id}_m^{\mathrm{rev}}$ in π consisting only of left-to-right minima. By Lemma 2.5 (c), this means that λ has at least $m-1$ distinct parts. Conversely, if λ has at least $m-1$ distinct parts, then the left-to-right minima of π form a decreasing sequence of length at least m. The generating function for the special case of $m = 4$ is easy to see. □

8. Conclusions and open problems

Inversion monotone sets. We presented the first proofs of inversion monotonicity in nontrivial cases, namely for the pair $\{1324, 231\}$, and through the construction method in Section 4. The next natural step would be to prove that a pair $\{1324, p\}$ with $p \in S_4$ is inversion monotone. Our injection for $\{1324, 231\}$ is too intricate to easily be adapted to wider classes, so new ideas are needed.

Problem 8.1. Prove that $\{1324, p\}$ is inversion monotone for some pattern $p \in S_4$.

Limit sequences. Our analysis of the limit sequences of pairs $\{1324, p\}$ relies on the identification of a decomposable 1324-avoider as a pair of partitions. The same technique is applicable to any collection of patterns containing 1324, as the problem reduces to enumerating restricted partitions. However, the limit sequences of other collections are often more difficult to determine. The limit sequence of the pattern 1243 is

$$1, 2, 5, 10, 20, 37, 66, 114, 193, 317, \ldots,$$

and this sequence is not in the OEIS. The problem is essentially equivalent to enumerating $I_k(1243)$, the indecomposable 1243-avoiding permutations with k inversions.

Problem 8.2. Determine the limit sequence of the pattern 1243.

In Proposition 5.1, we characterized the sets that have limit sequences: they are precisely the sets B that contain a pattern p such that $\mathrm{inv}(p) \leqslant 1$. But when do two sets have the same limit sequence?

Problem 8.3. Find necessary and sufficient conditions for two sets of patterns to have the same limit sequence.

Higher limit sequences. We were able to prove that the pair $\{1324, 1342\}$ has a secondary limit sequence (in the sense of Section 5) given by $(2 + 2x)C_{1324,1342}(x)$. The result follows from an analysis of the injection of Linusson and Verkama [LV25] used to prove a similar result for 1324. However, we observe secondary limit sequences also for the pairs $\{1324, p\}$ with p in

$$1243,\ 1432,\ 2341,\ 3421,\ 4321$$

without proofs. The patterns 1432 and 4321 are f-compatible (in the sense of Section 6), so there is hope of applying the same method as for 1342. The other three patterns are not f-compatible, and it is not clear why they have secondary limit sequences. Similarly, we have no explanation for the tertiary limit sequences of the remaining pairs.

Problem 8.4. Explain why every pair $\{1324, p\}$ with $p \in S_4$ has either a secondary or a tertiary limit sequence.

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References

[AAB11] M. Albert, M. Atkinson, and R. Brignall. "The enumeration of permutations avoiding 2143 and 4231". 2011. arXiv: 1108.0989 [math.CO].

[AAB12] M. Albert, M. Atkinson, and R. Brignall. "The Enumeration of Three Pattern Classes using Monotone Grid Classes". In: *Electron. J. Comb.* 19.3 (2012), P20.

[AAV09] M. H. Albert, M. D. Atkinson, and V. Vatter. "Counting 1324, 4231-Avoiding Permutations". In: *Electron. J. Comb.* 16.1 (2009), R136.

[AAV14] M. Albert, M. Atkinson, and V. Vatter. "Inflations of geometric grid classes: three case studies". In: *Australas. J. Comb.* 58 (2014), pp. 24–47.

[Alb+25] M. Albert, C. Bean, A. Claesson, É. Nadeau, J. Pantone, and H. Ulfarsson. "Combinatorial Exploration: An algorithmic framework for enumeration". In: *Mem. Am. Math. Soc.* (2025). To appear.

