Self-modified difference ascent sequences

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Abstract

Ascent sequences play a key role in the combinatorics of Fishburn structures. Difference ascent sequences are a natural generalization obtained by replacing ascents with d-ascents. We have recently extended the so-called hat map to difference ascent sequences, and self-modified difference ascent sequences are the fixed points under this map. We characterize self-modified difference ascent sequences and enumerate them in terms of certain generalized Fibonacci polynomials. Furthermore, we describe the corresponding subset of d-Fishburn permutations.

1 Introduction

Let n be a nonnegative integer and let $\alpha : [n] \to [n]$ be an endofunction, where $[n] = \{1, 2, \ldots, n\}$. For succinctness, we may identify α with the word $\alpha = a_1 \ldots a_n$, where $a_i = \alpha(i)$ for each $i \in [n]$. An index $i \in [n]$ is an *ascent* of α if $i = 1$ or $i \geq 2$ and $a_i > a_{i-1}$. We define the *ascent set* of α by

$$\operatorname{Asc} \alpha = \{i \in [n] \mid i \text{ is an ascent of } \alpha\}$$

and we let $\operatorname{asc} \alpha = \# \operatorname{Asc} \alpha$ denote the number of ascent in α. Our conventions differ from some others in the literature in that we are taking the indices of ascent tops, rather than bottoms, and that the first position is always an ascent.

We call α an *ascent sequence of length n* if for all $i \in [n]$ we have

$$a_i \leq 1 + \operatorname{asc}(a_1 \ldots a_{i-1}).$$

In particular, $a_1 \leq 1 + \operatorname{asc} \epsilon = 1$, where ϵ denotes the empty sequence. Since the entries of α are positive integers, this forces $a_1 = 1$. We let

$$\mathrm{A} = \{\alpha \mid \alpha \text{ is an ascent sequence}\}.$$

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As an example, the ascent sequences of length 3 are

$$111, 112, 121, 122, 123.$$

Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes, and Kitaev in 2010 [BMCDK10]. They played a crucial role in the enumeration of interval orders. In particular, the number of ascent sequences of length n is given by the nth Fishburn number (A022493 in the OEIS [OEI]), which is also the number of unlabeled interval orders of size n. Since then, ascent sequences have also been studied in their own right [CDD+13, DS11, MS14, KR17, CCEPG22, GK23, BP15, CMS14, Yan14, FJL+20, JS23]. Different variants and generalizations of ascent sequences have been proposed. Of particular interest to us are the weak ascent sequences [BCD23] and, more generally, the difference ascent sequences recently introduced by Dukes and Sagan [DS24].

Let d be a nonnegative integer. An index $i \in [n]$ is a *d-ascent* if $i = 1$ or $i \geq 2$ and

$$a_i > a_{i-1} - d.$$

As with ordinary ascents, we define the *d-ascent set* of α by

$$\operatorname{Asc}_d \alpha = \{i \in [n] \mid i \text{ is a } d\text{-ascent of } \alpha\}.$$

and we let $\operatorname{asc}_d \alpha = \# \operatorname{Asc}_d \alpha$ denote the number of d-ascent in α. Note that a 0-ascent is simply an ascent, while a 1-ascent ($a_i > a_{i-1} - 1$ or equivalently $a_i \geq a_{i-1}$) is what is called a *weak ascent*.

We call α a *d-ascent sequence* if for all $i \in [n]$ we have

$$a_i \leq 1 + \operatorname{asc}_d(a_1 \dots a_{i-1}).$$

Once again, the above restriction forces $a_1 = 1$. Let

$$\mathrm{A}_{d,n} = \{\alpha \mid \alpha \text{ is a } d\text{-ascent sequences of length } n\}$$

and

$$\mathrm{A}_d = \biguplus_{n \geq 0} \mathrm{A}_{d,n}\,.$$

Clearly, for $d = 0$ we recover the set of ascent sequences, while for $d = 1$ we obtain the set of *weak ascent sequences* of Bényi, Claesson, and Dukes [BCD23].

Given an ascent sequence α we can form the corresponding *modified ascent sequence* [BMCDK10] as follows: Scan the ascents of α from left to right; at each step, every element strictly to the left of and weakly larger than the current ascent is incremented by one. More formally, let

$$M(\alpha, j) = \alpha^+, \text{ where } \alpha^+(i) = a_i + \begin{cases} 1 & \text{if } i < j \text{ and } a_i \geq a_j, \\ 0 & \text{otherwise,} \end{cases}$$

and extend the definition of M to multiple indices $j_1, j_2, \dots, j_k$ by

$$M(\alpha, j_1, j_2, \dots, j_k) = M\big(M(\alpha, j_1, \dots, j_{k-1}), j_k\big).$$

Then

$$\hat{\alpha} = M(\alpha, \operatorname{Asc} \alpha),$$

where in this context $\operatorname{Asc} \alpha$ is the ascent list (rather than set) of α. For example, consider $\alpha = 121242232$. The ascent set of α is $\{1, 2, 4, 5, 8\}$; the ascent list is formed by listing those entries is increasing order, $\operatorname{Asc} \alpha = (1, 2, 4, 5, 8)$, and we compute $\hat{\alpha}$ as follows, where at each stage the entry governing the modification is underlined while the entries which are modified are bold:

$$\begin{aligned}
\alpha &= 121242232 \\
M(\alpha, 1) &= \underline{1}21242232 \\
M(\alpha, 1, 2) &= 1\underline{2}1242232 \\
M(\alpha, 1, 2, 4) &= 1\mathbf{3}1\underline{2}42232 \\
M(\alpha, 1, 2, 4, 5) &= 1312\underline{4}2232 \\
M(\alpha, 1, 2, 4, 5, 8) &= 1\mathbf{4}12\mathbf{5}22\underline{3}2 = \hat{\alpha}
\end{aligned}$$

Let

$$\hat{\mathrm{A}} = \{\hat{\alpha} \mid \alpha \in \mathrm{A}\}.$$

The construction described above can easily be inverted since $\operatorname{Asc} \alpha = \operatorname{Asc} \hat{\alpha}$. In other words, the mapping $\mathrm{A} \to \hat{\mathrm{A}}$ by $\alpha \mapsto \hat{\alpha}$ is a bijection.

We [CCS24] have recently extended the "hat map", $\alpha \mapsto \hat{\alpha}$, to d-ascent sequences: Let $d \geq 0$ and $\alpha \in \mathrm{A}_d$. Define *d-hat* of α as

$$\operatorname{hat}_d(\alpha) = M\big(\alpha, \operatorname{Asc}_d \alpha\big),$$

where $\operatorname{Asc}_d \alpha$ is the d-ascent list of α obtained by putting the set in increasing order, and let

$$\hat{\mathrm{A}}_d = \operatorname{hat}_d(\mathrm{A}_d)$$

denote the set of *modified d-ascent sequences*. As a special case, $\operatorname{hat}_0(\alpha) = \hat{\alpha}$, for each $\alpha \in \mathrm{A}_0$, and $\hat{\mathrm{A}}_0$ coincides with the set of modified ascent sequences defined by Bousquet-Mélou et al. [BMCDK10]. We will also use $\hat{\alpha}$ for $\operatorname{hat}_d(\alpha)$ if d is clear from context.

One may alternatively define the set $\hat{\mathrm{A}}_d$ recursively [CCS24]: Let $d \geq 0$ be a non-negative integer. Let $\hat{\mathrm{A}}_{d,0} = \{\epsilon\}$ and $\hat{\mathrm{A}}_{d,1} = \{1\}$. Suppose $n \geq 2$. Every $\hat{\alpha} \in \hat{\mathrm{A}}_{d,n}$ is of one of two forms depending on whether the last letter forms a d-ascent with the penultimate letter:

- $\hat{\alpha} = \hat{\beta} a$ and $1 \leq a \leq b - d$, or
- $\hat{\alpha} = \hat{\beta}^+ a$ and $b - d < a \leq 1 + \max \hat{\beta}$,

where $\hat{\beta} \in \hat{\mathrm{A}}_{d,n-1}$ and the last letter of $\hat{\beta}$ is b.

Let us recall a few more definitions and results from [CCS24] that we shall need below. An endofunction $\alpha = a_1 \ldots a_n$ is an *inversion sequence* if $a_i \leq i$ for each $i \in [n]$. Let I_n denote the set of endofunctions of length n and let $\mathrm{I} = \cup_{n \geq 0} \mathrm{I}_n$. We showed that

$$\mathrm{I} = \bigcup_{d \geq 0} \mathrm{A}_d,$$

and we defined the set $\hat{\mathrm{I}}$ of *modified inversion sequences* by

$$\hat{\mathrm{I}} = \bigcup_{d \geq 0} \hat{\mathrm{A}}_d \,. \tag{1}$$

A *Cayley permutation* is an endofunction α where $\operatorname{Im}\alpha = [k]$ for some $k \leq n$. Thus, a nonempty endofunction α is a Cayley permutation if it contains at least one copy of each integer between 1 and its maximum element. The set of Cayley permutations of length n is denoted by Cay_n.

