Modified difference ascent sequences and Fishburn structures

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Abstract

Ascent sequences and their modified version play a central role in the bijective framework relating several combinatorial structures counted by the Fishburn numbers. Ascent sequences are nonnegative integer sequences defined by imposing a bound on the growth of their entries in terms of the number of ascents contained in the corresponding prefix, while modified ascent sequences are the image of ascent sequences under the so-called hat map. By relaxing the notion of ascent, Dukes and Sagan have recently introduced difference ascent sequences. Here we define modified difference ascent sequences and study their combinatorial properties. Inversion sequences are a superset of the difference ascent sequences and we extend the hat map to this domain. Our extension depends on a parameter which we specialize to obtain a new set of permutations counted by the Fishburn numbers and characterized by a subdiagonality property.

1 Introduction

Fishburn structures is a collective term for combinatorial objects counted by the Fishburn numbers. These numbers appear as sequence A22493 in the OEIS [OEI] and the nth Fishburn number is defined as the coefficient of x^n in the series

$$\sum_{n\geq 0}\prod_{k=1}^{n}\bigl(1-(1-x)^k\bigr).$$

This generating function first appeared 2001 in a paper by Zagier [Zag01] concerned with bounds on the dimension of the space of Vassiliev's knot invariants. Eight years later, Bousquet-Mélou, Claesson, Dukes and Kitaev [BMCDK10] proved that this series also enumerates unlabeled interval orders, thus resolving a long standing open problem. Peter C. Fishburn pioneered the study of interval orders [Fis70a, Fis70b, Fis85] and it is in honor of him Claesson and Linusson [CL11] named the coefficients of Zagier's series.

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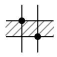
Bousquet-Mélou et al. [BMCDK10] laid the foundation of a bijective framework relating interval orders, Stoimenow matchings, and Fishburn permutations, defined by avoidance of a single bivincular pattern of length three. To link these objects, as well as to count them, they introduced an auxiliary set of sequences that embody their recursive structure more transparently, the ascent sequences. They defined them as certain nonnegative integer sequences whose growth of their entries is bounded by the number of ascents contained in the corresponding prefix. Research into Fishburn structures (sparked by the work of Bousquet-Mélou et al.) has blossomed over the last 15 years. The structures studied are mostly the ones previously mentioned but also include Fishburn matrices [Fis70b, DP10], descent correcting sequences [CL11] and inversion sequences avoiding the covincular pattern . Recently, Cerbai and Claesson [CC23a] introduced Fishburn trees and Fishburn covers to obtain simplified versions of the existing bijections.

The bijection relating ascent sequences with Fishburn permutations is easy to describe. Ascent sequences encode the recursive construction of Fishburn permutations by insertion of a new maximum element. On the other hand, their relation with (**2**+**2**)-free posets is better expressed in terms of a modified version, that is, their bijective image under the hat map. Roughly speaking, the hat map goes through the ascent tops of a given ascent sequence; at each step it increases by one all the entries in the corresponding prefix that are currently greater than or equal to the current ascent top. Modified ascent sequences interact better with Fishburn trees too, as they are simply obtained by reading the labels of Fishburn trees with the in-order traversal. Further, Fishburn trees arise from the max-decomposition of modified ascent sequences. In fact, even though they only appeared marginally in the original paper [BMCDK10], modified ascent sequences have recently assumed a key role in the understanding of Fishburn structures [CC23b, CC23a, Cera, Cerb].

In 2023, Bényi, Claesson and Dukes [BCD23] generalized ascent sequences to weak ascent sequences. They are defined analogously to the classical case, but (strict) ascents are replaced with weak ascents. In the spirit of the original framework, the authors provided bijections with several classes of matrices, posets and permutations. Among them, weak ascent sequences encode the active site construction of weak Fishburn permutations, a superset of Fishburn permutations defined by avoidance of a single bivincular pattern of length four.

By relaxing the bound on the growth of the rightmost entry further, that is, by replacing ascents or weak ascents with difference d ascents, Dukes and Sagan [DS23] arrived at d-ascent sequences. This allowed them to generalize the ascent and weak ascent constructions whose corresponding combinatorial objects now depended on the paramenter d. They also provided natural injections from d-ascent sequences to permutations avoiding a bivincular pattern of length $d+3$, leaving the problem of improving these maps to bijections open. This was done very recently by Zang and Zhou [ZZ], who introduced d-Fishburn permutations and proved that their recursive structure is embodied by d-ascent sequences in the same way as ascent sequences encode Fishburn permutations.

In this paper, we generalize the hat map to d-ascent sequences, obtaining modified d-ascent sequences in the process. We present a recursive construction of modified d-ascent sequences and use it to study their combinatorial properties. Our framework is in fact more flexible: it extends to inversion sequences, a superset of d-

ascent sequences. Further, our definition of the hat map depends on a parameter whose specific choices lead to interesting examples. Fishburn permutations are obtained [BMCDK10] by applying the Burge transpose [CC23b] to modified ascent sequences, and we prove that the same construction holds for modified d-ascent sequences and d-Fishburn permutations. Finally, we initiate the study of pattern avoidance on d-Fishburn permutations.

We start by giving the necessary tools and definitions in Section 2.

In Section 3, we introduce the d-hat map and use it to define the set of modified d-ascent sequences. We then provide a recursive description of modified d-ascent sequences and show in Proposition 3.3 that they are Cayley permutations whose d-ascent set is equal to the set of leftmost copies of the unmodified sequence.

Section 4 is devoted to the study of certain properties of the d-hat map. Our main result, Corollary 4.7, shows that d-hat is injective on modified d-ascent sequences. We then consider which statistics are preserved by d-hat in Section 4.2.

In Section 5, we define modified inversion sequences and the $\mathrm{hat}_{\max}$ map. We show that, under $\mathrm{hat}_{\max}$, a permutation corresponds bijectively to the inversion sequence recording its recursive construction by insertion of a new rightmost maximum value.

This approach is pushed further in Section 6. We restrict the $\mathrm{hat}_{\max}$ map to ascent sequences and weak descent sequences, characterizing the corresponding sets of permutations as those that are subdiagonal in a certain sense.

In Section 7, we prove that d-Fishburn permutations can be obtained as the bijective image of d-ascent sequences under the composition of the d-hat map with the Burge transpose, lifting a classical result by Bousquet-Mélou et al. [BMCDK10] to any $d \geq 0$.

In Section 8, we enumerate d-Fishburn permutations avoiding 231 using a bijection with certain Motzkin paths and the cluster method.

Section 9 contains some final remarks and suggestions for future work.

2 Preliminaries

For any nonnegative integer number n, let End_n be the set of *endofunctions*, $\alpha : [n] \to [n]$, where $[n] = \{1, 2, \dots, n\}$. We sometimes identify an endofunction α with the word $\alpha = a_1 \dots a_n$, where $a_i = \alpha(i)$ for each $i \in [n]$. We will use the convention that Greek letters will usually be used for sequences and the corresponding Roman letters will be used for their elements so, for example, a_i will be the ith element of α unless otherwise indicated. Let $\mathrm{End} = \cup_{n\geq 0} \mathrm{End}_n$. In general, given a definition of E_n (of elements of size n) we let $E = \cup_{n\geq 0} E_n$. Or, conversely, given a set E whose elements are equipped with a notion of size, we will denote by E_n the set of elements in E that have size n.

A *Cayley permutation* is an endofunction α where $\mathrm{Im}\,\alpha = [k]$, for some $k \leq n$. In other words, α is a Cayley permutation if it contains at least one copy of each integer between 1 and its maximum element. The set of Cayley permutations of length n is denoted by Cay_n. For example, $\mathrm{Cay}_1 = \{1\}$, $\mathrm{Cay}_2 = \{11, 12, 21\}$ and

$$\mathrm{Cay}_3 = \{111, 112, 121, 122, 123, 132, 211, 212, 213, 221, 231, 312, 321\}.$$

There is a well-known one-to-one correspondence between ballots (ordered set partitions) and Cayley permutations: The Cayley permutation $\alpha = a_1 \dots a_n$ encodes the ballot $B_1 \dots B_k$ where $k = \max \alpha$ and $i \in B_{a_i}$ for every $i \in [n]$.

An endofunction $\alpha \in \mathrm{End}_n$ is an *inversion sequence* if $a_i \leq i$ for each $i \in [n]$. We let I_n denote the set of inversion sequences of length n. For example,

$$\mathrm{I}_1 = \{1\}, \quad \mathrm{I}_2 = \{11, 12\}, \quad \mathrm{I}_3 = \{111, 112, 113, 121, 122, 123\}.$$

Let $\alpha : [n] \to [n]$ be an endofunction. We call $i \in [n]$ an *ascent* of α if $i = 1$ or $i \geq 2$ and

$$a_i > a_{i-1}.$$

We define the *ascent set* of α to be

$$\mathrm{Asc}\, \alpha = \{i \in [n] \mid i \text{ is an ascent of } \alpha\}$$

and

$$\mathrm{asc}\, \alpha = \#\, \mathrm{Asc}\, \alpha$$

where, for any set S, $\#S$ denotes the cardinality of S. Note that our conventions differ from some others in the literature in that we are taking the indices of ascent tops, rather than bottoms, and that 1 is always an ascent which is done for the purpose of simplifying the definition of an ascent sequence. It will sometimes be convenient to order $\mathrm{Asc}\, \alpha$ and other similar sets below increasingly to obtain the *ascent list*

$$\mathrm{Asc}\, \alpha = (i_1, i_2, \dots, i_k),$$

where $k = \mathrm{asc}\, \alpha$. Our notation will not distinguish between the set and its sequence.

From now on, let $\alpha_i = a_1 \dots a_i$ denote the prefix of α of length i. Call α an *ascent sequence* if for all $i \in [n]$ we have

$$a_i \leq 1 + \mathrm{asc}\, \alpha_{i-1}.$$

Note that when $i = 1$ we have $a_1 \leq 1 + \mathrm{asc}\, \epsilon = 1$, where ϵ denotes the empty sequence. Since the entries of α are positive integers, this forces $a_1 = 1$. Let A_0 be the set of ascent sequences and let $\mathrm{A}_{0,n}$ denote the set of ascent sequences of length n. For instance,

$$\mathrm{A}_{0,3} = \{111, 112, 121, 122, 123\}.$$

Clearly, every $\alpha \in \mathrm{A}_{0,n+1}$ is of the form $\alpha = \beta a$, where $\beta \in \mathrm{A}_{0,n}$ and $1 \leq a \leq 1 + \mathrm{asc}(\beta)$. Note that $\mathrm{A}_{0,n} \subseteq \mathrm{I}_n$. On the other hand, some ascent sequences are not Cayley permutations, the smallest example of which is 12124. Note also that we depart slightly from the original definition of ascent sequences [BMCDK10] and other papers on the topic in that our sequences use the positive, rather than nonnegative, integers. The reason is that we want to bring all the families of sequences considered in this paper under the umbrella of endofunctions of $[n]$ so as to relate them with Cayley permutations and inversion sequences.

The set $\hat{\mathrm{A}}_0$ of *modified ascent sequences* [BMCDK10] is the bijective image of A_0 under the $\alpha \mapsto \hat{\alpha}$ mapping, defined as follows. Given an ascent sequence α, let

$$M(\alpha, j) = \alpha^+, \text{ where } \alpha^+(i) = a_i + \begin{cases} 1 & \text{if } i < j \text{ and } a_i \geq a_j, \\ 0 & \text{otherwise,} \end{cases}$$

and extend the definition of M to multiple indices j_1,j_2,. . . ,j_k by

$$M(\alpha, j_1, j_2, \ldots, j_k) = M\big(M(\alpha, j_1, \ldots, j_{k-1}), j_k\big).$$

Then

$$\hat{\alpha} = M(\alpha, \operatorname{Asc}\alpha),$$

where in this context $\operatorname{Asc}\alpha$ is the ascent list of α. For example, if $\alpha = 121242232$, then $\operatorname{Asc}\alpha = (1, 2, 4, 5, 8)$ and we get the following where at each stage the entry governing the modification is underlined while the entries which are modified italicized:

$$\begin{aligned} \alpha &= 121242232 \\ M(\alpha,1) &= \underline{1}21242232 \\ M(\alpha,1,2) &= 1\underline{2}1242232 \\ M(\alpha,1,2,4) &= 1\mathbf{3}1\underline{2}42232 \\ M(\alpha,1,2,4,5) &= 1312\underline{4}2232 \\ M(\alpha,1,2,4,5,8) &= 1\mathbf{4}12\mathbf{5}22\underline{3}2 = \hat{\alpha} \end{aligned}$$

More informally, to determine $\hat{\alpha}$, we scan the ascents of α from left to right; at each step, every element strictly to the left of and weakly larger than the current ascent is incremented by one. The construction described above can easily be inverted since $\operatorname{Asc}\alpha = \operatorname{Asc}\hat{\alpha}$. Thus the mapping $\mathrm{A}_0 \to \hat{\mathrm{A}}_0$ by $\alpha \mapsto \hat{\alpha}$ is a bijection.

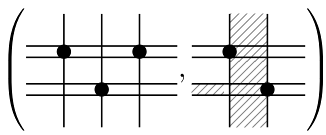
It is easy to turn this into a definition of $\hat{\mathrm{A}}_0$ which is recursive by length and will be given later (see Definition 3.2). Finally, in [CC23b] it was proved that

$$\hat{\mathrm{A}}_0 = \{\alpha \in \mathrm{Cay} \mid \operatorname{Asc}\alpha = \operatorname{nub}\alpha\}, \tag{1}$$

where

$$\operatorname{nub}\alpha = \{\min \alpha^{-1}(j) \mid 1 \le j \le \max\alpha\}$$

is the set of positions of leftmost copies. Equation (1) can be equivalently expressed in terms of Cayley-mesh patterns, introduced by the first author [Cer21], as

$\hat{\mathrm{A}}_0 \;=\; \mathrm{Cay}$  .

In the above pair of forbidden Cayley-mesh patterns, the leftmost one indicates an ascent that is not a leftmost copy; and the one on the right stands for a leftmost copy that is not an ascent. Unlike A_0, not every modified ascent sequence is an inversion sequence. For instance, the modified ascent sequence of $\alpha = 1212$ is $\hat{\alpha} = 1312$.