[Bev+20] D. Bevan, R. Brignall, A. Elvey Price, and J. Pantone. "A structural characterisation of Av(1324) and new bounds on its growth rate". In: *Eur. J. Comb.* 88 (2020). 103115.

[Bón98] M. Bóna. "The Permutation Classes Equinumerous to the Smooth Class". In: *Electron. J. Comb.* 5.1 (1998), R31.

[BTW88] P. Bak, C. Tang, and K. Wiesenfeld. "Self-organized criticality". In: *Phys. Rev. A* 38 (1 1988), pp. 364–374.

[CG02] S. Corteel and D. Gouyou-Beauchamps. "Enumeration of sand piles". In: *Discrete Math.* LaCIM 2000 Conference on Combinatorics, Computer Science and Applications 256.3 (2002), pp. 625–643.

[CGZ18] A. R. Conway, A. J. Guttmann, and P. Zinn-Justin. "1324-avoiding permutations revisited". In: *Adv. Appl. Math.* 96 (2018), pp. 312–333.

[Cha15] J. H. C. Chan. "An infinite family of inv-Wilf-equivalent permutation pairs". In: *Eur. J. Comb.* 44 (2015), pp. 57–76.

[CJS12] A. Claesson, V. Jelínek, and E. Steingrímsson. "Upper bounds for the Stanley-Wilf limit of 1324 and other layered patterns". In: *J. Comb. Theory Ser. A* 119.8 (2012), pp. 1680–1691.

[Dok+12] T. Dokos, T. Dwyer, B. P. Johnson, B. E. Sagan, and K. Selsor. "Permutation patterns and statistics". In: *Discrete Math.* 312.18 (2012), pp. 2760–2775.

[Fra+24] A. F. Franklín, A. Claesson, C. Bean, H. Ulfarsson, and J. Pantone. "Restricted Permutations Enumerated by Inversions". In: *Electron. Proc. Theor. Comput. Sci.* 403 (2024), pp. 96–100.

[Fra25] A. F. Franklín. “Pattern Avoiding Permutations Enumerated by Inversions”. In: *Discrete Math. Theor. Comput. Sci.* vol. 27:1, Permutation Patterns 2024 (Special issues 2025).

[LS90] V. Lakshmibai and B. Sandhya. “Criterion for smoothness of Schubert varieties in $\mathrm{Sl}(n)/B$”. In: *Proc. Indian Acad. Sci. (Math. Sci.)* 100.1 (1990), pp. 45–52.

[LV25] S. Linusson and E. Verkama. “Enumerating 1324-Avoiders with Few Inversions”. In: *Electron. J. Comb.* 32.3 (2025), P3.44.

[Min16] S. Miner. “Enumeration of several two-by-four classes”. 2016. arXiv: 1610.01908 [math.CO].

[MT04] A. Marcus and G. Tardos. “Excluded permutation matrices and the Stanley-Wilf conjecture”. In: *J. Comb. Theory Ser. A* 107.1 (2004), pp. 153–160.

[OEI26] OEIS Foundation Inc. “The On-Line Encyclopedia of Integer Sequences”. Published electronically at http://oeis.org. 2026.

[Pha99] T. H. D. Phan. “Structures ordonnees et dynamiques de piles de sable”. PhD thesis. Université Paris, 1999. 110 p.

[Sta12] R. P. Stanley. “Enumerative combinatorics. Vol. 1”. Second edition. Cambridge University Press, 2012, pp. xiv+626.

[Sta90] D. Stanton. “Unimodality and Young’s lattice”. In: *J. Comb. Theory Ser. A* 54.1 (1990), pp. 41–53.

[Vat12] V. Vatter. “Finding regular insertion encodings for permutation classes”. In: *J. Symb. Comput.* 47.3 (2012), pp. 259–265.