Proposition 1.1 ([CCS24]). *Given $d \geq 0$, let $\alpha \in \mathrm{A}_d$ and let $\hat{\alpha} = \mathrm{hat}_d(\alpha)$. Then*

(a) $\hat{\alpha}$ is a Cayley permutation,

(b) $\mathrm{Asc}_d\,\alpha = \mathrm{nub}\,\hat{\alpha}$,

(c) $\mathrm{Asc}_d\,\hat{\alpha} \subseteq \mathrm{nub}\,\hat{\alpha}$,

where

$$\mathrm{nub}\,\alpha = \{\min \alpha^{-1}(j) \mid 1 \leq j \leq \max \alpha\}$$

is the set of positions of leftmost occurrences.

2 Self-modified d-ascent sequences

Recall [BMCDK10] that an ascent sequence α is *self-modified* if $\alpha = \mathrm{hat}_0(\alpha)$. We extend this notion to d-ascent sequences by saying that $\alpha \in \mathrm{A}_d$ is *d-self-modified* if $\mathrm{hat}_d(\alpha) = \alpha$. Let

$$\widetilde{\mathrm{A}}_d = \{\alpha \in \mathrm{A}_d \mid \mathrm{hat}_d(\alpha) = \alpha\}$$

denote the set of d-self-modified d-ascent sequences; or, in short, self-modified d-ascent sequences. The main goal of this section is to obtain a better understanding of the sets $\widetilde{\mathrm{A}}_d$. Namely, in Theorem 2.5 we prove that $\widetilde{\mathrm{A}}_d = \mathrm{A}_d \cap \hat{\mathrm{A}}_d$. Further, we prove in Theorem 2.6 that $\widetilde{\mathrm{A}}_{d+1} \subseteq \widetilde{\mathrm{A}}_d$ for each $d \geq 0$, from which the curious chain of inclusions

$$\cdots \subseteq \widetilde{\mathrm{A}}_2 \subseteq \widetilde{\mathrm{A}}_1 \subseteq \widetilde{\mathrm{A}}_0 \subseteq \mathrm{A}_0 \subseteq \mathrm{A}_1 \subseteq \mathrm{A}_2 \subseteq \cdots \tag{2}$$

follows. We start with a couple of simple lemmas.

Let $\alpha = a_1 \dots a_n$ be an endofunction. An index $i \in [n]$ is a *left-right maximum* of α if $a_i > a_j$ for each $j \in [i-1]$. We will use the notation

$$\mathrm{lrMax}\,\alpha = \{i \in [n] \mid i \text{ is a left-right maximum of } \alpha\}.$$

It is easy to see that

$$\mathrm{lrMax}\,\alpha \subseteq \mathrm{nub}\,\alpha \quad \text{and} \quad \mathrm{lrMax}\,\alpha \subseteq \mathrm{Asc}_0\,\alpha, \tag{3}$$

two facts we will use often in this section.

Lemma 2.1. *For every $\alpha \in \mathrm{I}$ and $d \geq 0$, we have $\mathrm{lrMax}\,\alpha \subseteq \mathrm{Asc}_d\,\alpha$.*

Proof. By equation (3), we have

$$\operatorname{lrMax}\alpha \subseteq \operatorname{Asc}_0\alpha \subseteq \operatorname{Asc}_d\alpha, \tag{4}$$

where the last set containment follows by definition of d-ascent. □

Lemma 2.2. *Let $\beta \in \mathrm{A}_{d,n}$ and let $\hat{\beta} = \operatorname{hat}_d(\beta)$. Let $\hat{\beta}^+$ be obtained by increasing by one each entry of $\hat{\beta}$ greater than or equal to a, for some $a \geq 1$. If $\beta = \hat{\beta}^+$, then $\beta = \hat{\beta}$.*

Proof. For $i \in [n]$, denote by b_i, b_i' and b_i'' the ith entry of β, $\hat{\beta}$ and $\hat{\beta}^+$, respectively. By definition of $\hat{\beta}$ and $\hat{\beta}^+$, we have

$$b_i \leq b_i' \leq b_i''.$$

If $\beta = \hat{\beta}^+$, then $b_i = b_i' = b_i''$ for every $i \in [n]$, and our claim follows. □

Proposition 2.3. *Let $\alpha \in \mathrm{A}_d$. The following three statements are equivalent:*

(a) $\operatorname{hat}_d(\alpha) = \alpha$.

(b) $\operatorname{Asc}_d\alpha \subseteq \operatorname{lrMax}\alpha$.

(c) $\operatorname{Asc}_d\alpha \subseteq \operatorname{nub}\alpha$.

Proof. We prove (a) ⇔ (b) and (a) ⇔ (c).

Let us start by proving (a) ⇔ (b) by induction on the length of α. Our claim holds if α has length at most one. Let $\alpha \in \mathrm{A}_d$ and suppose that $\alpha = \beta a$, for some $\beta \in \mathrm{A}_{d,n}$, where $1 \leq a \leq 1 + \operatorname{asc}_d\beta$ and $n \geq 1$. Let $\hat{\alpha} = \operatorname{hat}_d(\alpha)$ and $\hat{\beta} = \operatorname{hat}_d(\beta)$. We shall consider two cases, according to whether or not a forms a d-ascent with the last letter b of β:

$$\hat{\alpha} = \begin{cases} \hat{\beta}a, & \text{if } 1 \leq a \leq b - d; \\ \hat{\beta}^+a, & \text{if } b - d < a \leq 1 + \operatorname{asc}_d\beta. \end{cases}$$

Initially, suppose that $\alpha = \hat{\alpha}$. We show that $\operatorname{Asc}_d\alpha \subseteq \operatorname{lrMax}\alpha$. If $a \leq b - d$, then $\beta = \hat{\beta}$ since $\alpha = \hat{\alpha}$. Moreover,

$$\operatorname{Asc}_d\alpha = \operatorname{Asc}_d\beta \subseteq \operatorname{lrMax}\beta = \operatorname{lrMax}\alpha,$$

where we used induction on β. Otherwise, suppose that $b - d < a$. Since we assumed $\alpha = \hat{\alpha}$, we have $\beta a = \hat{\beta}^+ a$, from which $\beta = \hat{\beta}^+$ follows. By Lemma 2.2, we have $\beta = \hat{\beta}$. Furthermore, since $\hat{\beta} = \hat{\beta}^+$, no entry of $\hat{\beta}$ is increased by one in going from $\hat{\beta}$ to $\hat{\beta}^+$. Thus, $a > c$ for each entry c of $\hat{\beta} = \beta$, i.e. $n + 1 \in \operatorname{lrMax}\alpha$. Finally,

$$\operatorname{Asc}_d\alpha = \operatorname{Asc}_d\beta \uplus \{n+1\} \subseteq \operatorname{lrMax}\beta \uplus \{n+1\} = \operatorname{lrMax}\alpha,$$

where $\operatorname{Asc}_d\beta \subseteq \operatorname{lrMax}\beta$ by induction.

To prove the converse, suppose that $\operatorname{Asc}_d\alpha \subseteq \operatorname{lrMax}\alpha$. Then

$$\operatorname{Asc}_d\beta = \operatorname{Asc}_d\alpha \cap [n] \subseteq \operatorname{lrMax}\alpha \cap [n] = \operatorname{lrMax}\beta,$$

and $\hat\beta = \beta$ follows by induction. Now, if $a \le b - d$, then

$$\hat\alpha = \hat\beta a = \beta a = \alpha,$$

as wanted. On the other hand, if $b - d < a$, then

$$n + 1 \in \operatorname{Asc}_d \alpha \subseteq \operatorname{lrMax} \alpha.$$

Since $n + 1 \in \operatorname{lrMax} \alpha$ and $\hat\beta = \beta$, no entry of $\hat\beta$ is increased by one in $\hat\beta^+$. Thus $\hat\beta^+ = \hat\beta$, and $\hat\alpha = \alpha$ follows immediately.

Let us now prove (a) $\Leftrightarrow$ (c). The proof is similar to the one just given so we shall keep the same notation. If $\hat\alpha = \alpha$, then

$$\operatorname{Asc}_d \alpha = \operatorname{nub} \hat\alpha = \operatorname{nub} \alpha, \tag{5}$$

where the first equality is item (b) of Proposition 1.1.

On the other hand, suppose that $\operatorname{Asc}_d \alpha \subseteq \operatorname{nub} \alpha$. Then

$$\operatorname{Asc}_d \beta = \operatorname{Asc}_d \alpha \cap [n] \subseteq \operatorname{nub} \alpha \cap [n] = \operatorname{nub} \beta.$$

and $\hat\beta = \beta$ by induction. Now, if $1 \le a \le b - d$, then $\hat\alpha = \hat\beta a = \beta a = \alpha$. Otherwise, suppose that $b - d < a \le 1 + \operatorname{asc}_d \beta$ and $\hat\alpha = \hat\beta^+ a = \beta^+ a$. Note that β and $\hat\alpha$ are Cayley permutations by item (a) of Proposition 1.1. Further, we have $n+1 \in \operatorname{Asc}_d \alpha \subseteq \operatorname{nub} \alpha$. That is, the last entry a is a leftmost copy in $\alpha = \beta a$. Since $\operatorname{Im} \beta = [k]$, where $k = \max \beta$, we must have $\operatorname{Im} \alpha = [k + 1]$ and $a = \max \beta + 1 = k + 1$. In particular, $\beta^+ = \beta$ and thus $\hat\alpha = \beta^+ a = \beta a = \alpha$. □

The previous proposition still holds if we replace the set inclusions with equalities in items (b) and (c), as we show next.