Dukes and Sagan [DS23] have recently introduced *difference d ascent sequences.* Let $\alpha \in \mathrm{End}_n$. Given a nonnegative number $d \ge 0$, we call $i \in [n]$ a *d-ascent* if $i = 1$ or $i \ge 2$ and

$$a_i > a_{i-1} - d.$$

As with ordinary ascents, we have the *d-ascent set* (or *list*)

$$\operatorname{Asc}_d \alpha = \{i \in [n] \mid i \text{ is a } d\text{-ascent of } \alpha\}.$$

and *d-ascent number*

$$\operatorname{asc}_d \alpha = \# \operatorname{Asc}_d \alpha.$$

Note that a 0-ascent is simply an ascent, while a 1-ascent is what is called a *weak ascent*:

$$a_i > a_{i-1} - 1 \iff a_i \geq a_{i-1}.$$

The analogue of the definition of an ascent sequence in the weak case is as expected. Call α a *d-ascent sequence* if for all $i \in [n]$ we have

$$a_i \leq 1 + \operatorname{asc}_d \alpha_{i-1}.$$

Once again, the above restriction forces $a_1 = 1$. From now on, denote by $\mathrm{A}_{d,n}$ the set of d-ascent sequences of length n. Clearly, for $d = 0$ we recover the set of ascent sequences, while for $d = 1$ we obtain the set of weak ascent sequences of Bényi et al. [BCD23]. Note also that $\operatorname{Asc}_d \alpha \subseteq \operatorname{Asc}_{d+1} \alpha$ for each d, from which the chain of containments

$$\mathrm{A}_0 \subseteq \mathrm{A}_1 \subseteq \mathrm{A}_2 \subseteq \mathrm{A}_3 \subseteq \dots \tag{2}$$

follows immediately.

We now connect d-ascent sequences and inversion sequences.

Lemma 2.1. *We have*

$$\mathrm{I} = \bigcup_{d \geq 0} \mathrm{A}_d\,.$$

Proof. We will prove that each side of the equality is contained in the other. We first show that $\mathrm{A}_d \subseteq \mathrm{I}$ for all $d \geq 0$ which will give one of the desired inclusions. If $\alpha = a_1 \dots a_n \in \mathrm{A}_d$ then $a_1 = 1$ as required for an inversion sequence. For $i \geq 2$, we have

$$a_i \leq 1 + \operatorname{asc}_d \alpha_{i-1} \leq 1 + (i-1) = i.$$

Thus $\alpha \in \mathrm{I}$.

For the other direction, it suffices to show that $\mathrm{I}_n \subseteq \mathrm{A}_{n,n}$. So take $\alpha = a_1 \dots a_n \in \mathrm{I}_n$. We have $a_1 = 1$ as needed. And for $i \geq 2$ we have $a_{i-1} \leq i - 1 \leq n - 1$. Hence $a_i > -1 \geq a_{i-1} - n$. Thus every index $i \geq 2$ is an n-ascent and so

$$a_i \leq i = 1 + \operatorname{asc}_n \alpha_{i-1}$$

showing that $\alpha \in \mathrm{A}_{n,n}$. ∎

We can now calculate the cardinality of certain $\mathrm{A}_{d,n}$.

Proposition 2.2. *For all $d \geq 0$ we have*

$$\#\,\mathrm{A}_{d,n} = \begin{cases} n! & \text{if } n \leq d+2, \\ (d+3)! - d! & \text{if } n = d+3. \end{cases}$$

Proof. By the previous lemma $\mathrm{A}_{d,n} \subseteq \mathrm{I}_n$. Since $\#I_n = n!$, to prove the first statement of the proposition, it suffices to show that if $n \leq d+2$ then every inversion sequence of length n is a d-ascent sequence.

Let $\alpha \in \mathrm{I}_n$ where $n \leq d+2$. We claim that for every proper prefix α_i, $i \leq d+1$, we have $\operatorname{Asc}_d \alpha_i = [i]$. Indeed, consider any element $a_j \in \alpha_i$. Then, since α is an inversion sequence,

$$a_{j-1} \leq j - 1 \leq i - 1 \leq d.$$

Also $a_j \geq 1$. So $a_{j-1} - a_j \leq d-1 < d$, which forces $j \in \operatorname{Asc}_d \alpha_i$, proving the claim. Now, for all $a_k \in \alpha$ we have

$$a_k \leq k = 1 + \operatorname{asc}_d \alpha_{k-1},$$

hence α is a d-ascent sequence, as desired.

To prove the second part of the proposition, we just need to show that when $n = d+3$ there are exactly $d!$ elements of I_{d+3} which are not d-ascent sequences.

Let $\alpha = a_1 \ldots a_{d+3}$ be such a sequence. We show that the last three entries of α are

$$a_{d+1}, a_{d+2}, a_{d+3} = d+1, 1, d+3,$$

while the prefix $\beta = a_1 \ldots a_d$ can be any inversion sequence of size d. If we had $\operatorname{Asc}_d(\beta a_{d+1} a_{d+2}) = [d+2]$, then, using an argument like that of the previous paragraph, we would have $\alpha \in \mathrm{A}_d$, which is a contradiction. On the other hand, it follows from the proof of the first part that $\operatorname{Asc}_d(\beta a_{d+1}) = [d+1]$. So it must be that $d+2 \notin \operatorname{Asc}_d(\beta a_{d+1} a_{d+2})$, i.e. $a_{d+2} \leq a_{d+1} - d$. Together with the fact that $a_{d+1} \leq d+1$ and $a_{d+2} \geq 1$, this forces

$$a_{d+1} = d+1 \quad \text{and} \quad a_{d+2} = 1.$$

Now, since we assumed that α is not a d-ascent sequence, but we know that its prefix $\beta a_{d+1} a_{d+2}$ is, it must be that

$$a_{d+3} > \operatorname{asc}(\beta a_{d+1} a_{d+2}) + 1 = d+2.$$

Since $\alpha \in \mathrm{I}_{d+3}$ we also have $a_{d+3} \leq d+3$. It follows that there is only one choice for the last element of α, namely $a_{d+3} = d+3$. In the end, we have $\alpha = \beta(d+1)1(d+3)$, where β is any inversion sequence of size d. Since there are $d!$ choices for such β, the proposition is proved. $\square$

3 Modified d-ascent sequences

We wish to extend the hat map $\alpha \mapsto \hat{\alpha}$, originally defined on A_0, to the set A_d. Let $\alpha \in \mathrm{A}_d$, for some $d \geq 0$. The *d-hat* of α is defined as

$$\operatorname{hat}_d(\alpha) = M\big(\alpha, \operatorname{Asc}_d \alpha\big),$$

where $\operatorname{Asc}_d \alpha$ is the d-ascent list of α. The d-hat map is a natural generalization of the hat map, obtained by replacing ascents with d-ascents. As a special case, we have $\operatorname{hat}_0(\alpha) = \hat{\alpha}$ for each $\alpha \in \mathrm{A}_0$. More generally, to compute $\operatorname{hat}_d(\alpha)$ scan the d-ascents of α from left to right; in each step, increment by one every element strictly to the left of and weakly larger than the current d-ascent. From now on, given $d \geq 0$, we let

$$\hat{\mathrm{A}}_d = \operatorname{hat}_d(\mathrm{A}_d)$$

denote the set of *modified d-ascent sequences*.

Let us set up some standard notation we shall use throughout the rest of this paper. We will consider d-ascent sequences $\alpha = \beta a$, where a is the last letter of α and β is a d-ascent sequence of size one less than α. If d is clear from context, we let $\hat{\alpha} = \operatorname{hat}_d(\alpha)$

and $\hat{\beta} = \mathrm{hat}_d(\beta)$. We also use "+" as a superscript that denotes the operation of adding one to the entries $c \geq a$ of a given sequence, where a is a threshold determined by the context. For instance, we denote by $\hat{\beta}^+ a$ the sequence obtained by adding one to each entry of $\hat{\beta}$ that is greater than or equal to a. Clearly, letting b denote the last letter of β, by definition of hat_d we have for every $n \geq 1$ and $\alpha \in \mathrm{A}_{d,n}$

$$\hat{\alpha} = \begin{cases} \hat{\beta} a & \text{if } a \leq b - d, \\ \hat{\beta}^+ a & \text{if } a > b - d. \end{cases} \tag{3}$$

Finally, we will denote the entries of the above sequences by

$$\begin{array}{lll} \alpha = a_1 \dots a_n, & \hat{\alpha} = a'_1 \dots a'_n, & \\ \beta = b_1 \dots b_{n-1}, & \hat{\beta} = b'_1 \dots b'_n, & \hat{\beta}^+ = b''_1 \dots b''_n, \end{array} \tag{4}$$

where n is the size of α. The behavior of hat_d on the last two letters of $\alpha \in \mathrm{A}_d$ is described more explicitly in the next lemma.

Lemma 3.1. *Let $\alpha = a_1 \dots a_n \in \mathrm{A}_d$, for some $d \geq 0$ and $n \geq 2$. Let $\mathrm{hat}_d(\alpha) = \hat{\alpha} = a'_1 \dots a'_n$. Then*

$$a'_{n-1}, a'_n = \begin{cases} a_{n-1} + 1, a_n & \text{if } a_{n-1} - d < a_n \leq a_{n-1}; \\ a_{n-1}, a_n & \text{else.} \end{cases}$$

Proof. We use induction on the size of α. Let $\alpha = \beta a_n$. The last element of $\hat{\alpha}$ is a_n by definition of hat_d. Similarly, the last letter of $\hat{\beta} = \mathrm{hat}_d(\beta)$ is a_{n-1}.

Suppose initially that $a_n > a_{n-1} - d$. Then n is a d-ascent and so $\hat{\alpha} = \hat{\beta}^+ a_n$. Now, if $a_{n-1} \geq a_n$ then $a'_{n-1} = a_{n-1} + 1$ and $\hat{\alpha}$ ends with $a_{n-1} + 1, a_n$. Otherwise, if $a_{n-1} < a_n$ then a_{n-1} will not be incremented and $\hat{\alpha}$ ends with a_{n-1}, a_n.

Finally, if $a_n \leq a_{n-1} - d$ then n is not a d-ascent. So in this case $\hat{\alpha} = \hat{\beta} a_n$ and the last two elements are a_{n-1}, a_n again. □

Our next goal is to provide a recursive definition of $\hat{\mathrm{A}}_d$ which does not depend on constructing A_d first. In the classical case, such a definition of $\hat{\mathrm{A}}_0$ is as follows [CC23b], where we use $\hat{\alpha}$ and $\hat{\beta}$ to denote generic elements of $\hat{\mathrm{A}}_d$. Note that this definition permits the computation of an element $\hat{\alpha}$ in $\hat{\mathrm{A}}_d$ directly from a given $\hat{\beta}$ in $\hat{\mathrm{A}}_d$ without needing to know α itself.

Definition 3.2. We have $\hat{\mathrm{A}}_{0,0} = \{\epsilon\}$ and $\hat{\mathrm{A}}_{0,1} = \{1\}$. Let $n \geq 2$. Then every $\hat{\alpha} \in \hat{\mathrm{A}}_{0,n}$ is of one of two forms depending on whether the last letter forms an ascent with the penultimate letter:

- $\hat{\alpha} = \hat{\beta} a$ and $1 \leq a \leq b$, or
- $\hat{\alpha} = \hat{\beta}^+ a$ and $b < a \leq 1 + \mathrm{asc}\, \hat{\beta}$,

where $\hat{\beta} \in \hat{\mathrm{A}}_{0,n-1}$ and the last letter of $\hat{\beta}$ is b.

We wish to highlight a detail that explains why the definition given above is consistent with letting $\hat{\alpha} = M\big(\alpha, \mathrm{Asc}\, \alpha\big)$. Given $\alpha \in \mathrm{A}$, to compute $\hat{\alpha}$ we increase entries in the current prefix if and only if we encounter an ascent of α. On the other hand, Definition 3.2 is stated directly in terms of the ascents of the modified sequence, i.e. in terms of $\operatorname{Asc}\hat{\beta}$. Since it is known [BMCDK10] that

$$\operatorname{Asc}\alpha = \operatorname{Asc}\operatorname{hat}_0(\alpha), \tag{5}$$

i.e. the ascent set is preserved under the hat map, these two approaches are in fact equivalent.

In the same spirit, we wish to give a recursive definition of $\hat{\mathrm{A}}_d$. The problem in generalizing Definition 3.2 is that in general the d-ascent set, as well as its cardinality, is not preserved under hat_d. For instance, for $d = 1$ we have $\operatorname{hat}_1(11) = 21$ and

$$\{1, 2\} = \operatorname{Asc}_1(11) \neq \operatorname{Asc}_1(21) = \{1\}.$$

A suggestion for an alternative approach comes from the classical case $d = 0$. Let $\alpha \in \mathrm{A}_0$ and let $\hat{\alpha} = \operatorname{hat}_0(\alpha)$. Then [BMCDK10]

$$\operatorname{Asc}\alpha = \operatorname{nub}\hat{\alpha} \quad \text{and} \quad \operatorname{asc}\beta = \max\hat{\alpha}. \tag{6}$$

In fact, the corresponding equalities hold for every $d \geq 0$, as we show in the next proposition.

Proposition 3.3. *Given $d \geq 0$, let $\alpha \in \mathrm{A}_d$ and let $\hat{\alpha} = \operatorname{hat}_d(\alpha)$. Then $\hat{\alpha}$ is a Cayley permutation with*

$$\operatorname{Asc}_d\alpha = \operatorname{nub}\hat{\alpha} \quad \textit{and} \quad \operatorname{asc}_d\alpha = \max\hat{\alpha}.$$

Proof. We use induction on the size of α. It is easy to see that the statement holds if α has length zero or one. Let $n \geq 2$ and let $\alpha \in \mathrm{A}_{d,n}$. As usual, let $\alpha = \beta a$, for some $\beta \in \mathrm{A}_{d,n-1}$ and $1 \leq a \leq 1 + \operatorname{asc}_d\beta$. By induction, $\hat{\beta} = \operatorname{hat}_d(\beta)$ is a Cayley permutation with $\operatorname{Asc}_d\beta = \operatorname{nub}\hat{\beta}$ and $\operatorname{asc}_d\beta = \max\hat{\beta}$. Following the definition of hat_d, we consider two possibities according to wheter or not a forms a d-ascent with the last letter b of β.