A. Data

Table 5. The values of $\mathrm{av}_n^k(1324, 1243)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	1	5	6	5	3	1									
5	1	1	2	7	12	18	20	15	9	4	1					
6	1	1	2	3	9	14	27	43	61	70	65	49	29	14	5	1
7	1	1	2	3	5	11	17	26	50	82	122	177	226	262	263	223
8	1	1	2	3	5	7	15	21	30	48	80	125	198	290	429	593
9	1	1	2	3	5	7	11	19	28	38	56	80	124	185	272	401
10	1	1	2	3	5	7	11	15	26	36	50	70	97	133	195	273
11	1	1	2	3	5	7	11	15	22	34	48	64	91	121	163	222
12	1	1	2	3	5	7	11	15	22	30	46	62	85	115	155	204
13	1	1	2	3	5	7	11	15	22	30	42	60	83	109	149	196
14	1	1	2	3	5	7	11	15	22	30	42	56	81	107	143	190
15	1	1	2	3	5	7	11	15	22	30	42	56	77	105	141	184

Table 6. The values of $\mathrm{av}_{n+1}^k(1324, 1243) - \mathrm{av}_n^k(1324, 1243)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1														
2	0	1	2	1												
3	0	−1	3	5	5	3	1									
4	0	0	−3	1	7	15	19	15	9	4	1					
5	0	0	0	−4	−3	−4	7	28	52	66	64	49	29	14	5	1
6	0	0	0	0	−4	−3	−10	−17	−11	12	57	128	197	248	258	222
7	0	0	0	0	0	−4	−2	−5	−20	−34	−42	−52	−28	28	166	370
8	0	0	0	0	0	0	−4	−2	−2	−10	−24	−45	−74	−105	−157	−192
9	0	0	0	0	0	0	0	−4	−2	−2	−6	−10	−27	−52	−77	−128
10	0	0	0	0	0	0	0	0	−4	−2	−2	−6	−6	−12	−32	−51
11	0	0	0	0	0	0	0	0	0	−4	−2	−2	−6	−6	−8	−18
12	0	0	0	0	0	0	0	0	0	0	−4	−2	−2	−6	−6	−8
13	0	0	0	0	0	0	0	0	0	0	0	−4	−2	−2	−6	−6
14	0	0	0	0	0	0	0	0	0	0	0	0	−4	−2	−2	−6

Table 7. The values of $\mathrm{av}_n^k(1324, 2143)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	4	6	5	3	1									
5	1	2	4	6	12	16	18	15	9	4	1					
6	1	2	4	6	10	18	26	36	50	60	58	46	29	14	5	1
7	1	2	4	6	10	14	28	36	52	70	104	135	168	200	212	193
8	1	2	4	6	10	14	22	38	52	70	96	130	184	245	310	400
9	1	2	4	6	10	14	22	30	54	70	96	126	174	224	318	403
10	1	2	4	6	10	14	22	30	44	72	96	126	170	224	294	386
11	1	2	4	6	10	14	22	30	44	60	98	126	170	220	294	378
12	1	2	4	6	10	14	22	30	44	60	84	128	170	220	290	378
13	1	2	4	6	10	14	22	30	44	60	84	112	172	220	290	374
14	1	2	4	6	10	14	22	30	44	60	84	112	154	222	290	374
15	1	2	4	6	10	14	22	30	44	60	84	112	154	202	292	374

Table 8. The values of $\mathrm{av}_{n+1}^k(1324, 2143) - \mathrm{av}_n^k(1324, 2143)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1														
2	0	1	2	1												
3	0	0	2	5	5	3	1									
4	0	0	0	0	7	13	17	15	9	4	1					
5	0	0	0	0	−2	2	8	21	41	56	57	46	29	14	5	1
6	0	0	0	0	0	−4	2	0	2	10	46	89	139	186	207	192
7	0	0	0	0	0	0	−6	2	0	0	−8	−5	16	45	98	207
8	0	0	0	0	0	0	0	−8	2	0	0	−4	−10	−21	8	3
9	0	0	0	0	0	0	0	0	−10	2	0	0	−4	0	−24	−17
10	0	0	0	0	0	0	0	0	0	−12	2	0	0	−4	0	−8
11	0	0	0	0	0	0	0	0	0	0	−14	2	0	0	−4	0
12	0	0	0	0	0	0	0	0	0	0	0	−16	2	0	0	−4
13	0	0	0	0	0	0	0	0	0	0	0	0	−18	2	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	−20	2	0