Corollary 2.4. *Let $\alpha \in \mathrm{A}_d$ and let $\hat\alpha = \operatorname{hat}_d(\alpha)$. Then*

$$\hat\alpha = \alpha \iff \operatorname{Asc}_d \alpha = \operatorname{lrMax} \alpha \iff \operatorname{Asc}_d \alpha = \operatorname{nub} \alpha.$$

Furthermore, if $\alpha \in \widetilde{\mathrm{A}}_d$ then $\operatorname{Asc}_d \alpha = \operatorname{Asc}_0 \alpha$.

Proof. To prove the first part it is enough, by Proposition 2.3, to show that $\hat\alpha = \alpha$ implies

$$\operatorname{Asc}_d \alpha = \operatorname{lrMax} \alpha = \operatorname{nub} \alpha,$$

where the inclusions $\operatorname{Asc}_d \alpha \subseteq \operatorname{lrMax} \alpha$ and $\operatorname{Asc}_d \alpha \subseteq \operatorname{nub} \alpha$ hold by the same proposition. The equality $\operatorname{Asc}_d \alpha = \operatorname{lrMax} \alpha$ now follows directly from Lemma 2.1. And $\operatorname{Asc}_d \alpha = \operatorname{nub} \alpha$ was proved in (5).

Let us now prove that $\operatorname{Asc}_d \alpha = \operatorname{Asc}_0 \alpha$ if $\alpha \in \widetilde{\mathrm{A}}_d$. Using equation (4) and Proposition 2.3 once more, we obtain

$$\operatorname{lrMax} \alpha \subseteq \operatorname{Asc}_0 \alpha \subseteq \operatorname{Asc}_d \alpha \subseteq \operatorname{lrMax} \alpha,$$

from which the desired equality follows. □

We are now ready for the promised characterization of $\widetilde{\mathrm{A}}_d$.

Theorem 2.5. *For each $d \geq 0$, we have $\widetilde{\mathrm{A}}_d = \mathrm{A}_d \cap \hat{\mathrm{A}}_d$.*

Proof. If $\alpha \in \widetilde{\mathrm{A}}_d$, then $\alpha \in \mathrm{A}_d$ and $\alpha = \mathrm{hat}_d(\alpha) \in \hat{\mathrm{A}}_d$ as well. On the other hand, suppose that $\alpha \in \mathrm{A}_d \cap \hat{\mathrm{A}}_d$. Since $\alpha \in \hat{\mathrm{A}}_d$, we have $\mathrm{Asc}_d\, \alpha \subseteq \mathrm{nub}\, \alpha$ by item (c) of Proposition 1.1. Then $\mathrm{hat}_d(\alpha) = \alpha$ by Proposition 2.3, i.e. we have $\alpha \in \widetilde{\mathrm{A}}_d$. □

We now have all the ingredients to prove equation (2), which is an immediate consequence of item (b) of the following theorem.

Theorem 2.6. *For every $d \geq 0$, we have:*

(a) $\widetilde{\mathrm{A}}_d \subseteq \widetilde{\mathrm{A}}_0$.

(b) $\widetilde{\mathrm{A}}_d \subseteq \widetilde{\mathrm{A}}_k$ *for each* $0 \leq k \leq d$. *In particular,* $\widetilde{\mathrm{A}}_{d+1} \subseteq \widetilde{\mathrm{A}}_d$.

(c) $\widetilde{\mathrm{A}}_d = \widetilde{\mathrm{A}}_0 \cap \hat{\mathrm{A}}_d$.

Proof. (a) Let $\alpha = a_1 \dots a_n \in \widetilde{\mathrm{A}}_d$, where $n \geq 1$. By Corollary 2.4, we have $\mathrm{Asc}_d\, \alpha = \mathrm{Asc}_0\, \alpha$. Therefore, we have $a_1 = 1$ and

$$a_{i+1} \leq 1 + \mathrm{asc}_d(a_1 \dots a_i) = 1 + \mathrm{asc}_0(a_1 \dots a_i)$$

for each $i \in [n-1]$, from which $\alpha \in \mathrm{A}_0$. Finally,

$$\mathrm{Asc}_0\, \alpha = \mathrm{Asc}_d\, \alpha = \mathrm{lrMax}\, \alpha$$

and $\mathrm{hat}_0(\alpha) = \alpha$ follows by Corollary 2.4 again.

(b) Let $\alpha \in \widetilde{\mathrm{A}}_d$ and $0 \leq k \leq d$. We prove that $\alpha \in \mathrm{A}_k$ and $\mathrm{hat}_k(\alpha) = \alpha$. Using item (a) and Theorem 2.5, we have that

$$\alpha \in \widetilde{\mathrm{A}}_d \subseteq \widetilde{\mathrm{A}}_0 = \mathrm{A}_0 \cap \hat{\mathrm{A}}_0 \subseteq \mathrm{A}_0 \subseteq \mathrm{A}_k\, .$$

Furthermore,

$$\begin{aligned} \mathrm{lrMax}\, \alpha &\subseteq \mathrm{Asc}_k\, \alpha && \text{(by Lemma 2.1)} \\ &\subseteq \mathrm{Asc}_d\, \alpha && \text{(since } k \leq d\text{)} \\ &= \mathrm{lrMax}\, \alpha && \text{(by Corollary 2.4).} \end{aligned}$$

Thus $\mathrm{lrMax}\, \alpha = \mathrm{Asc}_k\, \alpha$, and $\mathrm{hat}_k(\alpha) = \alpha$ follows by Corollary 2.4.

(c) We start with the inclusion $\widetilde{\mathrm{A}}_d \subseteq \widetilde{\mathrm{A}}_0 \cap \hat{\mathrm{A}}_d$. By Theorem 2.5,

$$\widetilde{\mathrm{A}}_d = \mathrm{A}_d \cap \hat{\mathrm{A}}_d \subseteq \hat{\mathrm{A}}_d\, .$$

Note also that $\widetilde{\mathrm{A}}_d \subseteq \widetilde{\mathrm{A}}_0$ by item (a) of this theorem, from which the desired inclusion follows. To prove the opposite inclusion, we can once again use Theorem 2.5 to get

$$\widetilde{\mathrm{A}}_0 = \mathrm{A}_0 \cap \hat{\mathrm{A}}_0 \subseteq \mathrm{A}_0 \subseteq \mathrm{A}_d$$

and

$$\widetilde{\mathrm{A}}_0 \cap \hat{\mathrm{A}}_d \subseteq \mathrm{A}_d \cap \hat{\mathrm{A}}_d = \widetilde{\mathrm{A}}_d,$$

which concludes the proof. □

The only inversion sequence that is d-self-modified for every $d \geq 0$ is the increasing permutation, as we will now show.

Proposition 2.7. *We have*

$$\bigcap_{d\geq 0} \widetilde{\mathrm{A}}_d = \{12\dots n \mid n \geq 0\}.$$

Proof. It follows immediately from the inductive description of hat_d that $12\dots n \in \widetilde{\mathrm{A}}_d$ for each $n, d \geq 0$. That is,

$$\{12\dots n \mid n \geq 0\} \subseteq \bigcap_{d\geq 0} \widetilde{\mathrm{A}}_d\,.$$

Conversely, suppose that $\alpha \in \widetilde{\mathrm{A}}_d$ for each $d \geq 0$. For a contradiction, suppose that $\alpha \in \mathrm{I}$ is not the increasing permutation; equivalently, let $\alpha = a_1 \dots a_n$ and suppose that there is some $i \in [n-1]$ such that $a_i \geq a_{i+1}$. Note that $i+1 \in \mathrm{Asc}_n\,\alpha = [n]$ and $a_i \geq a_{i+1}$. Therefore, the entry a_i is increased by one under the action of the map hat_n, which contradicts $\alpha \in \widetilde{\mathrm{A}}_n$. □

Given $\alpha \in \mathrm{I}$, let us recall [CCS24] the definition of the set

$$H(\alpha) = \{\mathrm{hat}_d(\alpha) \mid d \geq 0 \text{ and } \alpha \in \mathrm{A}_d\}$$

of all the d-hats of α. A seemingly more general notion of self-modified sequence arises by defining an inversion sequence α to be *self-modified* if $\alpha \in H(\alpha)$. The set $\tilde{\mathrm{I}}$ of *self-modified inversion sequences* is defined accordingly as

$$\tilde{\mathrm{I}} = \{\alpha \in \mathrm{I} \mid \alpha \in H(\alpha)\}.$$

It turns out that self-modified inversion sequences coincide with self-modified ascent sequences. Indeed, it is easy to see that

$$\tilde{\mathrm{I}} = \bigcup_{d\geq 0} \widetilde{\mathrm{A}}_d = \widetilde{\mathrm{A}}_0, \tag{6}$$

where the last equality follows by item (a) of Theorem 2.6.