- Suppose $a \leq b - d$. Then $\hat{\alpha} = \hat{\beta}a$. Note that $\hat{\alpha} \in \mathrm{Cay}_n$ since $\beta \in \mathrm{Cay}_{n-1}$ and $a \leq b - d \leq \max\beta$. Furthermore,

  $$\operatorname{Asc}_d\alpha = \operatorname{Asc}_d\beta = \operatorname{nub}\hat{\beta} = \operatorname{nub}\hat{\alpha}$$

  and

  $$\operatorname{asc}_d\alpha = \operatorname{asc}_d\beta = \max\hat{\beta} = \max\hat{\alpha}.$$

- Suppose $a > b - d$. Then $\hat{\alpha} = \hat{\beta}^+a$. Once again, it is easy to see that $\hat{\alpha} \in \mathrm{Cay}_n$ as follows.

  First note that by the definition of d-hat and induction we have

  $$a \leq \operatorname{asc}_d\beta + 1 = \max\hat{\beta} + 1$$

  If $a = \max\hat{\beta} + 1$, then $\hat{\beta}^+ = \hat{\beta}$ and

  $$\operatorname{Im}\hat{\alpha} = \operatorname{Im}\hat{\beta} \cup \{a\} = [\max\hat{\beta} + 1] = [\max\hat{\alpha}].$$

On the other hand, if $a \le \max \hat\beta$, then the only gap created in $\hat\beta^+$ (by lifting the entries $c \ge a$) is filled by a. More formally,

$$
\begin{aligned}
\operatorname{Im} \hat\alpha &= \operatorname{Im} \hat\beta^+ \cup \{a\} \\
&= \{1, 2, \dots, a-1\} \cup \{a+1, a+2, \dots, \max(\hat\beta)+1\} \cup \{a\} \\
&= [\max \hat\beta + 1] \\
&= [\max \hat\alpha].
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\operatorname{Asc}_d \alpha &= \operatorname{Asc}_d \beta \uplus \{n\} \\
&= \operatorname{nub} \hat\beta \uplus \{n\} \\
&= \operatorname{nub} \hat\beta^+ \uplus \{n\} \\
&= \operatorname{nub} \hat\alpha,
\end{aligned}
$$

where the last equality follows since a is a leftmost copy in $\hat\alpha$, and

$$\operatorname{asc}_d \alpha = \operatorname{asc}_d \beta + 1 = \max \hat\beta + 1 = \max \hat\alpha.$$

This finishes the proof of the proposition. □

The equality $\operatorname{asc}_d \beta = \max \operatorname{hat}_d(\beta)$ proved in Proposition 3.3 leads us to the following recursive definition of modified d-ascent sequences, where we use the same notational conventions as in Definition 3.2.

Definition 3.4. Let $d \ge 0$ be a nonnegative integer. Let $\hat{\mathrm{A}}_{d,0} = \{\epsilon\}$ and $\hat{\mathrm{A}}_{d,1} = \{1\}$. Then, suppose $n \ge 2$. Every $\hat\alpha \in \hat{\mathrm{A}}_{d,n}$ is of one of two forms depending on whether the last letter forms a d-ascent with the penultimate letter:

- $\hat\alpha = \hat\beta a$ and $1 \le a \le b - d$, or
- $\hat\alpha = \hat\beta^+ a$ and $b - d < a \le 1 + \max \hat\beta$,

where $\hat\beta \in \hat{\mathrm{A}}_{d,n-1}$ and the last letter of $\hat\beta$ is b.

The reader will immediately realize that the previous definition is obtained by replacing $\operatorname{asc} \hat\beta$ with $\max \hat\beta$ in Definition 3.2. When $d = 0$, the two definitions are equivalent by equation (6). Modified d-ascent sequences are built recursively by insertion of a new rightmost entry a, which is at most equal to one plus the current maximum; the parameter d determines the cases where the prefix is rescaled (by adding one to each entry $c \ge a$). For convenience, the analogous definitions of A_d and $\hat{\mathrm{A}}_d$ are illustrated below:

$$
\begin{aligned}
&(\mathrm{A}_d) \quad \alpha = \beta a,\ 1 \le a \le 1 + \operatorname{asc}_d \beta. \\
&(\hat{\mathrm{A}}_d) \quad \hat\alpha = \begin{cases} \hat\beta a, & 1 \le a \le b - d; \\ \hat\beta^+ a, & b - d < a \le 1 + \max \hat\beta. \end{cases}
\end{aligned}
$$

The equality $\operatorname{asc}_d \beta = \max \operatorname{hat}_d(\beta)$ acts as a bridge between the two definitions.

Let us end this section with a remark. In general, the set $\hat{\mathrm{A}}_d$ is not included in $\hat{\mathrm{A}}_{d+1}$. For instance, we have

$$\hat{\mathrm{A}}_0 = \{\mathrm{hat}_0(11), \mathrm{hat}_0(12)\} = \{11, 12\}$$

and

$$\hat{\mathrm{A}}_1 = \{\mathrm{hat}_1(11), \mathrm{hat}_1(12)\} = \{21, 12\}.$$

4 Properties of d-hat

We devote this section to the study of several aspects related to the d-hat map just introduced.

Recall from Proposition 3.3 that $\mathrm{nub}\,\mathrm{hat}_d(\alpha) = \mathrm{Asc}_d\,\alpha$. When $d = 0$, using equation (5) we obtain the equality

$$\mathrm{nub}\,\mathrm{hat}_0(\alpha) = \mathrm{Asc}\,\mathrm{hat}_0(\alpha)$$

characterizing $\hat{\mathrm{A}}$ as a subset of Cay (see equation (1)). Since we have established in Proposition 3.3 that $\hat{\mathrm{A}}_d \subseteq \mathrm{Cay}$ for every $d \geq 0$, a natural question arises:

Is there an analogous equality characterizing $\hat{\mathrm{A}}_d$ when $d \geq 1$?

As mentioned before Proposition 3.3, the equality $\mathrm{Asc}_d\,\mathrm{hat}_d(\alpha) = \mathrm{Asc}_d\,\alpha$ does not hold for $d \geq 1$. However, we show in Proposition 4.3 that one inclusion holds. First, a simple lemma.

Lemma 4.1. *Let $\beta \in \mathrm{End}$ and let β^+ be obtained from β by increasing by one each entry greater than or equal to a, for some $a \geq 0$. Then for all $d \geq 0$*

$$\mathrm{Asc}_d\,\beta^+ \subseteq \mathrm{Asc}_d\,\beta \quad \text{and} \quad \mathrm{Asc}\,\beta^+ = \mathrm{Asc}\,\beta.$$

Proof. Let $\beta = b_1 \dots b_n$ and $\beta^+ = b'_1 \dots b'_n$, where $b'_i = b_i$, if $b_i < a$, and $b'_i = b_i + 1$, if $b_i \geq a$. Note that the first position $i = 1$ is a d-ascent by definition. On the other hand, let $i \geq 2$ and suppose that $i \in \mathrm{Asc}_d\,\beta^+$. We show that $i \in \mathrm{Asc}_d\,\beta$. For a contradiction, suppose that $i \notin \mathrm{Asc}_d\,\beta$. More explicitly, we have

$$\begin{aligned} i \notin \mathrm{Asc}_d\,\beta &\iff b_i \leq b_{i-1} - d; \\ i \in \mathrm{Asc}_d\,\beta^+ &\iff b'_i > b'_{i-1} - d. \end{aligned}$$

Comparing the two inequalities forces $b'_i = b_i + 1$ and $b'_{i-1} = b_{i-1}$. Therefore, we have $b_{i-1} < a \leq b_i$ and

$$b_i \leq b_i + d \leq b_{i-1} < a \leq b_i,$$

which gives us the desired contradiction.

By the previous part of the proposition (and since an ascent is a 0-ascent), to prove the remaining equality $\mathrm{Asc}\,\beta^+ = \mathrm{Asc}\,\beta$ we only need to show that $\mathrm{Asc}\,\beta^+ \supseteq \mathrm{Asc}\,\beta$. Let $i \in \mathrm{Asc}\,\beta$. If $i = 1$, then $i \in \mathrm{Asc}\,\beta^+$. If instead $i \geq 2$, then $b_i > b_{i-1}$ and thus b_i will be increased in β^+ if b_{i-1} is increased. In any case, we have $b'_i > b'_{i+1}$, hence $i \in \mathrm{Asc}\,\beta^+$. This completes the proof. $\square$

Corollary 4.2. *Let $d \geq 0$. Suppose that $\alpha \in \mathrm{A}_d$ and let $\hat{\alpha} = \mathrm{hat}_d(\alpha)$. Then*

$$\mathrm{Asc}_d\, \hat{\alpha} \subseteq \mathrm{Asc}_d\, \alpha.$$

Proof. We use induction on the size n of α, taking the case $n \leq 1$ for granted. Assume $n \geq 2$. Let $\alpha = \beta a$, where $\beta \in \mathrm{A}_{d,n-1}$ and $1 \leq a \leq 1 + \mathrm{asc}_d\, \beta$, and let $\hat{\beta} = \mathrm{hat}_d(\beta)$. As usual, we consider two cases according to whether or not the last letter b of β forms a d-ascent with a.

Suppose first that $1 \leq a \leq b - d$. Then $\hat{\alpha} = \hat{\beta} a$, where $\mathrm{Asc}_d\, \beta \subseteq \mathrm{Asc}_d\, \hat{\beta}$ by induction. Now by Definition 3.4

$$\mathrm{Asc}_d\, \hat{\alpha} = \mathrm{Asc}_d\, \hat{\beta} \subseteq \mathrm{Asc}_d\, \beta = \mathrm{Asc}_d\, \alpha.$$

Otherwise, suppose that $b - d < a \leq 1 + \mathrm{asc}_d\, \beta$. Then $\hat{\alpha} = \hat{\beta}^+ a$. Now using Lemma 4.1 and induction we have

$$\mathrm{Asc}_d\, \hat{\alpha} \subseteq \mathrm{Asc}_d\, \hat{\beta}^+ \cup \{n\} \subseteq \mathrm{Asc}_d\, \hat{\beta} \cup \{n\} \subseteq \mathrm{Asc}_d\, \beta \cup \{n\} = \mathrm{Asc}_d\, \alpha.$$

This completes the demonstration. □

Combining Proposition 3.3 and Corollary 4.2 immediately gives the following result.

Proposition 4.3. *Let $d \geq 0$. We have, for any $\hat{\alpha} \in \hat{\mathrm{A}}_d$,*

$$\mathrm{Asc}_d\, \hat{\alpha} \subseteq \mathrm{nub}\, \hat{\alpha}.$$ □

4.1 Injectivity of hat_d

Our next goal is to prove that d-hat is injective on A_d for every $d \geq 0$. Let $\alpha \in \mathrm{I}$ be an inversion sequence. By Lemma 2.1, the quantity

$$\mathrm{dmin}\, \alpha = \min\{d \geq 0 \mid \alpha \in \mathrm{A}_d\}.$$

is a nonnegative integer for every α. Furthermore, by equation (2) if α is a d-ascent sequence for some d, then it is a k-ascent sequence for every $k \geq d$. It is natural to study the set

$$H(\alpha) = \{\mathrm{hat}_d(\alpha) \mid d \geq \mathrm{dmin}\, \alpha\}$$

of all the (meaningful) d-hats of α. Note that $H(\alpha) \subseteq \mathrm{Cay}$ by Proposition 3.3. Next, we show that $H(\alpha)$ is finite.

Lemma 4.4. *Let $\alpha \in \mathrm{I}_n$. Then $\mathrm{dmin}\, \alpha \leq n$. Further, we have $\mathrm{hat}_d(\alpha) = \mathrm{hat}_n(\alpha)$ for each $d \geq n$.*

Proof. Recall from the proof of Lemma 2.1 that $\mathrm{I}_n \subseteq \mathrm{A}_{n,n}$. The inequality $\mathrm{dmin}\, \alpha \leq n$ follows immediately. Finally, let $d \geq n$. Then

$$\mathrm{Asc}_d\, \alpha = \mathrm{Asc}_n\, \alpha = [n]$$

and the equality $\mathrm{hat}_d(\alpha) = \mathrm{hat}_n(\alpha)$ follows directly from the definition of d-hat. □

By Lemma 4.4, we have

$$H(\alpha) = \{\mathrm{hat}_d(\alpha) \mid \operatorname{dmin} \alpha \le d \le |\alpha|\},$$

from which the following corollary is obtained immediately.

Corollary 4.5. *Let α be an inversion sequence. Then $H(\alpha)$ is finite.* □

Let us now prove that the sets $H(\alpha)$ are disjoint. The injectivity of hat_d over A_d will immediately follow as a corollary.

Proposition 4.6. *Let α and σ be inversion sequences and suppose that $H(\alpha) \cap H(\sigma) \neq \emptyset$. Then $\alpha = \sigma$.*

Proof. We use induction on the size. The statement clearly holds for inversion sequences of size $n \le 1$, so suppose $n \ge 2$. Let α and σ be in I_n, with $H(\alpha) \cap H(\sigma) \neq \emptyset$. If $\gamma \in H(\alpha) \cap H(\sigma)$, then

$$\mathrm{hat}_d(\alpha) = \mathrm{hat}_k(\sigma) = \gamma,$$

for some $d \ge \operatorname{dmin} \alpha$ and $k \ge \operatorname{dmin} \sigma$. We prove that $\alpha = \sigma$. Denote by y the last letter of γ. Note that the last letters of α and σ are equal to y as well. That is, we have $\alpha = \beta y$ and $\sigma = \tau y$, where β and τ denote the corresponding prefixes of α and σ. We consider two cases, according to whether or not y is a leftmost copy in γ.