Table 9. The values of $\mathrm{av}_n^k(1324, 1342)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	4	6	5	3	1									
5	1	2	4	8	12	16	18	15	9	4	1					
6	1	2	4	8	14	22	32	44	54	60	58	46	29	14	5	1
7	1	2	4	8	14	24	38	56	80	110	142	175	204	220	218	193
8	1	2	4	8	14	24	40	62	92	134	188	256	338	431	534	642
9	1	2	4	8	14	24	40	64	98	146	212	300	416	564	746	967
10	1	2	4	8	14	24	40	64	100	152	224	324	460	640	876	1180
11	1	2	4	8	14	24	40	64	100	154	230	336	484	684	952	1308
12	1	2	4	8	14	24	40	64	100	154	232	342	496	708	996	1384
13	1	2	4	8	14	24	40	64	100	154	232	344	502	720	1020	1428
14	1	2	4	8	14	24	40	64	100	154	232	344	504	726	1032	1452
15	1	2	4	8	14	24	40	64	100	154	232	344	504	728	1038	1464

Table 10. The values of $\mathrm{av}_{n+1}^k(1324, 1342) - \mathrm{av}_n^k(1324, 1342)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1														
2	0	1	2	1												
3	0	0	2	5	5	3	1									
4	0	0	0	2	7	13	17	15	9	4	1					
5	0	0	0	0	2	6	14	29	45	56	57	46	29	14	5	1
6	0	0	0	0	0	2	6	12	26	50	84	129	175	206	213	192
7	0	0	0	0	0	0	2	6	12	24	46	81	134	211	316	449
8	0	0	0	0	0	0	0	2	6	12	24	44	78	133	212	325
9	0	0	0	0	0	0	0	0	2	6	12	24	44	76	130	213
10	0	0	0	0	0	0	0	0	0	2	6	12	24	44	76	128
11	0	0	0	0	0	0	0	0	0	0	2	6	12	24	44	76
12	0	0	0	0	0	0	0	0	0	0	0	2	6	12	24	44
13	0	0	0	0	0	0	0	0	0	0	0	0	2	6	12	24
14	0	0	0	0	0	0	0	0	0	0	0	0	0	2	6	12

Table 11. The values of $\mathrm{av}_n^k(1324, 1432)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	5	5	5	3	1									
5	1	2	5	9	13	15	17	13	9	4	1					
6	1	2	5	9	17	23	36	43	52	55	50	41	26	14	5	1
7	1	2	5	9	17	27	42	59	87	112	140	163	189	195	187	163
8	1	2	5	9	17	27	46	65	98	136	194	253	333	408	494	580
9	1	2	5	9	17	27	46	69	104	148	212	287	402	527	694	883
10	1	2	5	9	17	27	46	69	108	154	224	309	432	575	783	1026
11	1	2	5	9	17	27	46	69	108	158	230	321	454	613	833	1100
12	1	2	5	9	17	27	46	69	108	158	234	327	466	635	871	1162
13	1	2	5	9	17	27	46	69	108	158	234	331	472	647	893	1200
14	1	2	5	9	17	27	46	69	108	158	234	331	476	653	905	1222
15	1	2	5	9	17	27	46	69	108	158	234	331	476	657	911	1234