To end this section, we wish to provide two alternative characterizations of $\widetilde{\mathrm{A}}_0$. Recall that an endofunction $\alpha = a_1 \dots a_n$ is a *restricted growth function* if $a_1 = 1$ and

$$a_{i+1} \leq 1 + \max(a_1 \dots a_i)$$

for each $i \in [n-1]$. We let

$$\mathrm{RGF} = \{\alpha \mid \alpha \text{ is a restricted growth function}\}.$$

Note that $\mathrm{RGF} \subseteq \mathrm{A}_0$. There is a standard bijection between RGFs and set partitions of $[n]$ where the leftmost copies correspond to the minima of the blocks (nonempty subsets of $[n]$). Next, we show that restricted growth functions are Cayley permutations whose leftmost copies appear in increasing order.

Lemma 2.8. *We have*

$$\mathrm{RGF} = \{\alpha \in \mathrm{Cay} \mid \mathrm{nub}\,\alpha = \mathrm{lrMax}\,\alpha\}.$$

Proof. We use induction on the length n of α, where the cases $n = 0$ and $n = 1$ are trivial. Let $n \geq 2$ and let $\alpha = a_1 \dots a_{n-1}a = \beta a$, where $\beta = a_1 \dots a_{n-1}$ and $a \in [n]$. Initially, suppose that $\alpha \in \mathrm{RGF}$.

We first show that $\alpha \in \mathrm{Cay}$. By definition of RGF we have $\beta \in \mathrm{RGF}$. So by induction $\beta \in \mathrm{Cay}$, say with image $[k]$. If $a \leq k$ then α has the same image. Otherwise the RGF condition forces $a = k+1$ and α has image $[k+1]$. The proof that $\mathrm{nub}\,\alpha = \mathrm{lrMax}\,\alpha$ is similar: when $a \leq k$ then both nub and lrMax do not change in passing from β to α. And if $a = k+1$ then n is added to both sets.

On the other hand, suppose that $\alpha \in \mathrm{Cay}$ and $\mathrm{nub}\,\alpha = \mathrm{lrMax}\,\alpha$. We show that $\alpha \in \mathrm{RGF}$. If $n \in \mathrm{nub}\,\alpha = \mathrm{lrMax}\,\alpha$, then

$$\begin{aligned} [\max \alpha] &= \mathrm{Im}\,\alpha && \text{(since } \alpha \in \mathrm{Cay}) \\ &= \mathrm{Im}\,\beta \uplus \{a\} && \text{(since } n \in \mathrm{nub}\,\alpha) \end{aligned}$$

and also $a = \max \alpha$ since $n \in \mathrm{lrMax}\,\alpha$. Therefore, we have $a = 1 + \max \beta$ and $\mathrm{Im}\,\beta = [a-1]$. In particular, $\beta \in \mathrm{Cay}$ and

$$\mathrm{nub}\,\beta = \mathrm{nub}\,\alpha \cap [n-1] = \mathrm{lrMax}\,\alpha \cap [n-1] = \mathrm{lrMax}\,\beta.$$

By induction, we have $\beta \in \mathrm{RGF}$, and $\alpha \in \mathrm{RGF}$ follows as well since $a = 1 + \max \beta$. The case where $n \notin \mathrm{nub}\,\alpha = \mathrm{lrMax}\,\alpha$ can be proved in a similar fashion, and we leave the details to the reader. □

Next we characterize self-modified d-ascent sequences as those modified d-ascent sequences that are restricted growth functions.

Proposition 2.9. *For each $d \geq 0$, we have*

$$\widetilde{\mathrm{A}}_d = \hat{\mathrm{A}}_d \cap \mathrm{RGF}\,.$$

Proof. Let us start with the inclusion $\widetilde{\mathrm{A}}_d \subseteq \hat{\mathrm{A}}_d \cap \mathrm{RGF}$. Let $\alpha \in \widetilde{\mathrm{A}}_d$. Then $\alpha = \mathrm{hat}_d(\alpha) \in \hat{\mathrm{A}}_d$ and α is a Cayley permutation by item (a) of Proposition 1.1. Furthermore, by Corollary 2.4,

$$\mathrm{lrMax}\,\alpha = \mathrm{Asc}_d\,\alpha = \mathrm{nub}\,\alpha,$$

hence $\alpha \in \mathrm{RGF}$ follows by Lemma 2.8.

To prove the remaining inclusion, recall that $\mathrm{RGF} \subseteq \mathrm{A}_0$. Thus

$$\hat{\mathrm{A}}_d \cap \mathrm{RGF} \subseteq \hat{\mathrm{A}}_d \cap \mathrm{A}_0 \subseteq \hat{\mathrm{A}}_d \cap \mathrm{A}_d = \widetilde{\mathrm{A}}_d,$$

where the last equality is Theorem 2.5. □

Letting $d = 0$ in Proposition 2.9 yields

$$\widetilde{\mathrm{A}}_0 = \hat{\mathrm{A}}_0 \cap \mathrm{RGF}\,.$$

An alternative description of $\widetilde{\mathrm{A}}_0$ is showed in the next result.

Corollary 2.10. *We have*

$$\widetilde{\mathrm{A}}_0 = \hat{\mathrm{I}} \cap \mathrm{RGF}\,.$$

Proof. We have

$$\begin{aligned}
\widetilde{\mathrm{A}}_0 &= \bigcup_{d\geq 0} \widetilde{\mathrm{A}}_d && \text{(by equation (6))}\\
&= \bigcup_{d\geq 0} \Big(\hat{\mathrm{A}}_d \cap \mathrm{RGF}\Big) && \text{(by Proposition 2.9)}\\
&= \big(\bigcup_{d\geq 0} \hat{\mathrm{A}}_d\big) \cap \mathrm{RGF}\\
&= \hat{\mathrm{I}} \cap \mathrm{RGF} && \text{(by equation (1))},
\end{aligned}$$

finishing the proof. □

Theorem 2.5 characterizes self-modified d-ascent sequences as $\widetilde{\mathrm{A}}_d = \mathrm{A}_d \cap \hat{\mathrm{A}}_d$. It is easy to see that the inclusion $\tilde{\mathrm{I}} \subseteq \mathrm{I} \cap \hat{\mathrm{I}}$ holds as well. Indeed, using equation (6),

$$\tilde{\mathrm{I}} = \widetilde{\mathrm{A}}_0 = \mathrm{A}_0 \cap \hat{\mathrm{A}}_0 \subseteq \mathrm{I} \cap \hat{\mathrm{I}}\,.$$

However, the opposite inclusion does not hold. For instance, $11312 = \mathrm{hat}_0(11212)$ is a member of $\mathrm{I} \cap \hat{\mathrm{I}}$, but $11312 \notin H(11312) = \{31412, 43512\}$ and hence $11312 \notin \tilde{\mathrm{I}}$.

We end this section with one more remark. Self-modified inversion sequences are related to restricted growth functions through $\tilde{\mathrm{I}} = \hat{\mathrm{I}} \cap \mathrm{RGF}$. But the inclusion $\mathrm{RGF} \subseteq \hat{\mathrm{I}}$ does not hold: $1212 \in \mathrm{RGF}$, but $1212 \notin \hat{\mathrm{I}}$. One way to make it hold would be to alter the recursive definition of $\hat{\mathrm{A}}_d$ given in Section 1 by allowing $d = -\infty$ and letting the last letter a be chosen in the interval $[1 + \max \beta]$. This would give $\hat{\mathrm{A}}_{-\infty} = \mathrm{RGF}$.

3 Enumeration of self-modified d-ascent sequences

We aim to determine the generating function

$$\widetilde{\mathrm{A}}_d(q, x) = \sum_{\alpha} q^{\max(\alpha)} x^{|\alpha|},$$

where the sum ranges over all self-modified d-ascent sequences. Our solution will be in terms of certain Fibonacci polynomials which we introduce below.

The *Fibonacci numbers* F_n are defined by the second order recurrence relation $F_n = F_{n-1} + F_{n-2}$ with initial terms $F_0 = F_1 = 1$. Many generalizations of the Fibonacci numbers have been proposed. One may for instance consider the dth order recurrence relation $F_n = F_{n-1} + F_{n-d}$ with initial terms $F_0 = F_1 = \cdots = F_{d-1} = 1$. Or, generalizing in a different direction, one may consider Fibonacci polynomials such as those given by $F_0(x) = F_1(x) = 1$ and $F_n(x) = F_{n-1}(x) + xF_{n-2}(x)$. Combining these two ideas we define, for any fixed $d \geq 0$,

$$\begin{cases} F_{d,n}(x) = 1 & \text{for } n < d,\\ F_{d,n}(x) = F_{d,n-1}(x) + xF_{d,n-d}(x) & \text{for } n \geq d. \end{cases}$$

In generalizations of the Fibonacci sequence, like the one above, the "smallest" case typically corresponds to the (classical) Fibonacci recurrence, which in our definition is $d = 2$ (with $x = 1$). Note that we, however, also allow $d = 0$ and $d = 1$.