Initially, suppose that $n \notin \operatorname{nub} \gamma$. Recall by Proposition 3.3 that

$$\operatorname{Asc}_d \alpha = \operatorname{nub} \gamma = \operatorname{Asc}_k \sigma.$$

In particular, the last position n is neither a d-ascent in α, nor a k-ascent in σ. By definition of hat_d and hat_k, we have, respectively,

$$\gamma = \mathrm{hat}_d(\alpha) = \mathrm{hat}_d(\beta) y$$

and

$$\gamma = \mathrm{hat}_k(\sigma) = \mathrm{hat}_k(\tau) y.$$

This forces $\mathrm{hat}_d(\beta) = \mathrm{hat}_k(\tau)$ so that $H(\beta) \cap H(\tau) \neq \emptyset$. By induction, we have $\beta = \tau$ and consequently

$$\alpha = \beta y = \tau y = \sigma.$$

Finally, suppose that $n \in \operatorname{nub} \gamma$. The proof is similar to the previous case, the difference being that here the last position is a d-ascent in α, as well as a k-ascent in σ. Therefore,

$$\gamma = \mathrm{hat}_d(\alpha) = \mathrm{hat}_d(\beta)^+ y$$

and

$$\gamma = \mathrm{hat}_k(\sigma) = \mathrm{hat}_k(\tau)^+ y,$$

and thus $\mathrm{hat}_d(\beta)^+ = \mathrm{hat}_k(\tau)^+$. Since both $\mathrm{hat}_d(\beta)^+$ and $\mathrm{hat}_k(\tau)^+$ are obtained by rescaling entries $c \ge y$, we have $\mathrm{hat}_d(\beta) = \mathrm{hat}_k(\tau)$, and we can finish the proof as in the previous case. □

Corollary 4.7. *For each $d \ge 0$, we have a bijection* $\mathrm{hat}_d : \mathrm{A}_d \to \hat{\mathrm{A}}_d$. □

4.2 Statistics preserved by hat_d

Let us now turn our attention to which statistics are preserved by d-hat. Define the *weak descent set* of α to be

$$\mathrm{wDes}\,\alpha = \{i \geq 2 \mid a_i \leq a_{i-1}\}$$

We also say that i is a *right-left minimum index* of α if $a_i < a_j$ for all $i < j \leq n$. Further, the set of *right-left minima pairs* is

$$\mathrm{rlMinP}\,\alpha = \{(i, a_i) \mid i \text{ is a right-left minimum index of } \alpha\}.$$

The following lemma will be useful.

Lemma 4.8. *Let $\alpha = a_1 a_2 \dots a_n = \beta a_n$ where $n \geq 1$. Then*

$$\mathrm{rlMinP}(\alpha) = \mathrm{rlMinP}(a_1 \dots a_k) \uplus \{(n, a_n)\}$$

where $k \geq 1$ is the largest right-left minimum index of β such that $a_k < a_n$. If no such index exists then we let $k = 0$ so that $\mathrm{rlMinP}(a_1 \dots a_k) = \mathrm{rlMinP}(\emptyset) = \emptyset$.

Proof. Consider what happens in passing from $\mathrm{rlMinP}\,\beta$ to $\mathrm{rlMinP}\,\alpha$. Of course, (n, a_n) becomes a right-left minimum pair in $\mathrm{rlMinP}\,\alpha$ since a_n is the last element of the sequence. Furthermore, any right-left minimum values a_i of $\mathrm{rlMinP}\,\beta$ with $a_i > a_n$ will now have a smaller element to their right and so it will be removed in the transition to $\mathrm{rlMinP}\,\alpha$. The remaining pairs of $\mathrm{rlMinP}\,\beta$ will be preserved in $\mathrm{rlMinP}\,\alpha$. This is equivalent to our claim. □

Theorem 4.9. *Suppose $\alpha \in \mathrm{I}_n$. We have the following for all $\gamma \in H(\alpha)$:*

(a) $\mathrm{Asc}\,\gamma = \mathrm{Asc}\,\alpha$.

(b) $\mathrm{wDes}\,\gamma = \mathrm{wDes}\,\alpha$.

(c) $\mathrm{rlMinP}\,\gamma = \mathrm{rlMinP}\,\alpha$.

Proof. (a) We induct on n where the case $n \leq 1$ is trivial. Let $\alpha = \beta a$. Pick a d for which α is a d-ascent sequence and let $\hat{\alpha} = \mathrm{hat}_d(\alpha)$ and $\hat{\beta} = \mathrm{hat}_d(\beta)$. We follow our usual conventions (3) and (4) and denote by b, b' and b'' the last letter of β, $\hat{\beta}$ and $\hat{\beta}^+$, respectively. Note that $b' = b$ by Lemma 3.1. By induction, we have

$$\mathrm{Asc}\,\hat{\beta} = \mathrm{Asc}\,\beta.$$

There are now three cases. First suppose that $a \leq b - d$, so that $\hat{\alpha} = \hat{\beta} a$. From this, the induction hypothesis, and the fact that $a \leq b = b'$ we obtain

$$\mathrm{Asc}\,\hat{\alpha} = \mathrm{Asc}(\hat{\beta} a) = \mathrm{Asc}\,\hat{\beta} = \mathrm{Asc}\,\beta = \mathrm{Asc}(\beta a) = \mathrm{Asc}\,\alpha.$$

For the next two cases we will have $a > b - d$ so that n is a d-ascent and $\hat{\alpha} = \hat{\beta}^+ a$. If $a \leq b$, then

$$b'' = b' + 1 = b + 1 > a.$$

Thus, using Lemma 4.1,

$$\operatorname{Asc}\hat{\alpha} = \operatorname{Asc}(\hat{\beta}^+ a) = \operatorname{Asc}\hat{\beta}^+ = \operatorname{Asc}\hat{\beta} = \operatorname{Asc}\beta = \operatorname{Asc}\beta a = \operatorname{Asc}\alpha.$$

Finally, suppose that $a > b$. Then

$$b'' = b' = b < a$$

and, in a similar manner to the first case,

$$\operatorname{Asc}\hat{\alpha} = \operatorname{Asc}\hat{\beta}^+ \cup \{n\} = \operatorname{Asc}\beta \cup \{n\} = \operatorname{Asc}\alpha,$$

proving the first item.

(b) Directly from the definitions, for all inversion sequences α of length n we have $\operatorname{Asc}\alpha \uplus \operatorname{wDes}\alpha = [n]$. So this part follows immediately from (a).

(c) By induction

$$\operatorname{rlMinP}\hat{\beta} = \operatorname{rlMinP}\beta.$$

Again, we begin with the case $a \leq b - d$ so that $\hat{\alpha} = \hat{\beta}a$. By induction and the fact that both α and $\hat{\alpha}$ end in a, we see that the index k in Lemma 4.8 will be the same for both α and $\hat{\alpha}$. Thus, using the same lemma and the inductive hypothesis,

$$\begin{aligned}
\operatorname{rlMinP}\hat{\alpha} &= \operatorname{rlMinP}(b'_1 \dots b'_k) \uplus \{(n, a_n)\} \\
&= \operatorname{rlMinP}(b_1 \dots b_k) \uplus \{(n, a_n)\} \\
&= \operatorname{rlMinP}\alpha.
\end{aligned}$$

Now consider what happens when $a > b - d$ and $\hat{\alpha} = \hat{\beta}^+ a$. We must relate $\operatorname{rlMinP}\hat{\beta}$ and $\operatorname{rlMinP}\hat{\beta}^+$. By the way $\hat{\beta}^+$ is constructed from $\hat{\beta}$ we see that every pair $(i, b'_i) \in \operatorname{rlMinP}\hat{\beta}$ is either replaced by $(i, b'_i + 1) \in \operatorname{rlMinP}\hat{\beta}^+$ if $b'_i \geq a$ or remains as (i, b'_i) if $b'_i < a$. In particular, $\hat{\beta}^+ a$ and $\hat{\beta}a$ will have the same index k from Lemma 4.8. Moreover, due to our choice of k,

$$\operatorname{rlMinP}(b''_1 \dots b''_k) = \operatorname{rlMinP}(b'_1 \dots b'_k) = \operatorname{rlMinP}(b_1 \dots b_k).$$

The proof is now completed in a manner similar to the first case. □

5 Modified inversion sequences

Recall from Lemma 2.1 that $\mathrm{I} = \bigcup_{d \geq 0} \mathrm{A}_d$. We shall define the set $\hat{\mathrm{I}}$ of *modified inversion sequences* as

$$\hat{\mathrm{I}} = \bigcup_{d \geq 0} \hat{\mathrm{A}}_d\,. \tag{7}$$

An alternative way of arriving at $\hat{\mathrm{I}}$ is illustrated in the next result which follows easily from Lemma 2.1 and Proposition 4.6.

Proposition 5.1. *We have the disjoint union*

$$\hat{\mathrm{I}} = \biguplus_{\alpha \in \mathrm{I}} H(\alpha).$$ □

By Proposition 3.3, modified inversion sequences are Cayley permutations; that is, $\hat{\mathrm{I}} \subseteq \mathrm{Cay}$. Further, by Proposition 4.6 given any $\gamma \in \hat{\mathrm{I}}$ there is a unique $\alpha \in \mathrm{I}$ such that $\gamma = \mathrm{hat}_d(\alpha)$, for some $d \geq \operatorname{dmin} \alpha$. Note that such d is not unique, but α is. This allows us to define a map

$$h : \hat{\mathrm{I}} \to \mathrm{I}$$

by letting $h(\gamma)$ be the only inversion sequence α such that $\gamma \in H(\alpha)$. We wish to describe h more explicitly. First, let us recall [BMCDK10] an algorithm to define $h(\gamma)$ in the special case where γ is the modified ascent sequence of $\alpha \in \mathrm{A}_0$. Let $\gamma = g_1 \dots g_n$ and let $\operatorname{Asc} \gamma = (i_1, \dots, i_k)$. Then:

$$
\begin{aligned}
&\texttt{for } i = i_k, \dots, i_1 : \\
&\qquad \texttt{for } j = 1, \dots, i-1 : \\
&\qquad\qquad \texttt{if } g_j > g_i \texttt{ then } g_j := g_j - 1.
\end{aligned}
$$

The output of the above procedure is the desired ascent sequence α. Since $\alpha \in \mathrm{A}_0$, we have $\operatorname{Asc} \alpha = \operatorname{Asc} \gamma$. The previous algorithm goes through the 0-ascents of γ, from right to left, to determine the cases where the entries in the prefix need to be decreased. To define d-hat, we have replaced $\operatorname{Asc} \alpha$ with $\operatorname{Asc}_d \alpha$. By Proposition 3.3, we have $\operatorname{Asc}_d \alpha = \operatorname{nub} \gamma$. Therefore, by replacing $\operatorname{Asc} \gamma$ with $\operatorname{nub} \gamma$ in the algorithm just given we will obtain the desired generalization of h to the set $\hat{\mathrm{I}}$. Surprisingly, the definition does not depend on d. Instead of writing the algorithm explicitly, we shall give an equivalent, recursive description of h. Let $h(\epsilon) = \epsilon$, the empty sequence, and $h(1) = 1$. Suppose $n \geq 2$ and let $\gamma = g_1 \dots g_n \in \hat{\mathrm{I}}_n$. Let $\delta = g_1 \dots g_{n-1}$. Then

$$h(\gamma) = \begin{cases} h(\delta^-) g_n & \text{if } n \in \operatorname{nub} \gamma; \\ h(\delta) g_n & \text{otherwise,} \end{cases}$$

where δ^- is obtained from δ by decreasing by one each entry $c > g_n$. The map $h : \hat{\mathrm{I}} \to \mathrm{I}$ defined this way is bijective and

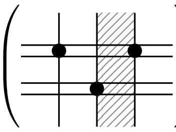
$$h \circ \mathrm{hat}_d(\alpha) = \alpha \quad \text{for every } \alpha \in \mathrm{A}_d.$$

We leave the details to the reader.

To obtain a deeper understanding of $\hat{\mathrm{I}}$, it would be interesting to characterize it as a subset of Cay in the same spirit of equation (1) for $\hat{\mathrm{A}}_0$. The following proposition is a first step in this direction.

Proposition 5.2. *Let* $\gamma \in \hat{\mathrm{I}}$. *Then* $\operatorname{Asc} \gamma \subseteq \operatorname{nub} \gamma$. *Thus,*

$$\hat{\mathrm{I}} \subseteq \mathrm{Cay}\left(\quad \right).$$



Proof. Since $\gamma \in \hat{\mathrm{I}}$, there exist $\alpha \in \mathrm{I}$ and $d \geq \operatorname{dmin} \alpha$ such that $\gamma = \mathrm{hat}_d(\alpha)$. In particular,

$$\operatorname{Asc}(\gamma) \subseteq \operatorname{Asc}_d(\gamma) \subseteq \operatorname{nub} \gamma,$$

where the last set containment is Proposition 4.3. $\square$

5.1 Maximal d-hat

Recall from Lemma 4.4 that $\operatorname{dmin}\alpha \leq n$ for each $\alpha \in \mathrm{I}_n$. By proposition 4.6, for each $n \geq 0$ we have an injection

$$\begin{aligned} \operatorname{hat}_n : \mathrm{I}_n &\longrightarrow \hat{\mathrm{I}}_n \\ \alpha &\longmapsto \operatorname{hat}_n(\alpha). \end{aligned}$$

Since—again by Lemma 4.4—applying d-hat gives the same result for every $d \geq n$, we will call *max-hat* the injection

$$\begin{aligned} \operatorname{hat}_{\max} : \mathrm{I} &\longrightarrow \hat{\mathrm{I}} \\ \alpha &\longmapsto \operatorname{hat}_{|\alpha|}(\alpha). \end{aligned}$$

The main goal of this subsection is to prove that $\operatorname{hat}_{\max}$ maps I bijectively to $\mathfrak{S}$. Namely, we show that $\operatorname{hat}_{\max}(\alpha)$ is the permutation whose recursive construction by insertion of a new rightmost entry is encoded by α.

We start with a simple lemma.