Table 12. The values of $\mathrm{av}_{n+1}^k(1324, 1432) - \mathrm{av}_n^k(1324, 1432)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1														
2	0	1	2	1												
3	0	0	3	4	5	3	1									
4	0	0	0	4	8	12	16	13	9	4	1					
5	0	0	0	0	4	8	19	30	43	51	49	41	26	14	5	1
6	0	0	0	0	0	4	6	16	35	57	90	122	163	181	182	162
7	0	0	0	0	0	0	4	6	11	24	54	90	144	213	307	417
8	0	0	0	0	0	0	0	4	6	12	18	34	69	119	200	303
9	0	0	0	0	0	0	0	0	4	6	12	22	30	48	89	143
10	0	0	0	0	0	0	0	0	0	4	6	12	22	38	50	74
11	0	0	0	0	0	0	0	0	0	0	4	6	12	22	38	62
12	0	0	0	0	0	0	0	0	0	0	0	4	6	12	22	38
13	0	0	0	0	0	0	0	0	0	0	0	0	4	6	12	22
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	6	12

Table 13. The values of $\mathrm{av}_n^k(1324, 4231)$ for $n, k \leqslant 15$.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	5	6	5	2	1									
5	1	2	5	10	16	18	16	10	5	2	1					
6	1	2	5	10	20	30	45	55	55	45	30	20	10	5	2	1
7	1	2	5	10	20	34	55	82	114	146	172	172	146	114	82	55
8	1	2	5	10	20	34	59	92	137	190	262	350	441	510	532	510
9	1	2	5	10	20	34	59	96	147	216	304	412	559	738	950	1188
10	1	2	5	10	20	34	59	96	151	226	332	462	627	842	1110	1448
11	1	2	5	10	20	34	59	96	151	230	342	492	681	924	1236	1618
12	1	2	5	10	20	34	59	96	151	230	346	502	713	982	1324	1768
13	1	2	5	10	20	34	59	96	151	230	346	506	723	1016	1386	1862
14	1	2	5	10	20	34	59	96	151	230	346	506	727	1026	1422	1928
15	1	2	5	10	20	34	59	96	151	230	346	506	727	1030	1432	1966

Table 14. The values of $\mathrm{av}_{n+1}^k(1324, 4231) - \mathrm{av}_n^k(1324, 4231)$ for $n, k \leqslant 15$.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1														
2	0	1	2	1												
3	0	0	3	5	5	2	1									
4	0	0	0	4	11	16	15	10	5	2	1					
5	0	0	0	0	4	12	29	45	50	43	29	20	10	5	2	1
6	0	0	0	0	0	4	10	27	59	101	142	152	136	109	80	54
7	0	0	0	0	0	0	4	10	23	44	90	178	295	396	450	455
8	0	0	0	0	0	0	0	4	10	26	42	62	118	228	418	678
9	0	0	0	0	0	0	0	0	4	10	28	50	68	104	160	260
10	0	0	0	0	0	0	0	0	0	4	10	30	54	82	126	170
11	0	0	0	0	0	0	0	0	0	0	4	10	32	58	88	150
12	0	0	0	0	0	0	0	0	0	0	0	4	10	34	62	94
13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	36	66
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	38

Table 15. The values of $\mathrm{av}_n^k(1324, 4321)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	5	6	5	3	0									
5	1	2	5	10	16	20	18	11	3	0	0					
6	1	2	5	10	20	32	49	61	65	50	26	10	1	0	0	0
7	1	2	5	10	20	36	59	90	130	168	192	189	153	96	48	15
8	1	2	5	10	20	36	63	100	153	218	307	394	483	525	531	477
9	1	2	5	10	20	36	63	104	163	242	349	478	640	820	1012	1177
10	1	2	5	10	20	36	63	104	167	252	373	524	720	946	1233	1530
11	1	2	5	10	20	36	63	104	167	256	383	548	766	1034	1365	1738
12	1	2	5	10	20	36	63	104	167	256	387	558	790	1080	1453	1882
13	1	2	5	10	20	36	63	104	167	256	387	562	800	1104	1499	1970
14	1	2	5	10	20	36	63	104	167	256	387	562	804	1114	1523	2016
15	1	2	5	10	20	36	63	104	167	256	387	562	804	1118	1533	2040