If $d = 0$, then $F_{0,n}(x) = F_{0,n-1}(x) + xF_{0,n}(x)$, which together with the initial condition $F_{0,-1}(x) = 1$ gives

$$F_{0,n}(x) = 1/(1-x)^{n+1} = (1 + x + x^2 + \cdots)^{n+1}.$$

In particular, $F_{0,n}(x)$ is a power series, while it is easy to see that $F_{d,n}(x)$ is a polynomial for any $d \geq 1$. For example, when $d = 1$ we have $F_{0,0}(x) = 1$ and $F_{1,n}(x) = (1 + x)F_{1,n-1}(x)$ for $n \geq 1$, and hence

$$F_{1,n}(x) = (1 + x)^n.$$

For any $d \geq 0$, define the generating function

$$F_d(x, y) = \sum_{n \geq 0} F_{d,n}(x)y^n.$$

Using standard techniques it follows from the recurrence relation for $F_{d,n}(x)$ that

$$F_d(x, y) = \frac{1}{1 - y - xy^d}. \tag{7}$$

Viewing this as a geometric series and applying the binomial theorem we find that

$$F_d(x, y) = \sum_{m \geq 0} \sum_{k=0}^{m} \binom{m}{k} x^k y^{(d-1)k+m}$$

and on extracting the coefficient of y^n we get

$$F_{d,n}(x) = \sum_{k \geq 0} \binom{n - (d-1)k}{k} x^k.$$

For reference, the first few polynomials for $d = 2$ are

$$\begin{aligned}
F_{2,0}(x) &= F_{2,1}(x) = 1;\\
F_{2,2}(x) &= 1 + x;\\
F_{2,3}(x) &= 1 + 2x;\\
F_{2,4}(x) &= 1 + 3x + x^2;\\
F_{2,5}(x) &= 1 + 4x + 3x^2;\\
F_{2,6}(x) &= 1 + 5x + 6x^2 + x^3.
\end{aligned}$$

Let us write $\mu \vDash n$ to indicate that μ is an integer composition of n. A well-known interpretation of the nth Fibonacci number, F_n, is the number of compositions $\mu \vDash n$ with parts in $\{1, 2\}$. Similarly, for $d \geq 2$, an interpretation of $F_{d,n}(1)$ is the number of compositions $\mu \vDash n$ with parts in $\{1, d\}$, and the polynomial $F_{d,n}(x)$ records the distribution of d-parts in such compositions. In symbols,

$$F_{d,n}(x) = \sum_{\substack{\mu=(m_1,\ldots,m_k) \vDash n \\ m_i \in \{1,d\}}} x^{|\mu|_d},$$

where $|\mu|_d = \#\{i : m_i = d\}$. This interpretation works for $d = 0$ as well; as previously noted $F_{0,n}(x)$ is power series rather than a polynomial in that case. For $d = 1$ a little extra care is needed. For our combinatorial interpretation to work we must regard the 1-parts as being distinct from the d-parts, even though $d = 1$ in this case. In other words, $F_{1,n}(x)$ records the distribution of $1'$-parts in compositions $\mu \vDash n$ with parts in $\{1, 1'\}$, where 1 and $1'$ denote two different kinds of parts, both of size 1.

The nth *Fibonacci factorial*, also called *Fibonorial* or *Fibotorial*, is defined by

$$F_n^! = F_1 F_2 \cdots F_n.$$

We note that sometime these terms are used for the version of the Fibonacci sequence where $F_0 = 0$ and $F_1 = 1$. In this manner we also define

$$F_{d,n}^!(x) = \prod_{i=0}^{n} F_{d,i}(x).$$

Note that the index i ranges from 0 to n rather than from 1 to n. This only makes a difference when $d = 0$ since $F_{0,0}(x) = 1/(1-x)$ while $F_{d,0}(x) = 1$ for $d \geq 1$. In particular,

$$F_{0,n}^!(x) = \left(\frac{1}{1-x}\right)^{\binom{n+2}{2}} \quad \text{and} \quad F_{1,n}^!(x) = (1+x)^{\binom{n+1}{2}}.$$

For $d \geq 2$ we do not have such simple formulas. As an illustration, the first few polynomials for $d = 2$ are

$$\begin{aligned}
&F_{2,0}^!(x) = F_{2,1}^!(x) = 1;\\
&F_{2,2}^!(x) = 1 + x;\\
&F_{2,3}^!(x) = 1 + 3x + 2x^2;\\
&F_{2,4}^!(x) = 1 + 6x + 12x^2 + 9x^3 + 2x^4;\\
&F_{2,5}^!(x) = 1 + 10x + 39x^2 + 75x^3 + 74x^4 + 35x^5 + 6x^6.
\end{aligned}$$

For any $d \geq 0$, define the generating function

$$F_d^!(x, y) = \sum_{n \geq 0} F_{d,n}^!(x) y^n.$$

Also, define

$$K_{d,n}(x) = \sum_{\mu} x^{\ell(\mu)} \quad \text{and} \quad K_d(x, y) = \sum_{n \geq 0} K_{d,n}(x) y^n,$$

where the former sum ranges over all integer compositions μ of n into parts of size d or larger, and $\ell(\mu)$ denotes the number of parts of μ.

Lemma 3.1. *We have*

$$K_d(x, y) = \frac{1-y}{1-y-xy^d}.$$

Proof. The generating function for single parts of size at least d is $y^d/(1-y)$. Thus

$$\begin{aligned}
\sum_{n \geq 0} K_{d,n}(x) y^n &= \sum_{k \geq 0} \left(\frac{xy^d}{1-y}\right)^k \\
&= \frac{1}{1 - \dfrac{xy^d}{1-y}} = \frac{1-y}{1-y-xy^d}.
\end{aligned}$$

□

Lemma 3.2. *For any $d \geq 0$, we have*

(a) $F_{d,n}(x) = F_{d,n-1}(x) + K_{d,n}(x)$ *for $n \geq 1$, and*

(b) $F_{d,n}(x) = K_{d,0}(x) + K_{d,1}(x) + \cdots + K_{d,n}(x)$ *for $n \geq 0$.*

Proof. Identity (b) is obtained by repeated application of identity (a), so let us focus on (a). An immediate consequence of Lemma 3.1 and identity (7) is

$$K_d(x, y) = (1 - y)F_d(x, y),$$

from which (a) follows by identifying coefficients. While this settles the claimed identity, let us also provide a, perhaps more elucidating, combinatorial proof. Assume $n \geq 1$ and let $\mu = (m_1, m_2, \ldots, m_k) \vDash n$ be such that $m_i \in \{1, d\}$ for each $i \in [k]$. If $m_k = 1$ then we map μ to the composition $\mu' = (m_1, m_2, \ldots, m_{k-1}) \vDash n-1$ obtained from μ by removing its last part. This procedure is trivially reversible. If $m_k = d$, then we need to map μ (in a reversible way) to a composition ν with parts of size at least d. Moreover, ν should have as many parts as μ has d-parts. Having spelled out these criteria, the map now presents itself: Assume that m_i is the first d-part of μ; that is, $m_1 = \cdots = m_{i-1} = 1$ and $m_i = d$. We simply sum these up to get the first part $n_1 = i - 1 + d$ of ν; the second part n_2 is obtained by applying the same procedure to $(m_{i+1}, m_{i+2}, \ldots, m_k)$; and so on. For instance, $\mu = (1, 3, 3, 1, 1, 3)$, where $d = 3$ and $n = 12$, gets mapped to $\nu = (4, 3, 5)$. □

Let us say that a sequence of numbers $c_1c_2 \ldots c_\ell$ is *decreasing with pace d* if the difference between consecutive elements is at least d; that is, if $c_j - c_{j+1} \geq d$ for $1 \leq j < \ell$.

Lemma 3.3. *Any $\alpha \in \mathrm{A}_d$ is self-modified if and only if it can be written*

$$\alpha = 1B_1 2B_2 \ldots kB_k,$$

where $k = \max \alpha$ and each factor iB_i is decreasing with pace d.

Proof. Let $\alpha \in \widetilde{\mathrm{A}}_d$ with $\max \alpha = k$ be given. By Proposition 2.9, α is an RGF and so can be written

$$\alpha = 1B_1 2B_2 \ldots kB_k$$

where $\operatorname{nub} \alpha$ is the set of positions of the elements $1, 2, \ldots, k$ which are not in the B_i. But from Corollary 2.4 we have $\operatorname{Asc}_d \alpha = \operatorname{nub} \alpha$. Thus each factor iB_i must be void of d-ascents; that is, each iB_i is decreasing with pace d.

Conversely, if $\alpha = 1B_1 2B_2 \ldots kB_k$, where $k = \max \alpha$ and each factor iB_i is decreasing with pace d, then $\operatorname{nub} \alpha = \operatorname{Asc}_d \alpha$ and hence α is self-modified by Corollary 2.4. □

The following theorem reveals a striking relationship between Fibonacci factorials and the number of self-modified d-ascent sequences. Recall the generating function $\widetilde{\mathrm{A}}_d(q, x) = \sum_\alpha q^{\max(\alpha)} x^{|\alpha|}$, where the sum ranges over all $\alpha \in \widetilde{\mathrm{A}}_d$.