Lemma 5.3. *Let $\alpha \in \mathrm{I}_n$. Suppose that $\alpha = \beta a$, for some $\beta \in \mathrm{I}_{n-1}$ and $1 \leq a \leq n$. Then*

$$\operatorname{hat}_{\max}(\alpha) = \operatorname{hat}_{\max}(\beta)^{+} a.$$

Proof. We have:

$$\begin{aligned} \operatorname{hat}_{\max}(\alpha) &= \operatorname{hat}_n(\beta a) \\ &= \operatorname{hat}_{n-1}(\beta)^{+} a && (\text{since } \operatorname{Asc}_n \alpha = [n]) \\ &= \operatorname{hat}_{\max}(\beta)^{+} a && (\text{by Lemma 4.4}). \end{aligned}$$

This concludes the proof. □

Lemma 5.4. *Let $\alpha \in \mathrm{I}$. Then $\operatorname{hat}_{\max}(\alpha) \in \mathfrak{S}$.*

Proof. We use induction on the size n of α, where the case $n \leq 1$ is easy to prove. Let $n \geq 2$. Let $\alpha = \beta a$, for some $\beta \in \mathrm{I}_{n-1}$ and $a \in [n]$. By Lemma 5.3, we have $\operatorname{hat}_{\max}(\alpha) = \operatorname{hat}_{\max}(\beta)^{+} a$, which is clearly a permutation since $\operatorname{hat}_{\max}(\beta) \in \mathfrak{S}_{n-1}$ by induction. □

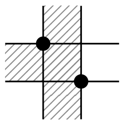
Corollary 5.5. *We have a size-preserving bijection $\operatorname{hat}_{\max} : \mathrm{I} \to \mathfrak{S}$.*

Proof. By Proposition 4.6 and Lemma 5.4, the map $\operatorname{hat}_n : \mathrm{I}_n \to \mathfrak{S}_n$ is injective for every $n \geq 0$. The theorem follows since it is well known that I_n and $\mathfrak{S}_n$ are equinumerous. □

The behavior of $\operatorname{hat}_{\max}$ on I can be summarized by saying that α encodes the construction of $\operatorname{hat}_{\max}(\alpha)$ by insertion of a new rightmost entry. More specifically, when we modify α under $\operatorname{hat}_{\max}$, at each step we increase by one all the entries in the current prefix that are greater than or equal to the current rightmost one. This step-by-step process is illustrated below for $\alpha = 1224315$:

$$\begin{aligned} 1 &\longmapsto 1 \\ 12 &\longmapsto 1\underline{2} \\ 122 &\longmapsto 1\mathbf{3}\underline{2} \\ 1224 &\longmapsto 132\underline{4} \\ 12243 &\longmapsto 1\mathbf{4}2\mathbf{5}\underline{3} \\ 122431 &\longmapsto \mathbf{25364}\underline{1} \\ 1224315 &\longmapsto 2\mathbf{637}41\underline{5} \ = \ \mathrm{hat}_{\max}(\alpha) \end{aligned}$$

We end this section with a simple remark. A *flat step* in $\alpha = a_1 \dots a_n \in \mathrm{End}$ is a pair of consecutive equal entries $a_i = a_{i+1}$. Let $\alpha \in \mathrm{I}$ and let $\gamma = \mathrm{hat}_{\max}(\alpha)$. It is easy to see that $a_i = a_{i+1}$ is a flat step in α if and only if in γ we have $g_i > g_{i+1}$ and no entries g_j, $j < i$, satisfy $g_{i+1} < g_j < g_i$. Define the mesh pattern $\mathfrak{a}$ accordingly as

$\mathfrak{a} =$  .

The next proposition follows immediately.

Proposition 5.6. *The map* $\mathrm{hat}_{\max}$ *restricts to a bijection between inversion sequences with no flat steps and permutations avoiding* $\mathfrak{a}$. $\square$

6 Subdiagonal permutations

Recall from Subsection 5.1 that an inversion sequence α encodes the construction of $\mathrm{hat}_{\max}(\alpha)$ by insertion of a new rightmost entry. In this section, we restrict $\mathrm{hat}_{\max}$ to the set of ascent sequences and characterize the resulting set of permutations which have the following subdiagonal property.

Any permutation $\pi \in \mathfrak{S}$ factors uniquely by maximal increasing runs as $\pi = B_1 B_2 \dots B_k$, where $k = n + 1 - \operatorname{asc}\alpha$. We say that π is

ir-superdiagonal,	if $c \geq i$ for each $c \in B_i$;
ir-subdiagonal,	if $c \leq n + 1 - i$ for each $c \in B_i$,

where the prefix "ir" denotes that π is decomposed by "increasing runs". Clearly, two analogous notions are obtained by replacing maximal increasing runs with maximal decreasing runs; that is, if $\pi = C_1 \dots C_k$, where now the blocks C_i are maximally decreasing, we say that π is

dr-superdiagonal,	if $c \geq i$ for each $c \in C_i$;
dr-subdiagonal,	if $c \leq n + 1 - i$ for each $c \in C_i$.

It is easy to see that π is ir-subdiagonal if and only if its complement is dr-superdiagonal; similarly, it is dr-subdiagonal if and only if its complement is ir-superdiagonal. So it is no restriction to only consider subdiagonal permutations, denoted by

$$\mathrm{D}^{\nearrow} = \{\pi : \pi \text{ is ir-subdiagonal}\};$$
$$\mathrm{D}^{\searrow} = \{\pi : \pi \text{ is dr-subdiagonal}\}.$$

In the following two subsections, we shall prove that

$$\mathrm{hat}_{\max}(\mathrm{A}_0) = \mathrm{D}^{\nearrow} \quad \text{and} \quad \mathrm{hat}_{\max}(\mathrm{wD}) = \mathrm{D}^{\searrow},$$

where wD denotes the set of weak descent sequences, defined later. As a result of what was observed in Subsection 5.1, ascent sequences encode the recursive construction of ir-subdiagonal permutations by successive insertions of a new rightmost entry. And weak descent sequences encode dr-subdiagonal permutations in the same way. This construction is reminiscent of the way ascent sequences encode Fishburn permutations [BMCDK10], the difference being that in the case of Fishburn permutations a new maximum is inserted at each step. Note that we have not been able to find bivincular patterns characterizing $\mathrm{D}^{\nearrow}$ and $\mathrm{D}^{\searrow}$. Finally, we define an isomorphism between two generating trees for weak descent sequences and *primitive ascent sequences*, defined as those ascent sequences that have no flat steps.

6.1 ir-subdiagonal permutations

Throughout this section, we let $\pi = B_1 \dots B_k$ be the decomposition of a given permutation π into maximal increasing runs. If c is an entry of B_j, $1 \le j \le k$, we let $\mathrm{ind}_\pi(c) = j$ denote the index of the block of π that contains c. Letting $\pi = p_1 \dots p_n$ and $\pi_i = p_1 \dots p_i$, it is easy to see that

$$\pi \in \mathrm{D}^{\nearrow} \iff p_i \le |\pi| + 1 - \mathrm{ind}_\pi(p_i)$$

for each i, where

$$\mathrm{ind}_\pi(p_i) = i + 1 - \mathrm{asc}\,\pi_i. \tag{8}$$

The next lemma shows that ir-subdiagonal permutations and ascent sequences share a similar recursive structure.

Lemma 6.1. *Let $\pi = p_1 \dots p_n \in \mathrm{D}^{\nearrow}$ and let $a \in [n+1]$. Then*

$$\pi^+ a \in \mathrm{D}^{\nearrow} \iff a \le 1 + \mathrm{asc}\,\pi.$$

Proof. We start by showing that entries in the prefix π^+ satisfy the subdiagonality constraint in $\pi^+ a$ if $\pi \in \mathrm{D}^{\nearrow}$. Let c be an entry in π and denote by c^+ the corresponding entry in π^+. Observe that $\mathrm{Asc}\,\pi = \mathrm{Asc}\,\pi^+$. Using equation (8), we obtain the equality $\mathrm{ind}_{\pi^+ a}(c^+) = \mathrm{ind}_\pi(c)$. Using this equality and the fact that $\pi \in \mathrm{D}^{\nearrow}$ gives

$$c^+ \le c + 1 \le |\pi| + 1 - \mathrm{ind}_\pi(c) + 1 = |\pi^+ a| + 1 - \mathrm{ind}_{\pi^+ a}(c^+).$$

Next we consider the last entry a. If $a < p_n$, then both $\pi^+ a$ is ir-subdiagonal and $a \le 1 + \mathrm{asc}\,\pi$. Indeed, we have $\mathrm{ind}_{\pi^+ a}(a) = \mathrm{ind}_\pi(p_n) + 1$ and $\pi \in \mathrm{D}^{\nearrow}$ so that

$$a < p_n \le n + 1 - \mathrm{ind}_\pi(p_n) = n + 2 - \mathrm{ind}_{\pi^+ a}(a) = |\pi^+ a| + 1 - \mathrm{ind}_{\pi^+ a}(a).$$

Similarly, using equation (8),

$$a < p_n \le n+1-\mathrm{ind}_\pi(p_n) = n+1-(n+1-\mathrm{asc}\,\pi) < 1+\mathrm{asc}\,\pi.$$

Finally, suppose that $a > p_n$. Then $\mathrm{ind}_{\pi^+a}(a) = \mathrm{ind}_\pi(p_n)$. Now $\pi^+a \in \mathrm{D}^{\nearrow}$ if and only if $a \le |\pi^+a| + 1 - \mathrm{ind}_{\pi^+a}(a)$. But by equation (8), this last inequality is equivalent to

$$a \le n+2-\mathrm{ind}_\pi(p_n) = n+2-(n+1-\mathrm{asc}\,\pi) = 1+\mathrm{asc}\,\pi,$$

as desired. □

Theorem 6.2. *Let* $\alpha \in \mathrm{I}$ *and let* $\hat{\alpha} = \mathrm{hat}_{\max}(\alpha)$. *Then*

$$\alpha \in \mathrm{A}_0 \iff \hat{\alpha} \in \mathrm{D}^{\nearrow}.$$

Therefore, $\mathrm{hat}_{\max}$ *restricts to a bijection from* A_0 *to* $\mathrm{D}^{\nearrow}$.

Proof. We use induction on the size of α where the result is clear for size at most one. Let $\alpha = \beta a$, for some $\beta \in \mathrm{I}_n$. By Lemma 5.3, we have $\hat{\alpha} = \hat{\beta}^+a$. Using induction, we have that $\beta \in \mathrm{A}_0$ if and only if $\hat{\beta} \in \mathrm{D}^{\nearrow}$. Now using Theorem 4.9 we have

$$\alpha \in \mathrm{A}_0 \iff a \le 1+\mathrm{asc}\,\beta = 1+\mathrm{asc}\,\hat{\beta}.$$

But, by the lemma just proved, the inequality is equivalent to $\hat{\alpha} = \hat{\beta}^+a \in \mathrm{D}^{\nearrow}$ as desired. □

By Theorems 4.6 and 6.2, the set $\mathrm{D}^{\nearrow}$ of ir-subdiagonal permutations is the bijective image of the set A_0 of ascent sequences under $\mathrm{hat}_{\max}$. Furthermore, by Proposition 5.6 and the previous result, primitive ascent sequences are in bijection with ir-subdiagonal permutations avoiding $\mathfrak{a}$. The next corollary follows immediately.

Corollary 6.3. *For each* $n \ge 0$*, the number of ir-subdiagonal permutations of size* n *is equal to the* n*th Fishburn number, that is, the number of ascent sequences of length* n*. Furthermore, the number of ir-subdiagonal permutations avoiding* $\mathfrak{a}$ *is equal to the number of primitive ascent sequences (see also A138265 [OEI]).* □

6.2 dr-subdiagonal permutations and weak descent sequences

Recall that the set of weak descents of $\alpha \in \mathrm{End}$ is

$$\mathrm{wDes}\,\alpha = \{i \ge 2 \mid a_i \le a_{i-1}\}$$

Note that $[n] = \mathrm{wDes}\,\alpha \uplus \mathrm{Asc}\,\alpha$ for every $\alpha \in \mathrm{End}_n$; that is, every $i \in [n]$ is either a weak descent or a strict ascent. The set wD of *weak descent sequences* is defined as

$$\mathrm{wD}_n = \{\alpha \in \mathrm{I}_n \mid a_1 = 1 \text{ and } a_i \le 1+\mathrm{wdes}\,\alpha_{i-1} \text{ for each } i \in [n]\},$$

where $\mathrm{wdes}\,\alpha = |\,\mathrm{wDes}\,\alpha|$.

The next result is a counterpart of Theorem 6.2 and states that $\alpha \in \mathrm{I}$ is a weak descent sequence if and only if $\mathrm{hat}_{\max}(\alpha)$ is dr-subdiagonal. Its proof is obtained by simply replicating the steps of Lemma 6.1 and Theorem 6.2, and is thus omitted.

Theorem 6.4. *Let $\pi \in \mathrm{D}^{\searrow}$. Then $\pi^{+}a \in \mathrm{D}^{\searrow}$ if and only if $a \le 1 + \mathrm{wdes}\,\pi$. Furthermore, if $\alpha \in \mathrm{I}$ and $\hat{\alpha} = \mathrm{hat}_{\max}(\alpha)$, then*

$$\alpha \in \mathrm{wD} \iff \hat{\alpha} \in \mathrm{D}^{\searrow}$$

and $\mathrm{hat}_{\max}$ *restricts to a bijection from* wD *to* $\mathrm{D}^{\searrow}$. □

We wish to prove that weak descent sequences (and thus dr-subdiagonal permutations) are equinumerous with primitive ascent sequences. A generating tree for ascent sequences is encoded by the following generating rule, where the pair (a, ℓ) keeps track of the number of ascents, a, and the last letter, ℓ:

$$\begin{cases} \text{Root: } (1,1) \\ (a,\ell) \longrightarrow (a,1)(a,2)\dots(a,\ell-1)(a,\ell)(a+1,\ell+1)(a+1,\ell+2)\dots(a+1,a+1). \end{cases}$$

The above rule encodes the standard construction of ascent sequences by insertion of a new rightmost entry. The root $(1, 1)$ corresponds to the only ascent sequence of size one, namely the single letter word 1. Further, if $\alpha \in \mathrm{A}_0$ has a ascents and last letter ℓ, then it produces $a + 1$ children by insertion of a new rightmost entry $i \in [a + 1]$. If $i \le \ell$, then the number of ascents remains the same; otherwise, if $i > \ell$, then a new ascent is created. To obtain a generating rule for primitive ascent sequences, we remove the child (a, ℓ) corresponding to a flat step and obtain:

$$\Omega : \begin{cases} \text{Root: } (1,1) \\ (a,\ell) \longrightarrow (a,1)(a,2)\dots(a,\ell-1)(a+1,\ell+1)(a+1,\ell+2)\dots(a+1,a+1). \end{cases}$$

A generating rule for weak descent sequences that uses the number of weak descents w and the last letter u as parameters is now obtained similarly as:

$$\Theta : \begin{cases} \text{Root: } (0,1) \\ (w,u) \longrightarrow (w+1,1)(w+1,2)\dots(w+1,u)(w,u+1)(w,u+2)\dots(w,w+1). \end{cases}$$

In this case, inserting $i \in [w + 1]$ creates a new weak descent if and only if $i \le u$.