Table 16. The values of $\mathrm{av}_{n+1}^k(1324, 4321) - \mathrm{av}_n^k(1324, 4321)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1														
2	0	1	2	1												
3	0	0	3	5	5	3	0									
4	0	0	0	4	11	17	18	11	3	0	0					
5	0	0	0	0	4	12	31	50	62	50	26	10	1	0	0	0
6	0	0	0	0	0	4	10	29	65	118	166	179	152	96	48	15
7	0	0	0	0	0	0	4	10	23	50	115	205	330	429	483	462
8	0	0	0	0	0	0	0	4	10	24	42	84	157	295	481	700
9	0	0	0	0	0	0	0	0	4	10	24	46	80	126	221	353
10	0	0	0	0	0	0	0	0	0	4	10	24	46	88	132	208
11	0	0	0	0	0	0	0	0	0	0	4	10	24	46	88	144
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	46	88
13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	46
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24

Table 17. The values of $\mathrm{av}_n^k(1324, 2341)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	5	5	5	3	1									
5	1	2	5	8	13	16	15	13	9	4	1					
6	1	2	5	8	16	23	32	39	48	51	44	37	26	14	5	1
7	1	2	5	8	16	26	39	56	77	98	114	131	150	161	155	133
8	1	2	5	8	16	26	42	63	94	129	171	214	266	319	369	411
9	1	2	5	8	16	26	42	66	101	146	202	274	366	464	574	693
10	1	2	5	8	16	26	42	66	104	153	219	305	426	568	739	940
11	1	2	5	8	16	26	42	66	104	156	226	322	457	628	843	1110
12	1	2	5	8	16	26	42	66	104	156	229	329	474	659	903	1214
13	1	2	5	8	16	26	42	66	104	156	229	332	481	676	934	1274
14	1	2	5	8	16	26	42	66	104	156	229	332	484	683	951	1305
15	1	2	5	8	16	26	42	66	104	156	229	332	484	686	958	1322

Table 18. The values of $\mathrm{av}_{n+1}^k(1324, 2341) - \mathrm{av}_n^k(1324, 2341)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1														
2	0	1	2	1												
3	0	0	3	4	5	3	1									
4	0	0	0	3	8	13	14	13	9	4	1					
5	0	0	0	0	3	7	17	26	39	47	43	37	26	14	5	1
6	0	0	0	0	0	3	7	17	29	47	70	94	124	147	150	132
7	0	0	0	0	0	0	3	7	17	31	57	83	116	158	214	278
8	0	0	0	0	0	0	0	3	7	17	31	60	100	145	205	282
9	0	0	0	0	0	0	0	0	3	7	17	31	60	104	165	247
10	0	0	0	0	0	0	0	0	0	3	7	17	31	60	104	170
11	0	0	0	0	0	0	0	0	0	0	3	7	17	31	60	104
12	0	0	0	0	0	0	0	0	0	0	0	3	7	17	31	60
13	0	0	0	0	0	0	0	0	0	0	0	0	3	7	17	31
14	0	0	0	0	0	0	0	0	0	0	0	0	0	3	7	17

Table 19. The values of $\mathrm{av}_n^k(1324, 2413)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	5	5	5	3	1									
5	1	2	5	10	12	15	16	13	9	4	1					
6	1	2	5	10	20	25	33	42	49	47	47	39	26	14	5	1
7	1	2	5	10	20	36	51	69	86	110	132	146	155	163	157	141
8	1	2	5	10	20	36	65	93	135	178	223	276	336	388	442	483
9	1	2	5	10	20	36	65	110	165	241	335	444	557	690	826	980
10	1	2	5	10	20	36	65	110	185	277	413	582	803	1056	1347	1671
11	1	2	5	10	20	36	65	110	185	300	455	675	971	1354	1837	2428
12	1	2	5	10	20	36	65	110	185	300	481	723	1079	1552	2195	3014
13	1	2	5	10	20	36	65	110	185	300	481	752	1133	1675	2423	3432
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1735	2561	3690
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2627	3843