Theorem 3.4. *For any $d \geq 0$,*

$$\widetilde{\mathrm{A}}_d(q, x) = 1 + qxF_d^!(x, qx).$$

$d\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	2	5	14	43	143	510	1936	7775	32869	145665	674338
1	1	1	1	2	4	10	27	81	262	910	3363	13150	54135
2	1	1	1	1	2	4	9	23	64	194	629	2177	7982
3	1	1	1	1	1	2	4	9	22	58	167	515	1698
4	1	1	1	1	1	1	2	4	9	22	57	158	467
5	1	1	1	1	1	1	1	2	4	9	22	57	157
6	1	1	1	1	1	1	1	1	2	4	9	22	57

Table 1: The number of self-modified d-ascent sequences of length n

Proof. Let $\alpha \in \widetilde{\mathrm{A}}_d$. In accordance with Lemma 3.3, write $\alpha = 1B_1 2B_2 \dots kB_k$, in which $k = \max \alpha$ and each factor iB_i is decreasing with pace d. Assume that the length of iB_i is ℓ and write

$$iB_i = c_1 c_2 \dots c_\ell.$$

Let $m_j = c_j - c_{j+1}$ for $j \in [\ell - 1]$ and let $m_\ell = c_\ell$. Note that $iB_i = c_1 c_2 \dots c_\ell$ can be recovered from the composition

$$\mu = (m_1, m_2, \dots, m_\ell) \vDash i$$

by letting $c_j = m_j + \cdots + m_\ell$. In other words, μ encodes the factor iB_i, and the length of iB_i equals the number of parts of μ. If we exclude the last part of μ and let $\mu' = (m_1, \dots, m_{\ell-1})$, then we have a composition each part of which is at least d, and the possible choices for μ' are recorded by $K_{d,m}(x)$, where $m = m_1 + \cdots + m_{\ell-1}$. The last part, m_ℓ, of μ can be any integer between 1 and i and hence the possible choices for μ are recorded by

$$xK_{d,0}(x) + xK_{d,1}(x) + \cdots + xK_{d,i-1}(x).$$

By item (b) of Lemma 3.2, this expression simplifies to $xF_{d,i-1}(x)$ and hence the generating function for $\alpha \in \widetilde{\mathrm{A}}_d$ with $k = \max \alpha$ is

$$q^k \prod_{i=1}^{k} xF_{d,i-1}(x) = q^k x^k F^!_{d,k-1}(x).$$

The claim now follows by summing over k. □

Theorem 3.4 allows us to easily tabulate the cardinalities of $\widetilde{\mathrm{A}}_{d,n}$; see Table 1.

For $d = 0$ and $d = 1$ we even get explicit general expressions. For $d = 0$ we rediscover a formula originally given by Bousquet-Mélou et al. [BMCDK10]:

$$\begin{aligned}\widetilde{\mathrm{A}}_0(q,x) &= 1 + qxF^!_0(x, qx) \\ &= 1 + \sum_{k\geq 1} (qx)^k F^!_{0,k-1}(x) = \sum_{k\geq 0} \frac{(qx)^k}{(1-x)^{\binom{k+1}{2}}}.\end{aligned}$$

$d\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	2	5	14	43	143	510	1936	7775	32869	145665	674338
1	1	1	2	4	10	27	81	262	910	3363	13150	54135	233671
2	1	1	2	4	9	23	64	194	629	2177	7982	30871	125402
3	1	1	2	4	9	22	58	167	515	1698	5925	21810	84310
4	1	1	2	4	9	22	57	158	467	1474	4934	17448	64847
5	1	1	2	4	9	22	57	157	454	1387	4476	15243	54606
6	1	1	2	4	9	22	57	157	453	1369	4321	14293	49570
7	1	1	2	4	9	22	57	157	453	1368	4297	14027	47615
8	1	1	2	4	9	22	57	157	453	1368	4296	13996	47178
9	1	1	2	4	9	22	57	157	453	1368	4296	13995	47139
10	1	1	2	4	9	22	57	157	453	1368	4296	13995	47138

Table 2: The coefficients of $\big(\widetilde{\mathrm{A}}_d(1,x) - [d]_x\big)/x^d$

For $d = 1$ (self-modified weak ascent sequences) we get the following formula:

$$\begin{aligned}\widetilde{\mathrm{A}}_1(q,x) &= 1 + qxF_1^!(x,qx)\\ &= 1 + \sum_{k\geq 1}(qx)^k F_{1,k-1}^!(x) = \sum_{k\geq 0}(qx)^k(1+x)^{\binom{k}{2}}.\end{aligned}$$

Note that the diagonals of Table 1 each appear to tend to some constant. To bring out this pattern let us skip the first d ones of each row and left adjust. In other words we are interested in the coefficients of $\big(\widetilde{\mathrm{A}}_d(1,x)-[d]_x\big)/x^d$, where $[d]_x = 1+x+\cdots+x^{d-1}$; we display those coefficients in Table 2. What can be said about the sequence that is emerging as $d \to \infty$? Formally, this limit should be understood as follows. Let $G_0(x)$, $G_1(x)$, $G_2(x)$, ... be a sequence of power series. Then

$$\lim_{k\to\infty} G_k(x) = \sum_{n\geq 0} c_n x^n$$

if, given a nonnegative integer n, there is a corresponding K such that the coefficient of x^n in $G_k(x)$ is c_n for all $k \geq K$.

Theorem 3.5. *We have*

$$\lim_{d\to\infty}\frac{1}{(qx)^d}\left(\widetilde{\mathrm{A}}_d(q,x) - [d]_{qx}\right) = \sum_{k\geq 0}(qx)^k(1+x)(1+2x)\cdots(1+kx).$$

Let $R(q,x)$ be the generating function on the right-hand side of the claimed identity, and let $r_n(q)$ be the coefficient of x^n in $R(q,x)$. The sequence $(r_n(1))_{n\geq 0}$ starts

$$1, 1, 2, 4, 9, 22, 57, 157, 453, 1368, 4296, 13995, 47138, 163779, 585741,$$

which agrees with the last row of Table 2. There is a corresponding entry in the OEIS, namely A124380. No combinatorial interpretation is given in that entry, but it is now straightforward to give one.

Lemma 3.6. *Let $\mathcal{R}_n$ be the set of restricted growth functions $\alpha = 1B_1 2B_2 \dots kB_k$ such that, for each $i \in [k]$, either B_i is empty or $B_i = b_i$ is a single letter with $b_i \le i$. With $r_n(q)$ defined as above, we have*

$$r_n(q) = \sum_{\alpha \in \mathcal{R}_n} q^{\max \alpha}.$$

Proof. The following generating function for the Stirling numbers of the second kind, $S(n,k) = \#\{\alpha \in \mathrm{RGF}_n : \max \alpha = k\}$, is well-known:

$$\sum_{n \ge 0} x^n \sum_{\alpha \in \mathrm{RGF}_n} q^{\max \alpha} = \sum_{k \ge 0} \frac{(qx)^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

A proof goes as follows. Let $\alpha \in \mathrm{RGF}_n$ with $\max \alpha = k$ be given. Factor $\alpha = 1B_1 2B_2 \dots kB_k$ in which each letter of B_i is at most i. The possible choices of B_i are encoded in the generating function $1/(1-ix) = 1 + ix + i^2x^2 + \cdots$ and hence the possible choices of α are encoded by $x^k/((1-x)(1-2x)\cdots(1-kx))$. For $\alpha \in \mathcal{R}_n$ with $\max \alpha = k$ the same approach applies, but now $|B_i| \le 1$ and the choices of B_i are encoded by $1 + ix$. □

Proof of Theorem 3.5. Letting $f_{d,n}(q)$ denote the coefficient of x^n in $\widetilde{\mathrm{A}}_d(q,x)$ we find that the coefficient of x^n in $(\widetilde{\mathrm{A}}_d(q,x) - [d]_{qx})/(qx)^d$ is $f_{d,n+d}(q)/q^d$. Thus, it suffices to prove that, for any $d \ge 0$ and $n \le d$,

$$f_{d,n+d}(q) = q^d r_n(q).$$

Assume $n \le d$ and let $\alpha \in \widetilde{\mathrm{A}}_{d,n+d}$ with $k = \max \alpha$ be given. Write $\alpha = 1B_1 2B_2 \dots kB_k$, where each iB_i is decreasing with pace d. Note that B_i is empty whenever $i \le d$. Moreover, since $n \le d$ we have that B_{d+i} is empty or $B_{d+i} = b_{d+i}$ is a singleton with $b_{d+i} \in [i]$ for $i \ge 1$. The desired bijection from $\widetilde{\mathrm{A}}_{d,n+d}$ onto $\mathcal{R}_n$ is now provided by

$$1\,2\,\dots\,d\,(d+1)B_{d+1}\,(d+2)B_{d+2}\,\dots\,kB_k \;\mapsto\; 1B_{d+1}\,2B_{d+2}\,\dots\,(k-d)B_k.$$

For instance, if $d = 4$ and $n = 3$, then

$$1234516 \mapsto 112, \quad 1234561 \mapsto 121, \quad 1234562 \mapsto 122, \quad 1234567 \mapsto 123.$$

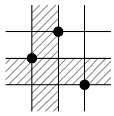
Finally, if $\alpha \mapsto \beta$ in this way, then $\max \alpha = d + \max \beta$, which conludes the proof. □

4 Self-modified d-Fishburn permutations

Let $\mathfrak{S}$ denote the set of all permutations $\pi = p_1 p_2 \dots p_n$ of $[n]$ for all $n \ge 0$. Inspired by a question of Dukes and Sagan [DS24], Zang and Zhou [ZZ] have recently introduced the set $\mathfrak{F}_d$ of *d-Fishburn permutations.* They are a generalization of *Fishburn permutations* [BMCDK10] where the recursive structure of $\mathfrak{F}_d$ is encoded by A_d, for every $d \ge 0$. The bijection between $\mathfrak{F}_d$ and A_d defined this way is denoted by

$$\Phi_d : \mathrm{A}_d \to \mathfrak{F}_d\,.$$

Fishburn permutations are defined as $\mathfrak{F}_0 = \mathfrak{S}(\mathfrak{f})$, where $\mathfrak{f}$ is the following (bivincular) mesh pattern [BMCDK10, BC11]:

$\mathfrak{f} =$  ,

that is, $\mathfrak{F}_0$ is the set of all $\pi \in \mathfrak{S}$ which do not contain a subsequence $p_i p_j p_k$ with $j = i + 1 < k$ and $p_k + 1 = p_i < p_j$.