To show that primitive ascent sequences and weak descent sequences are equinumerous, we shall give a bijection between the generating trees encoded by the rules Ω and Θ. Namely, we show that Ω and Θ are equivalent under the linear transformation

$$\begin{cases} w = a-1 \\ u = a-\ell+1 \end{cases} \iff \begin{cases} a-w = 1 \\ \ell+u = a+1. \end{cases} \tag{9}$$

Indeed, the root $(a, \ell) = (1, 1)$ is mapped to $(w, u) = (0, 1)$. Further, assume that (a, ℓ) is mapped to (w, u), i.e. that $w = a - 1$ and $u = a + 1 - \ell$. Then the children of (a, ℓ) are mapped bijectively to the children of (w, u), since

$$\begin{aligned} (a,1) &\mapsto (a-1,a) = (w,w+1) \\ (a,2) &\mapsto (a-1,a-1) = (w,w) \\ &\vdots \\ (a,\ell-1) &\mapsto (a-1,a-\ell+2) = (w,u+1) \\ (a+1,\ell+1) &\mapsto (a,a-\ell+1) = (w+1,u) \\ (a+1,\ell+2) &\mapsto (a,a-\ell) = (w+1,u-1) \\ &\vdots \\ (a+1,a+1) &\mapsto (a,1) = (w+1,1). \end{aligned}$$

As a result, we obtain a bijection between the generating tree of primitive ascent sequences, encoded by Ω, and the generating tree of wD, encoded by Θ. The next result follows immediately.

Corollary 6.5. *For each $n \geq 0$, the number of weak descent sequences of size n is equal to the is equal to the number of primitive ascent sequences of size n.* $\square$

7 Difference Fishburn permutations

Prompted by a question in [DS23], Zang and Zhou [ZZ] have recently introduced *d-permutations*, defined as follows. Fix $d \geq 0$ and let $\pi = p_1 \dots p_n$ be a permutation of $\mathfrak{S}_n$, with $n \geq 1$. We denote by $\pi^{(k)}$ the subsequence of π which contains the elements $[k]$. For example, if $\pi = 641523$ then $\pi^{(4)} = 4123$. The following procedure defines the *d-active elements* of π:

- Set 1 to be a d-active element.
- For $k = 2, 3, \dots, n$, let k be d-inactive if k is to the left of $k-1$ in π and there exist at least d elements of $\pi^{(k)}$ between k and $k-1$ that are d-active. Otherwise, k is said to be d-active.

Returning to our example $\pi = 641523$ with $d = 2$ we compute the d-active elements as follows, where such elements are set in boldface. By the initial condition

$$\pi^{(1)} = \mathbf{1}.$$

Next

$$\pi^{(2)} = \mathbf{12}.$$

since 2 is to the right of 1 and so will be active. Similarly

$$\pi^{(3)} = \mathbf{123}.$$

Now

$$\pi^{(4)} = 4\mathbf{123}$$

with 4 not active since the number of active elements between it and 3 is $2 \geq d$. Clearly

$$\pi^{(5)} = 4\mathbf{1523}.$$

Finally

$$\pi = \pi^{(6)} = \mathbf{6}4\mathbf{1523}$$

where 6 is active since the number of active elements between it and 5 is $1 < d$.

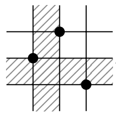
Let $\operatorname{Act}_d \pi$ denote the set of d-active elements of π. Furthermore, denote by $\operatorname{AscBot} \pi$ the set

$$\operatorname{AscBot} \pi = \{p_i \in [n-1] : p_i < p_{i+1}\}$$

of *ascent bottoms* of π. Note that these are elements of π rather than positions. Then, π is said to be a *d-permutation* if

$$\operatorname{AscBot} \pi \subseteq \operatorname{Act}_d \pi,$$

and we denote by F_d the set of d-Fishburn permutations. Recall that Fishburn permutations [BMCDK10] are defined as those permutations avoiding the bivincular pattern

$\mathfrak{f} =$  .

We wish to give an alternative definition of d-Fishburn permutations that is reminiscent of the classical case $d = 0$. We say that a permutation π contains the *d-Fishburn pattern*, $\mathfrak{f}_d$, if it contains an occurrence $p_i p_{i+1} p_j$ of $\mathfrak{f}$ where p_i is d-inactive. The other two elements p_{i+1} and p_j can be either d-active or d-inactive. With a slight abuse, we will use the suggestive notation $\mathfrak{S}(\mathfrak{f}_d)$ to denote the set of permutations that do not contain $\mathfrak{f}_d$.

Proposition 7.1. *For every $d \geq 0$,*

$$\mathrm{F}_d = \mathfrak{S}(\mathfrak{f}_d).$$

Proof. Let $\pi \in \mathfrak{S}$. We show that π contains $\mathfrak{f}_d$ if and only if $\operatorname{AscBot}\pi \not\subseteq \operatorname{Act}_d \pi$. Initially, suppose that π contains an occurrence $p_i p_{i+1} p_j$ of $\mathfrak{f}_d$. Then $p_i \in \operatorname{AscBot}\pi$ and p_i is not d-active. Thus, $\operatorname{AscBot}\pi \not\subseteq \operatorname{Act}_d \pi$, as wanted. On the other hand, suppose that $\operatorname{AscBot}\pi \not\subseteq \operatorname{Act}_d \pi$. That is, there is an entry p_i such that $p_i \in \operatorname{AscBot}$ and $p_i \notin \operatorname{Act}_d \pi$. Note that $p_i < p_{i+1}$. Further, since $p_i \notin \operatorname{Act}_d \pi$, by definition of d-active site we have $p_i - 1 = p_j$, for some $j > i$. Finally, the triple $p_i p_{i+1} p_j$ is an occurrence of $\mathfrak{f}_d$, finishing the proof. $\square$

Zang and Zhou proved that F_0 coincides with the set of Fishburn permutations, while F_1 is equal to the set of weak Fishburn permutations introduced by Bényi et al. [BCD23]. Further, they showed that F_d is tree-like in the following sense.

Consider a set Π of permutations. As usual, let $\Pi_n = \Pi \cap \mathfrak{S}_n$. Say that Π is *tree-like* if $\Pi_0 = \{\epsilon\}$ (where ϵ is the empty permutation) and, for $n \geq 1$, every $\pi \in \Pi_n$ is obtained by inserting n into a site of some $\rho \in \Pi_{n-1}$, called the *parent* of π. The spaces between letters of ρ into which n can be inserted are called the *active sites with respect to* Π, and all other sites of ρ are said to be *inactive*. Active sites are labled $1, 2, \dots$ from left to right. The active sites of $\pi \in \mathrm{F}_d$ are called *d-active sites* and are the site before π as well as the sites which lie just after a d-active element.

Finally, Zang and Zhou generalized the classical encoding of Fishburn permutations by ascent sequences to d-Fishburn permutations and d-ascent sequences; that is, they defined bijections

$$\Phi_d : \mathrm{A}_d \longrightarrow \mathrm{F}_d$$

by letting, recursively,

- $\Phi_d(\epsilon) = \epsilon$, and
- for $n \geq 1$ if $\alpha = \beta a \in \mathrm{A}_{d,n}$ then $\Phi_d(\alpha)$ is the result of inserting n into the active site active site of $\Phi_d(\beta)$ labeled a.

7.1 Burge transpose and Fishburn permutations

The set of *Burge words* is defined as

$$\mathrm{Bur}_n = \left\{ \binom{u}{\alpha} : u \in \mathrm{WI}_n,\ \alpha \in \mathrm{Cay}_n,\ \mathrm{wDes}(u) \subseteq \mathrm{wDes}(\alpha) \right\},$$

where WI_n is the subset of Cay_n consisting of the weakly increasing Cayley permutations. We define a transposition operation T on Bur_n as follows [CC23b]. To compute the *Burge transpose* w^T of $w = \binom{u}{\alpha} \in \mathrm{Bur}_n$, turn each column of w upside down and then sort the columns in ascending order with respect to the top entry, breaking ties by sorting in weakly decreasing order with respect to the bottom entry. Observe that T is an involution on Bur_n. Now, let $\mathrm{id}_n = 12\dots n$ be the identity permutation. Since id_n has no weak descents, $\binom{\mathrm{id}_n}{\alpha}$ is a Burge word for every $\alpha \in \mathrm{Cay}_n$. Thus, for any $\alpha \in \mathrm{Cay}_n$, we can always pick id_n as the top row, and we get a map $\mathtt{t} : \mathrm{Cay}_n \to \mathfrak{S}_n$, defined by

$$\binom{\mathrm{id}}{\alpha}^T = \binom{\mathrm{sort}(\alpha)}{\mathtt{t}(\alpha)},$$

for any $\alpha \in \mathrm{Cay}$, where $\mathrm{sort}(\alpha)$ is obtained by sorting the entries of α in weakly increasing order. If $\pi \in \mathfrak{S}$ is a permutation, then $\mathtt{t}(\pi) = \pi^{-1}$ (and thus $\mathtt{t}$ is surjective). Note that the map $\mathtt{t}$ was originally [CC23b] denoted by the letter γ.

One of the main advantages of modified ascent sequences and the Burge transpose is that they give a non-recursive description of the bijection $\Phi_0 : \mathrm{A}_0 \to \mathrm{F}_0$. Indeed [BMCDK10], if $\hat{\alpha} = \mathrm{hat}_0(\alpha)$ is the modified ascent sequence of α, then $\mathtt{t}(\hat{\alpha})$ is the Fishburn permutation corresponding to α under Φ_0. With the ascent sequence $\alpha = 121242232$ of the example before Lemma 2.1, we obtain $\hat{\alpha} = 141252232$ and

$$\begin{aligned}\binom{\mathrm{id}}{\hat{\alpha}}^T &= \begin{pmatrix}1&2&3&4&5&6&7&8&9\\1&4&1&2&5&2&2&3&2\end{pmatrix}^T\\ &= \begin{pmatrix}1&1&2&2&2&2&3&4&5\\3&1&9&7&6&4&8&2&5\end{pmatrix} = \binom{\mathrm{sort}(\hat{\alpha})}{\mathtt{t}(\hat{\alpha})},\end{aligned}$$

where $\mathtt{t}(\hat{\alpha}) = \Phi_0(\alpha)$.

We wish to use the d-hat map to generalize the above construction to every $d \geq 0$. That is, we shall prove that the diagram

$$\begin{tikzcd} \mathrm{A}_d \arrow[r, "\Phi_d"] \arrow[dr, "\mathrm{hat}_d"'] & \mathrm{F}_d \\ & \hat{\mathrm{A}}_d \arrow[u, "\mathtt{t}"'] \end{tikzcd} \tag{10}$$

commutes for every $d \geq 0$ and that all the arrows are size-preserving bijections. To this end, it will be convenient to let

$$\mathrm{Im}_d = \mathtt{t}\big(\mathrm{hat}_d(\mathrm{A}_d)\big).$$

Our proof of the commutativity of diagram (10) proceeds in the following steps: First we show that $\mathtt{t}$ is injective so that the composition with hat_d is an injective map.

Next we demonstrate that Im_d has a tree-like structure and describe the active sites of its permutations. As a corollary, we obtain $\mathrm{Im}_d = \mathrm{F}_d$. The equality $\Phi_d = \mathtt{t} \circ \mathrm{hat}_d$ then follows by showing that $\mathtt{t} \circ \mathrm{hat}_d$ has a recursive description that is identical to the one given for Φ_d in terms of active sites.

Let us start by proving that $\mathtt{t}$ is injective. For the rest of this section, given a d-ascent sequence α, we let $\hat{\alpha} = \mathrm{hat}_d(\alpha)$ denote the d-hat of α.

Proposition 7.2. *For all $d, n \geq 0$, the map $\mathtt{t} : \hat{\mathrm{A}}_{d,n} \to \mathfrak{S}_n$ is injective.*

Proof. We fix $d \geq 0$ and use induction on n. Let us set up the following notation for the rest of the proof. We will consider two d-ascent sequences α and ω in $\mathrm{A}_{d,n}$, where

$$\alpha = \beta a \quad \text{and} \quad \omega = \tau w,$$

for some β and τ in $\mathrm{A}_{d,n-1}$. We also let a, b, w and t denote the last letter of α, β, ω and τ, respectively. By definition of hat_d, we have

$$\hat{\alpha} = \begin{cases} \hat{\beta} a & \text{if } a \leq b - d; \\ \hat{\beta}^{+} a & \text{if } a > b - d; \end{cases}$$

and

$$\hat{\omega} = \begin{cases} \hat{\tau} w & \text{if } w \leq t - d; \\ \hat{\tau}^{+} w & \text{if } w > t - d. \end{cases}$$

We assume $\hat{\alpha} \neq \hat{\omega}$ and prove that $\mathtt{t}(\hat{\alpha}) \neq \mathtt{t}(\hat{\omega})$.

Assume first that $\beta \neq \tau$. Since the map hat_d is injective by Proposition 4.7, we have $\hat{\beta} \neq \hat{\tau}$ and by induction, $\mathtt{t}(\hat{\beta}) \neq \mathtt{t}(\hat{\tau})$. Note that sorting $\binom{\mathrm{id}}{\hat{\beta}}^T$ and $\binom{\mathrm{id}}{\hat{\beta}+}^T$ yields the same permutation in the lower row and similarly for $\hat{\tau}$ and $\hat{\tau}^{+}$. So, whichever case of the hat_d map we are in, we will be placing n into the two different permutations $\mathtt{t}(\hat{\beta})$ and $\mathtt{t}(\hat{\tau})$ to compute $\mathtt{t}(\hat{\alpha})$ and $\mathtt{t}(\hat{\omega})$. This must result in distinct permutations, as desired.

Now assume that $\beta = \tau$. Since $\alpha \neq \omega$ it must be that $a \neq w$ and we can assume, without loss of generality, that $a < w$. If $a \leq b - d$, then $\binom{\mathrm{id}}{\hat{\alpha}}^T$ is computed from $\binom{\mathrm{id}}{\hat{\beta}}^T$ by placing the column $\binom{a}{n}$ at the beginning of the list of columns with top entry a. Note that such columns must exist because of the given inequality. If $a > b - d$ then $\binom{\mathrm{id}}{\hat{\alpha}}^T$ is computed from $\binom{\mathrm{id}}{\hat{\beta}+}^T$ by inserting the column $\binom{a}{n}$ between the columns with top entry $a - 1$ and those with top entry $a + 1$ (if there are any of the latter). In this case there will be no other columns with top entry a. It is now a simple matter of checking to show that in all possible cases the fact that $a < b$ will force n to be in $\mathtt{t}(\hat{\alpha})$ strictly to the left of its appearance in $\mathtt{t}(\hat{\omega})$. This completes the proof of injectivity. $\square$

Corollary 7.3. *For every $d \geq 0$, the map $\mathtt{t} \circ \mathrm{hat}_d : \mathrm{A}_d \to \mathrm{Im}_d$ is a bijection.*

Proof. Our claim follows directly from Proposition 4.7, Proposition 7.2 and the definition of Im_d. $\square$

Lemma 7.4. *For all $d \geq 0$, the set Im_d is tree-like.*

Proof. Clearly $\epsilon \in \mathrm{Im}_{d,0}$, so let $n \geq 1$. Pick $\pi \in \mathrm{Im}_{d,n}$ and suppose $\pi = \mathtt{t}(\hat{\alpha})$, for some $\alpha \in \mathrm{A}_d$. Suppose that $\alpha = \beta a$ and consider $\rho = \mathtt{t}(\hat{\beta})$. We claim that π is obtained by inserting n in some site of ρ which will prove the theorem. As usual, there are two cases depending on whether $\hat{\alpha} = \hat{\beta} a$ or $\hat{\alpha} = \hat{\beta}^+ a$.