Table 20. The values of $\mathrm{av}_{n+1}^k(1324, 2413) - \mathrm{av}_n^k(1324, 2413)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1														
2	0	1	2	1												
3	0	0	3	4	5	3	1									
4	0	0	0	5	7	12	15	13	9	4	1					
5	0	0	0	0	8	10	17	29	40	43	46	39	26	14	5	1
6	0	0	0	0	0	11	18	27	37	63	85	107	129	149	152	140
7	0	0	0	0	0	0	14	24	49	68	91	130	181	225	285	342
8	0	0	0	0	0	0	0	17	30	63	112	168	221	302	384	497
9	0	0	0	0	0	0	0	0	20	36	78	138	246	366	521	691
10	0	0	0	0	0	0	0	0	0	23	42	93	168	298	490	757
11	0	0	0	0	0	0	0	0	0	0	26	48	108	198	358	586
12	0	0	0	0	0	0	0	0	0	0	0	29	54	123	228	418
13	0	0	0	0	0	0	0	0	0	0	0	0	32	60	138	258
14	0	0	0	0	0	0	0	0	0	0	0	0	0	35	66	153

Table 21. The values of $\mathrm{av}_n^k(1324, 2431)$ for $n, k \leqslant 15$.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	5	6	4	3	1									
5	1	2	5	10	15	17	15	11	7	4	1					
6	1	2	5	10	19	30	43	50	50	49	39	29	19	11	5	1
7	1	2	5	10	19	34	55	80	114	140	153	165	161	150	132	105
8	1	2	5	10	19	34	59	93	140	202	278	352	420	476	518	544
9	1	2	5	10	19	34	59	97	154	231	335	468	639	823	1014	1218
10	1	2	5	10	19	34	59	97	158	246	366	534	754	1033	1393	1799
11	1	2	5	10	19	34	59	97	158	250	382	567	825	1166	1615	2187
12	1	2	5	10	19	34	59	97	158	250	386	584	860	1242	1758	2440
13	1	2	5	10	19	34	59	97	158	250	386	588	878	1279	1839	2593
14	1	2	5	10	19	34	59	97	158	250	386	588	882	1298	1878	2679
15	1	2	5	10	19	34	59	97	158	250	386	588	882	1302	1898	2720

Table 22. The values of $\mathrm{av}_{n+1}^k(1324, 2431) - \mathrm{av}_n^k(1324, 2431)$ for $n, k \leqslant 15$.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1														
2	0	1	2	1												
3	0	0	3	5	4	3	1									
4	0	0	0	4	11	14	14	11	7	4	1					
5	0	0	0	0	4	13	28	39	43	45	38	29	19	11	5	1
6	0	0	0	0	0	4	12	30	64	91	114	136	142	139	127	104
7	0	0	0	0	0	0	4	13	26	62	125	187	259	326	386	439
8	0	0	0	0	0	0	0	4	14	29	57	116	219	347	496	674
9	0	0	0	0	0	0	0	0	4	15	31	66	115	210	379	581
10	0	0	0	0	0	0	0	0	0	4	16	33	71	133	222	388
11	0	0	0	0	0	0	0	0	0	0	4	17	35	76	143	253
12	0	0	0	0	0	0	0	0	0	0	0	4	18	37	81	153
13	0	0	0	0	0	0	0	0	0	0	0	0	4	19	39	86
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	20	41