We [CCS24] gave an alternative definition of $\mathfrak{F}_d$ that is reminiscent of the classical case $d = 0$, which we now recall. First, we describe a procedure to determine the *d-active* and *d-inactive* [ZZ] elements of a given permutation $\pi = p_1 \dots p_n$. Let $\pi^{(k)}$ denote the restriction of π to the elements of $[k]$.

- Set 1 to be a d-active element.
- For $k = 2, 3, \dots, n$, let k be d-inactive if k is to the left of $k - 1$ in π and there exist at least d elements of $\pi^{(k)}$ between k and $k - 1$ that are d-active. Otherwise, k is said to be d-active.

We say that a permutation π contains the *d-Fishburn pattern* [CCS24], denoted by $\mathfrak{f}_d$, if it contains an occurrence $p_i p_j p_k$ of $\mathfrak{f}$ where p_i is d-inactive. The other two elements p_j and p_k can be either d-active or d-inactive. With slight abuse of notation, we let $\mathfrak{S}(\mathfrak{f}_d)$ be the set of permutations that do not contain $\mathfrak{f}_d$. Finally, for every $d \geq 0$ we have that [CCS24, Prop. 7.1]

$$\mathfrak{F}_d = \mathfrak{S}(\mathfrak{f}_d).$$

In the same paper, we showed that d-Fishburn permutations can be alternatively obtained as the bijective image of d-ascent sequences under the composition of the d-hat map with the *Burge transpose* [CC23], lifting a classical result by Bousquet-Mélou et al. [BMCDK10] to any $d \geq 0$. As we will build on this alternative description of $\mathfrak{F}_d$, we wish to recall the necessary tools and definitions.

Given an endofunction $\alpha = a_1 \dots a_n$, let

$$\mathrm{wDes}\,\alpha = \{i \geq 2 \mid a_i \leq a_{i-1}\}$$

denote the set of *weak descents* of α. The set of *Burge words* [CC23] is defined as

$$\mathrm{Bur}_n = \left\{ \binom{u}{\alpha} : u \in \mathrm{WI}_n,\ \alpha \in \mathrm{Cay}_n,\ \mathrm{wDes}(u) \subseteq \mathrm{wDes}(\alpha) \right\},$$

where WI_n is the subset of Cay_n consisting of the weakly increasing Cayley permutations. The *Burge transpose* is a transposition operation T on Bur_n defined as follows. Given $w = \binom{u}{\alpha} \in \mathrm{Bur}_n$, to compute w^T turn each column of w upside down and then sort the columns in weakly increasing order with respect to the top entry, breaking ties by sorting in weakly decreasing order with respect to the bottom entry. Observe that T is an involution on Bur_n. Furthermore, by picking $\mathrm{id}_n = 12 \dots n$ as the top row, we obtain a map $\mathtt{t} : \mathrm{Cay}_n \to \mathfrak{S}_n$ defined by[1]

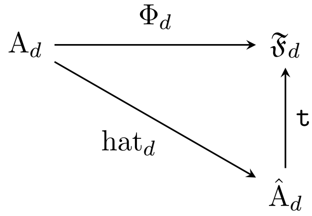
$$\binom{\mathrm{id}_n}{\alpha}^T = \binom{\mathrm{sort}(\alpha)}{\mathtt{t}(\alpha)},$$

[1]The map $\mathtt{t}$ was originally [CC23] denoted by the letter γ.

for any $\alpha \in \mathrm{Cay}$, where $\mathrm{sort}(\alpha)$ is obtained by sorting the entries of α in weakly increasing order. As a special case, if α is a permutation, then $\mathtt{t}(\alpha) = \alpha^{-1}$; note that this proves that $\mathtt{t}$ is surjective. Finally, for every $d \geq 0$ we have [CCS24, Thm. 7.10]

$$\Phi_d = \mathtt{t} \circ \mathrm{hat}_d.$$

In other words, the diagram



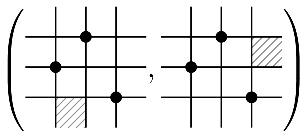
(8)

commutes for every $d \geq 0$ and all the arrows are size-preserving bijections.

For the rest of this section, let $d \geq 0$ be a fixed nonnegative integer. Denote by

$$\widetilde{\mathfrak{F}}_d = \Phi_d(\widetilde{\mathrm{A}}_d)$$

the set of d-Fishburn permutations corresponding to self-modified d-ascent sequences under the bijection Φ_d. Bousquet-Mélou et al. [BMCDK10] proved that $\widetilde{\mathfrak{F}}_0 = \mathfrak{S}(3\bar{1}52\bar{4})$, where a permutation π avoids the barred pattern $3\bar{1}52\bar{4}$ if every occurrence of the pattern 231 plays the role of 352 in an occurrence of 31524. Formally, for every $i < j < k$ such that $p_k < p_i < p_j$, there exist $\ell \in \{i+1, i+2, \dots, j-1\}$ and $m > k$ such that $p_i p_\ell p_j p_k p_m$ is an occurrence of 31524. More visually, in terms of mesh patterns,

$\widetilde{\mathfrak{F}}_0 = \mathfrak{S}$  .

In the same spirit, we wish to characterize $\widetilde{\mathfrak{F}}_d$. From now on, we say that a permutation $\pi = p_1 \dots p_n$ *contains* $\mathfrak{s}_d$ if there are two indices $i < j$ such that

- $p_i = p_j + 1$; and
- $\mathrm{asc}(p_i p_{i+1} \dots p_j) \leq d$.

Furthermore, with slight abuse of notation, we write $\mathfrak{S}(\mathfrak{s}_d)$ to denote the set of permutations that do not contain $\mathfrak{s}_d$. Our goal is to prove that

$$\widetilde{\mathfrak{F}}_d = \mathfrak{S}(3\bar{1}52\bar{4}, \mathfrak{s}_d) = \widetilde{\mathfrak{F}}_0(\mathfrak{s}_d). \tag{9}$$

Observe that

$$\widetilde{\mathfrak{F}}_d = \Phi_d(\widetilde{\mathrm{A}}_d) = \mathtt{t}\big(\mathrm{hat}_d(\widetilde{\mathrm{A}}_d)\big) = \mathtt{t}(\widetilde{\mathrm{A}}_d), \tag{10}$$

where $\mathrm{hat}_d(\widetilde{\mathrm{A}}_d) = \widetilde{\mathrm{A}}_d$ by definition of self-modified d-ascent sequence. In other words, permutations in $\widetilde{\mathfrak{F}}_d$ are precisely those that are obtained by applying the Burge transpose to some self-modified d-ascent sequence in $\widetilde{\mathrm{A}}_d$. Furthermore, by Lemma 3.3 every $\alpha \in \widetilde{\mathrm{A}}_d$ can be written $\alpha = 1B_1 2B_2 \dots kB_k$, where $k = \max \alpha$ and each factor iB_i is decreasing with pace d.

Since we know from Proposition 2.9 that $\widetilde{\mathrm{A}}_d \subseteq \mathrm{RGF}$, it will be useful to describe the ascents of permutations in $\mathtt{t}(\mathrm{RGF})$.

Lemma 4.1. *Let $\alpha \in \mathrm{RGF}$, $k = \max \alpha$, and*

$$\binom{\mathrm{id}}{\alpha}^T = \binom{\mathrm{sort}(\alpha)}{\pi} = \begin{pmatrix} 1\dots 1 & 2\dots 2 & \dots & k\dots k \\ p_1\dots p_{i_1} & p_{i_1+1}\dots p_{i_2} & \dots & p_{i_{k-1}+1}\dots p_{i_k} \end{pmatrix}.$$

Then:

(a) $\mathrm{Asc}\,\pi = \{1, i_1+1, i_2+1, \dots, i_{k-1}+1\}$.