Suppose first that $\hat{\alpha} = \hat{\beta} a$. As in the proof of Proposition 7.2, $\binom{\mathrm{id}}{\hat{\alpha}}^T$ is obtained from $\binom{\mathrm{id}}{\hat{\beta}}^T$ by inserting the column $\binom{a}{n}$ at the beginning of the columns with top entry a. This means that $\pi = \mathtt{t}(\hat{\alpha})$ is obtained from $\rho = \mathtt{t}(\hat{\beta})$ by inserting n in the corresponding site, which proves the claim in this case.

Consider the second case. Here $\binom{\mathrm{id}}{\hat{\alpha}}^T$ is obtained from $\binom{\mathrm{id}}{\hat{\beta}^+}^T$ by inserting the column $\binom{a}{n}$ between the columns with upper entry $a-1$ and those with upper entry $a+1$, there being no columns with upper entry a. But since $\hat{\beta}$ and $\hat{\beta}^+$ are order isomorphic, it follows again that insertion of n in the corresponding site of ρ yields π. $\square$

Since Im_d is tree-like, it is natural to want a description of the active sites of $\rho \in \mathrm{Im}_d$. By Corollary 7.3, the map $\mathtt{t} \circ \mathrm{hat}_d : \mathrm{A}_d \to \mathrm{Im}_d$ is a bijection. If $\beta = (\mathtt{t} \circ \mathrm{hat}_d)^{-1}(\rho)$ then one could consider all d-ascent sequences of the form $\alpha = \beta a$ for $1 \leq a \leq \mathrm{asc}_d\, \beta + 1$. Computing the permutations $\pi = \mathtt{t}(\hat{\alpha})$ for each such α and comparing with ρ would accomplish this task. But it would be nice to have a description of the active sites which can be read off from the permutation itself as one would do for pattern-avoidance classes. To this end, given $\rho \in \mathrm{Im}_d$, let $\beta \in \mathrm{A}_d$ be its preimage. Now

$$\binom{\mathrm{id}}{\hat{\beta}}^T = \binom{\mathrm{sort}(\hat{\beta})}{\rho}.$$

We shall use the active sites of ρ to define a labeling of the sites of $\mathrm{sort}(\hat{\beta})$ by letting a site of $\mathrm{sort}(\hat{\beta})$ be active if and only if the corresponding site of ρ is active. In this context, active sites refer to the tree-like structure of Im_d established in Lemma 7.4, and should not be confused with the active sites with respect to F_d. We will first describe the active sites of $\mathrm{sort}(\hat{\beta})$. To do so we need the concept of a *run* in a sequence which is a maximum factor (subsequence of consecutive elements) consisting of equal elements.

Lemma 7.5. *Suppose* $\beta \in \mathrm{A}_{d,n-1}$.

(*a*) *The elements of the runs of* $\mathrm{sort}(\hat{\beta})$ *are* $1, 2, \ldots, \mathrm{asc}_d\, \beta$ *from left to right.*

(*b*) *The active sites of* $\mathrm{sort}(\hat{\beta})$ *are the sites before, after, or between its runs.*

(*c*) *The number of active sites of* $\mathtt{t}(\hat{\beta})$ *is equal to* $\mathrm{asc}_d\, \beta + 1$.

Proof. (*a*) This follows directly from Proposition 3.3.

(*b*) Suppose $\alpha = \beta a$ where $1 \leq a \leq \mathrm{asc}_d\, \beta + 1$. If a does not create a d-ascent then $\binom{\mathrm{id}}{\hat{\alpha}}^T$ is obtained from $\binom{\mathrm{id}}{\hat{\beta}}^T$ by inserting the column $\binom{a}{n}$ at the beginning of the run of a's in $\mathrm{sort}(\hat{\beta})$. This will be the site before $\mathrm{sort}(\hat{\beta})$ or the site between the run of $(a-1)$'s and the run of a's. Now suppose a does cause a d-ascent so that $\hat{\alpha} = \hat{\beta}^+ a$. Note that the runs of $\mathrm{sort}(\hat{\beta})$ and $\mathrm{sort}(\hat{\beta}^+)$ are the same except that the entries in the latter which are greater than or equal to a have been increased by one. Now $\binom{\mathrm{id}}{\hat{\alpha}}^T$ is obtained from $\binom{\mathrm{id}}{\hat\beta}^T$ by inserting the column $\binom{a}{n}$ after the run of $(a-1)$'s in $\mathrm{sort}(\hat\beta^+)$. So this will either be between the runs for $a-1$ and $a+1$ or at the end. This shows that the sites before, after, or between the runs are indeed active.

To see that these are the only active sites, note that $|\,\mathrm{Im}_{d,n}\,|$ is the sum of the number of active sites over all elements of $\mathrm{Im}_{d,n-1}$. Since $\mathrm{Im}_{d,n}$ is in bijection with $\mathrm{A}_{d,n}$ we have that $|\,\mathrm{Im}_{d,n}\,|$ is also the sum of $\mathrm{asc}_d\,\beta+1$ over all $\beta \in A_{d,n-1}$. But in the previous paragraph we showed that there are at least $\mathrm{asc}_d\,\beta+1$ active sites in every $\mathrm{sort}(\hat\beta)$. Since the two sums are equal, we must have exactly $\mathrm{asc}_d\,\beta+1$ active sites in every $\mathrm{sort}(\hat\beta)$. Thus there can be no others.

(c) From Item (b), the number of active sites of $\mathtt{t}(\hat\beta)$ is equal to one plus the number of runs of $\hat\beta$. Our claim follows immediately since there are exactly $\mathrm{asc}_d\,\beta$ runs by Item (a). □

Next we show that the last letter of a d-ascent sequence determines the active site where the maximum of the corresponding permutation in Im_d is inserted.

Lemma 7.6. *Let $d \geq 0$. Let $\alpha \in \mathrm{A}_{d,n}$ and let $\pi = \mathtt{t}(\hat\alpha) \in \mathrm{Im}_{d,n}$. Then π is obtained by inserting n in the ath active site of its parent, where a is the last letter of α.*

Proof. Suppose that $\alpha = \beta a$, for some $\beta \in \mathrm{A}_{d,n-1}$. As observed in the proof of Item (b) of Lemma 7.5, the active sites of $\mathrm{sort}(\hat\beta)$ are the sites before, after, or between its runs. Since the column $\binom{a}{n}$ is inserted at the beginning of the run of a's in $\mathrm{sort}(\hat\beta)$, or after the last run if no run of a's exists, it follows immediately that n is inserted in the ath active site of its parent. □

We now wish to express the active sites of $\pi \in \mathrm{Im}_{d,n}$ in terms of its parent $\rho \in \mathrm{Im}_{d,n-1}$. We will call the sites of ρ which remain between the same two element in π *common*. In addition, there will be two new sites before and after n in π. The following criterion is similar to the one [DS23] for the avoidance class of the bivincular pattern $\sigma_d = (d+2)|(d+3)12\ldots\overline{d(d+1)}$.

Lemma 7.7. *Suppose $\pi \in \mathrm{Im}_{d,n}$ has parent $\rho \in \mathrm{Im}_{d,n-1}$. Then each common site is either active in both π and ρ or inactive in both. Also, the site before n is always active in π. For the site after n, let s and t be the number of active sites before n in π and before $n-1$ in ρ, respectively. Then the site after n is active if and only if*

$$s > t - d.$$

Proof. Let $\alpha = (\mathtt{t} \circ \mathrm{hat}_d)^{-1}(\pi) = \beta a$ and let β have last element b. From the active sites of ρ we can determine $\mathrm{sort}(\hat\beta)$. More precisely, from Lemma 7.5 one can construct $\mathrm{sort}(\hat\beta)$ by filling in the elements between the ith and $(i+1)$st active sites with i's for each $i \geq 1$. Moreover, by Lemma 7.6 the number of active sites before $n-1$ is the last letter of β.

Now consider what happens when the column $\binom{a}{n}$ is added to $\binom{\mathrm{id}}{\hat\beta}^T$. Again we see from the proof of Lemma 7.5, that wherever this column is inserted, it becomes the beginning of a run of a's. Now using Item (b) of the lemma, we see that all the common sites retain their character and that the site to the left of n must be active.

Finally, look at the site to the right of n. From the definition of s and t as well as the observation at the end of the first paragraph of this proof, we have $s = a$ and $t = b$. Furthermore, since we only count active sites before n, we can determine s just from knowing the sites of ρ and the position of n in π. So if $s \leq t - d$ then $a \leq b - d$ and a does not create a d-ascent. It follows that $\binom{a}{n}$ is placed at the beginning of run of other a's. So, the site to its right will not be active since it does not begin a run. On the other hand, if $s > t - d$ then a similar argument shows that the column is inserted as a run of a's having only one element. This forces the site to its right to be active and finishes the proof. □

To prove that $\mathrm{Im}_d = \mathrm{F}_d$, we relate active sites with respect to Im_d with active sites with respect to F_d. To avoid confusion, we will call a site F_d-*active* if it is active with respect to F_d, and Im_d-*active* if it is active with respect to Im_d. We will also need the following lemma by Zang and Zhou.

Lemma 7.8. *[ZZ, Lemma 2.5] Let $d \geq 0$ and $n \geq 1$. Let $\pi \in \mathfrak{S}_n$ and let ρ be obtained by removing n from π. Then $\pi \in \mathrm{F}_{d,n}$ if and only if $\rho \in \mathrm{F}_{d,n-1}$ and n is placed before ρ or after some d-active element of ρ.* □

By Lemma 7.8, the F_d-active sites of $\rho \in \mathrm{F}_d$ are precisely those positions that follow a d-active element of ρ, together with the position before the leftmost entry.

Theorem 7.9. *For any $d, n \geq 0$,*

$$F_{d,n} = \mathrm{Im}_{d,n}\,.$$

Furthermore, a site of $\pi \in \mathrm{F}_{d,n} = \mathrm{Im}_{d,n}$ is F_d-active if and only if it is Im_d-active.

Proof. We use induction on n, where the claim holds for $n \leq 1$. Let $n \geq 2$ and assume that $\mathrm{F}_{d,n-1} = \mathrm{Im}_{d,n-1}$. By induction, given $\rho \in \mathrm{F}_{d,n-1} = \mathrm{Im}_{d,n-1}$, a site of ρ is F_d-active if and only if it is Im_d-active. Since both Im_d and F_d are tree-like, by Lemmas 7.4 and 7.8, respectively, the equality $\mathrm{F}_{d,n} = \mathrm{Im}_{d,n}$ follows immediately.

Let us now consider a permutation $\pi \in \mathrm{F}_{d,n} = \mathrm{Im}_{d,n}$. We have to show that a site of π is F_d-active if and only if it is Im_d-active. The site before the leftmost entry is active in both cases by item (b) of Lemma 7.5 and by Lemma 7.8. Now, let $\rho \in \mathrm{F}_{d,n-1} = \mathrm{Im}_{d,n-1}$ be the father of π. By Lemma 7.7 each common site is Im_d-active in π if and only if it is Im_d-active in ρ; and the new site before n is Im_d-active. Similarly, by definition of d-active entry and Lemma 7.8, each of these sites is F_d-active in π if and only if it is F_d-active in ρ, and n is always placed in an F_d active site which is directly after an F_d-active element. Since by induction F_d-active and Im_d-active sites of ρ coincide, the desired claim holds for every common site, as well as for the new site before n.

To finish the proof of the theorem, we only need to consider the new site after n. Using the same notation as in Lemma 7.7, let s and t be the number of Im_d-active sites before n in π and before $n-1$ in ρ, respectively. By this lemma, the site after n is Im_d-active if and only if $s > t - d$. If n appears to the right of $n-1$ in π, then n is F_d-active. Moreover, we have $s \geq t + 1$ since the site before n is Im_d-active. Thus

$$s \geq t + 1 > t \geq t - d$$

and the site after n is Im_d-active, as desired. On the other hand, suppose that n appears to the left of $n-1$. Write

$$\pi = g_1 \dots g_i \; n \; g_{i+1} \dots g_j \; (n-1) \; g_{j+1} \dots g_{n-1},$$

for some $i \leq j$. We have

$$\begin{aligned} t &= \#\ \mathrm{Im}_d\text{-active sites before } (n-1) \text{ in } \rho \\ &= \#\ \mathrm{F}_d\text{-active sites before } (n-1) \text{ in } \rho \end{aligned}$$

by induction, and

$$\begin{aligned} s &= \#\ \mathrm{Im}_d\text{-active sites before } n \text{ in } \pi \\ &= \#\ \mathrm{Im}_d\text{-active sites before } g_{i+1} \text{ in } \rho \\ &= \#\ \mathrm{F}_d\text{-active sites before } g_{i+1} \text{ in } \rho \end{aligned}$$

where the last step is again by induction. Therefore,

$$\begin{aligned} t - s &= \#\ \mathrm{F}_d\text{-active sites between } g_{i+1} \text{ and } n-1 \text{ in } \rho \\ &= \#\ \mathrm{F}_d\text{-active entries in } g_{i+1} \dots g_j \text{ in } \rho, \end{aligned}$$

where at the last step we used Lemma 7.8. Finally, by Lemma 7.7, the site after n is Im_d-active if and only if $s > t - d$. Rearranging terms gives $t - s < d$ which is equivalent to n being a d-active element by the definition of d-active entries. In turn, this is equivalent to the site after n being F_d-active by Lemma 7.8. This completes the proof. □

Theorem 7.10. *For any $d \geq 0$,*

$$\Phi_d = \mathtt{t} \circ \mathrm{hat}_d.$$

Proof. We have established in Theorem 7.9 that the maps Φ_d and $\mathtt{t} \circ \mathrm{hat}_d$ have the same image $\mathrm{F}_d = \mathrm{Im}_d$. Let us prove inductively that $\Phi_d = \mathtt{t} \circ \mathrm{hat}_d$. Let $\alpha = \beta a \in \mathrm{A}_{d,n}$, where $\beta \in \mathrm{A}_{d,n-1}$ and $1 \leq a \leq 1 + \operatorname{asc}\beta$. By induction, we have

$$\Phi_d(\beta) = \mathtt{t}(\mathrm{hat}_d(\beta)) =: \rho.$$

Again by Theorem 7.9, a site of ρ is F_d-active if and only if it is Im_d-active. Moreover, the last letter of α determines the label of the active site where n is inserted both under Φ_d, by definition, and under $\mathtt{t} \circ \mathrm{hat}_d$, by Lemma 7.6. Thus $\Phi_d(\alpha) = \mathtt{t}(\mathrm{hat}_d(\alpha))$, finishing the proof. □

8 Pattern avoidance in F_d

The introduction and characterization of the d-Fishburn permutations opens the door to pattern avoidance results parameterized by d. As an illustration, we shall study one such instance in some depth, namely the case of d-Fishburn permutations avoiding the classical pattern 213. First, recall the bivincular pattern

$$\sigma_d = (d+2)|(d+3)12 \dots \overline{d(d+1)}.$$

Zang and Zhou [ZZ, Theorem 2.4] proved that

$$\mathrm{F}_d \subseteq \mathfrak{S}(\sigma_d) \tag{11}$$

for every $d \geq 0$, where for $d = 0, 1$ equality holds.