Table 23. The values of $\mathrm{av}_n^k(1324, 3412)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	5	6	4	3	1									
5	1	2	5	10	14	16	16	11	6	4	1					
6	1	2	5	10	18	30	37	46	46	45	38	27	16	8	5	1
7	1	2	5	10	18	34	53	74	98	118	134	139	134	123	106	86
8	1	2	5	10	18	34	57	92	130	184	237	294	349	373	400	407
9	1	2	5	10	18	34	57	96	150	220	313	422	551	690	826	945
10	1	2	5	10	18	34	57	96	154	242	353	508	699	934	1223	1526
11	1	2	5	10	18	34	57	96	154	246	377	552	795	1102	1503	1998
12	1	2	5	10	18	34	57	96	154	246	381	578	843	1208	1691	2314
13	1	2	5	10	18	34	57	96	154	246	381	582	871	1260	1807	2522
14	1	2	5	10	18	34	57	96	154	246	381	582	875	1290	1863	2648
15	1	2	5	10	18	34	57	96	154	246	381	582	875	1294	1895	2708

Table 24. The values of $\mathrm{av}_{n+1}^k(1324, 3412) - \mathrm{av}_n^k(1324, 3412)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1														
2	0	1	2	1												
3	0	0	3	5	4	3	1									
4	0	0	0	4	10	13	15	11	6	4	1					
5	0	0	0	0	4	14	21	35	40	41	37	27	16	8	5	1
6	0	0	0	0	0	4	16	28	52	73	96	112	118	115	101	85
7	0	0	0	0	0	0	4	18	32	66	103	155	215	250	294	321
8	0	0	0	0	0	0	0	4	20	36	76	128	202	317	426	538
9	0	0	0	0	0	0	0	0	4	22	40	86	148	244	397	581
10	0	0	0	0	0	0	0	0	0	4	24	44	96	168	280	472
11	0	0	0	0	0	0	0	0	0	0	4	26	48	106	188	316
12	0	0	0	0	0	0	0	0	0	0	0	4	28	52	116	208
13	0	0	0	0	0	0	0	0	0	0	0	0	4	30	56	126
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	32	60

Table 25. The values of $\mathrm{av}_n^k(1324, 3421)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	2	5	6	5	2	1									
5	1	2	5	10	16	18	16	10	5	2	1					
6	1	2	5	10	20	30	47	55	53	43	31	20	10	5	2	1
7	1	2	5	10	20	34	57	84	122	152	162	160	138	111	80	55
8	1	2	5	10	20	34	61	94	145	208	292	369	437	471	478	453
9	1	2	5	10	20	34	61	98	155	232	340	470	645	830	1003	1174
10	1	2	5	10	20	34	61	98	159	242	364	522	740	1003	1344	1725
11	1	2	5	10	20	34	61	98	159	246	374	546	792	1106	1517	2034
12	1	2	5	10	20	34	61	98	159	246	378	556	816	1158	1620	2219
13	1	2	5	10	20	34	61	98	159	246	378	560	826	1182	1672	2322
14	1	2	5	10	20	34	61	98	159	246	378	560	830	1192	1696	2374
15	1	2	5	10	20	34	61	98	159	246	378	560	830	1196	1706	2398

Table 26. The values of $\mathrm{av}_{n+1}^k(1324, 3421) - \mathrm{av}_n^k(1324, 3421)$ for $n, k \leqslant 15$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1														
2	0	1	2	1												
3	0	0	3	5	5	2	1									
4	0	0	0	4	11	16	15	10	5	2	1					
5	0	0	0	0	4	12	31	45	48	41	30	20	10	5	2	1
6	0	0	0	0	0	4	10	29	69	109	131	140	128	106	78	54
7	0	0	0	0	0	0	4	10	23	56	130	209	299	360	398	398
8	0	0	0	0	0	0	0	4	10	24	48	101	208	359	525	721
9	0	0	0	0	0	0	0	0	4	10	24	52	95	173	341	551
10	0	0	0	0	0	0	0	0	0	4	10	24	52	103	173	309
11	0	0	0	0	0	0	0	0	0	0	4	10	24	52	103	185
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	52	103
13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	52
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24

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