(b) For each $i < j$, we have

$$\mathrm{asc}(p_i \dots p_j) = \ell_j - \ell_i + 1,$$

where ℓ_i and ℓ_j are the entries above p_i and p_j in $\mathrm{sort}(\alpha)$*, respectively.*

Proof. To prove (a), let $j \in [k]$. By definition of Burge transpose, the entry p_{i_j} below the rightmost copy of j in $\mathrm{sort}(\alpha)$ is the index of the leftmost copy of j in α. Since $\alpha \in \mathrm{RGF}$, the leftmost copy of j precedes each copy of $j+1$ in α; that is, $p_{i_j} < p_{i_j}+1$ for each $j \in [k-1]$, and thus $\mathrm{Asc}\,\pi \supseteq \{1, i_1+1, i_2+1, \dots, i_{k-1}+1\}$. The other inclusion is trivial because each factor $p_{i_j+1} \dots p_{i_{j+1}}$ is weakly decreasing by definition of Burge transpose.

We shall now prove (b). By the first item, the ascents of π are precisely those entries that are below the leftmost copy of some $j \in [k]$ in $\mathrm{sort}(\alpha)$. Since the first position is an ascent by definition, the sequence $p_i \dots p_j$ contains exactly one ascent for each number from ℓ_i (the entry above p_i) to ℓ_j (the entry above p_j); that is, we have

$$\mathrm{asc}(p_i \dots p_j) = \ell_j - \ell_i + 1. \qquad \square$$

Theorem 4.2. *We have*

$$\widetilde{\mathfrak{F}}_d = \widetilde{\mathfrak{F}}_0(\mathfrak{s}_d).$$

Proof. We showed in item (a) of Theorem 2.6 that $\widetilde{\mathrm{A}}_d \subseteq \widetilde{\mathrm{A}}_0$. Using this and equation (10) we obtain

$$\widetilde{\mathfrak{F}}_d = \mathtt{t}(\widetilde{\mathrm{A}}_d) \subseteq \mathtt{t}(\widetilde{\mathrm{A}}_0) = \widetilde{\mathfrak{F}}_0\,.$$

Now, let $\pi \in \widetilde{\mathfrak{F}}_0$. To obtain the desired equality, it suffices to prove that π contains $\mathfrak{s}_d$ if and only if $\pi \notin \widetilde{\mathfrak{F}}_d$. We will only provide the details of the forward direction as the demonstration of the converse can be obtained by just reversing each step.

Since $\pi \in \widetilde{\mathfrak{F}}_0 = \mathtt{t}(\widetilde{\mathrm{A}}_0)$, there is some $\alpha \in \widetilde{\mathrm{A}}_0$ such that

$$\binom{\mathrm{id}}{\alpha}^T = \binom{\mathrm{sort}(\alpha)}{\pi}.$$

Furthermore, by Lemma 3.3, we have

$$\alpha = 1B_1 2B_2 \dots kB_k,$$

where $k = \max \alpha$ and each factor iB_i is decreasing with pace zero, i.e. is weakly decreasing. By definition, π containing $\mathfrak{s}_d$ means that there are indices $i < j$ with $p_i = p_j + 1$ and $\mathrm{asc}(p_i \dots p_j) \leq d$. Now applying item (b) of Lemma 4.1 (which can be done since α is an RGF) shows that the ascent inequality implies $\ell_j - \ell_i < d$ where we are using the notation of the lemma. But $p_i = p_j + 1$ so that

$$\binom{\mathrm{id}}{\alpha} = \begin{pmatrix} \dots p_j \; p_i \dots \\ \dots \ell_j \; \ell_i \dots \end{pmatrix}.$$

Since $i < j$ we have that $\ell_j \geq \ell_i$ and this forces the pair $\ell_j \ell_i$ to be part of the same factor kB_k above. On the other hand $\ell_j - \ell_i < d$ so that factor can not be decreasing with pace d. Using Lemma 3.3 again, we see that $\alpha \notin \widetilde{\mathrm{A}}_d$. Thus, since $\mathtt{t}$ is injective, $\pi = \mathtt{t}(\alpha) \notin \mathtt{t}(\widetilde{\mathrm{A}}_d) = \widetilde{\mathfrak{F}}_d$ as desired. □

5 Pattern avoidance in $\widetilde{\mathfrak{F}}_d$

The first two authors have developed a theory of transport of patterns between Fishburn permutations and modified ascent sequences [CC23]. We recently showed that the same machinery also applies to d-Fishburn permutations and modified d-ascent sequences [CCS24, Thm. 8.6]. More explicitly, for any $d \geq 0$ and permutation τ,

$$\mathtt{t} : \hat{\mathrm{A}}_d(B_\tau) \longrightarrow \mathfrak{F}_d(\tau)$$

is a size-preserving bijection, where B_τ is a finite set called the *Fishburn basis* of τ, $\hat{\mathrm{A}}_d(B_\tau)$ is the set of modified d-ascent sequences avoiding every pattern in B_τ, and $F_d(\tau)$ is the set of d-Fishburn permutations avoiding τ. Moreover, there is a constructive procedure to compute B_τ [CC23].

Since the map $\mathtt{t}$ is injective on $\hat{\mathrm{A}}_d$ [CCS24, Cor. 4.7], it is also injective on $\widetilde{\mathrm{A}}_d = \mathrm{A}_d \cap \hat{\mathrm{A}}_d$. The following is a corollary to the previous discussion and the transport theorem [CCS24, Thm. 8.6].

Corollary 5.1. *For any $d \geq 0$ and permutation τ, the map $\mathtt{t} : \widetilde{\mathrm{A}}_d(B_\tau) \to \widetilde{\mathfrak{F}}_d(\tau)$ is a size-preserving bijection. In particular, $\#\widetilde{\mathfrak{F}}_{d,n}(\tau) = \#\widetilde{\mathrm{A}}_{d,n}(B_\tau)$.*

For instance, let us again consider the pattern 213. Recall [CC23] that $B_{213} = \{112, 213\}$ and hence $\#\widetilde{\mathfrak{F}}_d(213) = \#\widetilde{\mathrm{A}}_d(112, 213)$. We will enumerate these sets below.

Lemma 5.2. *Let $\alpha \in \widetilde{\mathrm{A}}_d$ and write $\alpha = 1B_1 2B_2 \dots kB_k$ as in Lemma 3.3. Then α avoids 112 and 213 if and only if $B_1 = B_2 = \cdots = B_{k-1} = \emptyset$.*

Proof. Assume that $B_1 = B_2 = \cdots = B_{k-1} = \emptyset$ so that $\alpha = 12 \dots kB_k$. Since the last two letters of 112 form an ascent and kB_k is decreasing (with pace d), any occurrence of 112 would have to wholly reside in the prefix $12 \dots k$ of α, but that is clearly impossible. Similarly, α avoids 213. To prove the converse, assume that $B_i \neq \emptyset$ for some $i \in [k-1]$ and let b be the first letter of B_i. If $b = i$ (this can only happen if $d = 0$), then ibk is an occurrence of 112 in α. Otherwise, $b < i$ and ibk is an occurrence of 213 in α. □

Proposition 5.3. *We have*

$$F_{d+1,n-1}(q) = \sum_{\alpha \in \widetilde{\mathrm{A}}_{d,n}(112,213)} q^{\operatorname{wdes} \alpha} = \sum_{\pi \in \widetilde{\mathfrak{F}}_{d,n}(213)} q^{\operatorname{ides} \pi},$$

where $\operatorname{wdes} \alpha = \# \operatorname{wDes} \alpha$ *is the number of weak descents of* α *and* $\operatorname{ides} \pi = \operatorname{des}(\pi^{-1})$ *is the number of inverse descents of* π.

Proof. Let $\alpha \in \widetilde{\mathrm{A}}_d(112, 213)$, $k = \max \alpha$ and write $\alpha = 12 \dots kB_k$ as in Lemma 5.2. By following the same reasoning as in the proof of Theorem 3.4, we find that

$$\sum_{\alpha} q^{\max \alpha} x^{|\alpha|} = 1 + \sum_{k \geq 1} q^k x^k F_{d,k-1}(x) = 1 + qxF_d(x, qx),$$

in which α ranges over $\widetilde{\mathrm{A}}_d(112, 213)$. Using the explicit formula (7) it is easy to verify that $F_{d+1}(q, x) = F_d(qx, x)$. Also, by Lemma 3.3, $\operatorname{wdes} \alpha + \max \alpha = |\alpha|$. Putting this all together we have

$$\sum_{\alpha} q^{\operatorname{wdes} \alpha} x^{|\alpha|} = \sum_{\alpha} q^{-\max \alpha} (qx)^{|\alpha|} = 1 + xF_d(qx, x) = 1 + xF_{d+1}(q, x).$$

The first equality of our proposition follows by identifying coefficients in this identity.

Now, the Fishburn basis of 213 is $\{112, 213\}$ and, by Corollary 5.1, the Burge transpose `t` is a bijection between $\widetilde{\mathrm{A}}_{d,n}(112, 213)$ and $\widetilde{\mathfrak{F}}_{d,n}(213)$. More specifically [CC24, Lemma 5.2], a sequence $\alpha \in \widetilde{\mathrm{A}}_{d,n}(112, 213)$ with k weak descents is mapped under `t` to a permutation $\pi \in \widetilde{\mathfrak{F}}_{d,n}(213)$ with k inverse descents, and hence the second equality follows. □

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