Proposition 8.1. *We have* $\mathrm{F}_d(213) = \mathfrak{S}(\sigma_d, 213)$.

Proof. The inclusion $\mathrm{F}_d(213) \subseteq \mathfrak{S}(\sigma_d, 213)$ follows from (11). For the same reason, if $d \leq 1$ we obtain the desired equality. Now let $d \geq 2$. We shall prove the remaining inclusion $\mathfrak{S}(\sigma_d, 213) \subseteq \mathrm{F}_d(213)$. Let $\pi \in \mathfrak{S}_d(\sigma, 213)$. For a contradiction, suppose that $\pi \notin \mathrm{F}_d$. That is, π contains an occurrence $p_i p_{i+1} p_j$ of $\mathfrak{f}$ where p_i is not a d-active element. Since p_i is not d-active, there are at least d entries $p_{u_1}, \dots, p_{u_d}$, $u_1 < u_2 < \cdots < u_d$, between p_{i+1} and p_j that are smaller than p_j (and d-active). Further, since $d \geq 2$, these must be in increasing order or else they would create an occurrence of 213 with p_j. Thus we have obtained an occurrence $p_i p_{i+1} p_{u_1} \dots p_{u_d} p_j$ of σ_d, which is impossible. □

In order to enumerate $\mathrm{F}_d(213)$, we show that $\mathfrak{S}_n(\sigma_d, 213)$ is in bijection with the set of Dyck paths of semilength n that do not contain $\mathtt{DDU}^{d+1}$ as a factor. Let us start by defining a bijection ϕ from $\mathfrak{S}_n(213)$ to Dyck paths of semilength n. It is simply a tilted version of what is sometimes called [CK08] the *standard bijection* from 132-avoiding permutations to Dyck paths. Any non-empty permutation $\pi \in \mathfrak{S}(213)$ decomposes uniquely as

$$\pi = p_1 L R,$$

where all the entries in L are larger than p_1, and all the entries in R are smaller than p_1. Then ϕ is defined recursively by mapping the empty permutation to the empty path and letting

$$\phi(\pi) = \phi(p_1 L R) = \mathtt{U}\phi(L)\mathtt{D}\phi(R),$$

where here we abuse notation and use the same letter L for the permutation that is order isomorphic to L. Under the bijection ϕ, the value of the first letter determines the first return to the x-axis.

We show that ϕ restricts to a bijection from $\mathfrak{S}(\sigma_d, 213)$ to Dyck paths avoiding $\mathtt{DDU}^{d+1}$ as a factor, for every $d \geq 0$. First a lemma whose easy proof is omitted.

Lemma 8.2. *Let $\pi \in \mathfrak{S}_n(213)$ and let $\rho = \phi(\pi)$ be the corresponding Dyck path. Then*

$$p_1 < p_2 < \cdots < p_k \iff \mathtt{U}^k \text{ is a prefix of } \rho. \qquad \square$$

Lemma 8.3. *Let $\pi \in \mathfrak{S}_n(213)$ and let $\rho = \phi(\pi)$ be the corresponding Dyck path. Then, for any $d \geq 0$,*

$$\pi \text{ contains } \sigma_d \iff \mathtt{DDU}^{d+1} \text{ is a factor of } \rho.$$

Proof. We use induction on n, where $n = 0$ and $n = 1$ are trivial. Assume our claim holds for $n - 1$ where $n \geq 2$ and let $\pi = p_1 L R \in \mathfrak{S}_n(213)$. Initially, suppose that π contains an occurrence $p_i p_{i+1} p_{u_1} \dots p_{u_d} p_j$ of σ_d. If either $p_j \in L$ or $p_i \in R$, then we can conclude that ρ contains a factor $\mathtt{DDU}^{d+1}$ by induction. Otherwise, since entries in L are larger than entries in R, it must necessarily be that $p_{i+1} \in L$ while p_{u_1} is contained in R. Moreover, since $p_j = p_i - 1$, we have $i = 1$ and p_j is the largest entry in R. Now, since L is not empty, the path $\mathtt{U}\phi(L)\mathtt{D}$ ends with $\mathtt{DD}$. Furthermore, since π avoids 213, all the entries preceding p_j in R are in increasing order. Taking p_j into account, (at least) the first $d+1$ entries of R are in increasing order. Using Lemma 8.2, it follows that $\phi(R)$ starts with $\mathtt{U}^{d+1}$. Hence the last two steps of $\mathtt{U}\phi(L)\mathtt{D}$ form a factor $\mathtt{DDU}^{d+1}$ with the first $d+1$ steps of $\phi(R)$, as wanted.

On the other hand, suppose that ρ contains a factor $\mathtt{DDU}^{d+1}$. We will show that π contains σ_d. Similarly to the argument in the previous paragraph, if the whole factor $\mathtt{DDU}^{d+1}$ is contained in either $\phi(L)$ or $\phi(R)$, then we can conclude the proof by induction. Otherwise, it must be that the last two steps of $\mathtt{U}\phi(L)\mathtt{D}$ are $\mathtt{DD}$ and the first $d+1$ steps of $\phi(R)$ are $\mathtt{U}^{d+1}$. Since $\phi(L)$ is not empty, we have $p_1 < p_2$. Using Lemma 8.2 once again, we have that the first $d+1$ entries of R, say $p_{u_1}, \dots, p_{u_d}, p_{u_{d+1}}$, are in increasing order. Finally, the maximum entry of R is equal to $p_1 - 1$, and we obtain the desired occurrence $p_1 p_2 p_{u_1} \dots p_{u_d}(p_i - 1)$ of σ_d in π. □

For any fixed $d \geq 0$, we shall derive a generating function for the numbers $\#\,\mathrm{F}_{d,n}(213)$. By the preceding proposition we can achieve this by counting Dyck paths having no $\mathtt{DDU}^{d+1}$ factor. In fact, we shall derive a generating function for the distribution of the number of $\mathtt{DDU}^{d+1}$ factors over Dyck paths. Let us start with the case $d = 0$. In the spirit of the cluster method [GJ79, Wan11], consider Dyck paths in which a subset of the $\mathtt{DDU}$ factors have been marked. For instance,

$$\rho = \mathtt{UUDU}\underline{\mathtt{DDU}}\mathtt{UUDDU}\underline{\mathtt{DDU}}\mathtt{D}$$

has three $\mathtt{DDU}$ factors, two of which have been marked (underlined). Let us encode ρ as a word ρ' over the alphabet $\{\mathtt{U}, \mathtt{D}, \mathtt{D}'\}$ by replacing each marked $\mathtt{DDU}$ factor with a $\mathtt{D}'$. In our example we have

$$\rho' = \mathtt{UUDUD'UUDDUD'D}.$$

Note that ρ' represents a marked Dyck path if and only if ρ' itself is a Dyck path, when interpreting $\mathtt{D}'$ as $\mathtt{D}$, and the height at which any $\mathtt{D}'$ step starts is at least two.

Let $\mathcal{P}_0 \in \mathbb{Q}\langle \mathtt{U}, \mathtt{D}, \mathtt{D}' \rangle$ be the formal sum of Dyck paths with two sorts of down steps, $\mathtt{D}$ and $\mathtt{D}'$. By the usual first return decomposition $\mathcal{P}_0$ satisfies

$$\mathcal{P}_0 = 1 + \mathtt{U}\mathcal{P}_0\mathtt{D}\mathcal{P}_0 + \mathtt{U}\mathcal{P}_0\mathtt{D}'\mathcal{P}_0.$$

Let $\mathcal{Q}_0 \in \mathbb{Q}\langle \mathtt{U}, \mathtt{D}, \mathtt{D}' \rangle$ be the formal sum of the subset of the paths encoded in $\mathcal{P}_0$ defined by requiring that the height at which any $\mathtt{D}'$ step starts is at least two. Then

$$\mathcal{Q}_0 = (\mathtt{U}\mathcal{P}_0\mathtt{D})^*,$$

where we use the (Kleene star) convention $\mathcal{F}^* = 1 + \mathcal{F} + \mathcal{F}^2 + \dots$. Define the map $\varphi : \mathbb{Q}\langle \mathtt{U}, \mathtt{D}, \mathtt{D}' \rangle \to \mathbb{Q}[q, x]$ by $\mathtt{U} \mapsto x$, $\mathtt{D} \mapsto 1$, $\mathtt{D}' \mapsto qx$ and extending by linearity. Now, letting $P_0(q, x) = \varphi(\mathcal{P}_0)$ and $Q_0(q, x) = \varphi(\mathcal{Q}_0)$, we get the functional equations:

$$\begin{aligned} P_0(q, x) &= 1 + xP_0(q, x)^2 + qx^2 P_0(q, x)^2; \\ Q_0(q, x) &= 1/(1 - xP_0(q, x)). \end{aligned}$$

Note that

$$\sum_{\rho} (1+q)^{\mathtt{DDU}(\rho)}\, x^{|\rho|} \;=\; Q_0(q,x),$$

where the sum ranges over all Dyck paths, $|\rho|$ is the semilength of ρ, and $\mathtt{DDU}(\rho)$ is short for the number of DDU factors in ρ. Indeed, the power series $Q_0(q,x)$ counts Dyck paths with respect to semilength and number of marked DDU factors, but so does the left-hand side: For each of the DDU factors there is a choice to be made, mark it (with a q) or leave it unmarked. Thus,

$$Q_0(q-1,x) \;=\; \sum_{\rho} q^{\mathtt{DDU}(\rho)}\, x^{|\rho|} \tag{12}$$

is the generating function we seek. In particular, $Q_0(-1,x)$ is the generating function for Dyck paths with no DDU factors.

A similar analysis applies when $d \geq 1$. In this case we consider Dyck paths ρ in which a subset of the $\mathtt{DDU}^{d+1}$ factors are marked, and we encode such a path by a word ρ' over the alphabet $\{\mathtt{U},\mathtt{U}',\mathtt{D}\}$, where $\mathtt{U}'$ represents a marked $\mathtt{DDU}^{d+1}$ factor. In this way, ρ' represents a marked Dyck path if and only if ρ' itself is a Dyck path, when interpreting $\mathtt{U}'$ as $\mathtt{U}^{d-1}$, and the height at which any $\mathtt{U}'$ step starts is at least two. As the reader may have noticed, for the preceding description to make sense in the special case $d = 1$ we need to view $\mathtt{U}^0$ as a level-step and in this case we are really dealing with Motzkin paths rather than Dyck paths. However, the equations describing the resulting language hold uniformly for any $d \geq 1$ and this is the reason for not separating out $d = 1$ as a special case.

Let $\mathcal{P}_d \in \mathbb{Q}\langle \mathtt{U},\mathtt{U}',\mathtt{D}\rangle$ be be the formal sum of Dyck paths with two sorts of up steps, U and $\mathtt{U}'$, where each $\mathtt{U}'$ can be thought of representing $\mathtt{DDU}^{d+1}$ and thus each such step contributes $d-1$ to the height of the path. By a simple extension of the first return decomposition we find that

$$\mathcal{P}_d = 1 + \mathtt{U}\mathcal{P}_d\mathtt{D}\mathcal{P}_d + \mathtt{U}'\mathcal{P}_d(\mathtt{D}\mathcal{P}_d)^{d-1}.$$

Let $\mathcal{Q}_d \in \mathbb{Q}\langle \mathtt{U},\mathtt{U}',\mathtt{D}\rangle$ be the formal sum of the subset of the paths encoded in $\mathcal{P}_d$ defined by requiring that the height at which any $\mathtt{U}'$ step starts is at least two. Then

$$\mathcal{Q}_d = \big(\mathtt{U}(\mathtt{U}\mathcal{P}_d\mathtt{D})^*\mathtt{D}\big)^*.$$

Define $\varphi : \mathbb{Q}\langle \mathtt{U},\mathtt{U}',\mathtt{D}\rangle \to \mathbb{Q}[q,x]$ by $\mathtt{U} \mapsto x$, $\mathtt{D} \mapsto 1$ and $\mathtt{U}' \mapsto qx^{d+1}$. Then, with $P_d(q,x) = \varphi(\mathcal{P}_d)$ and $Q_d(q,x) = \varphi(\mathcal{Q}_d)$, we have

$$\begin{aligned} P_d(q,x) &= 1 + xP_d(q,x)^2 + qx^{d+1}P_d(q,x)^{d-1}; \\ Q_d(q,x) &= \cfrac{1}{1 - \cfrac{x}{1 - xP_d(q,x)}}. \end{aligned}$$

By following the same line of reasoning as were used to demonstrate identity (12) we arrive the following result.

Proposition 8.4. *For any $d \geq 0$,*

$$\sum_{\rho} q^{\mathtt{DDU}^{d+1}(\rho)}\, x^{|\rho|} \;=\; Q_d(q-1,x),$$

where the sum ranges over all Dyck paths, $|\rho|$ is the semilength of ρ, and $\mathtt{DDU}^{d+1}(\rho)$ is short for the number of $\mathtt{DDU}^{d+1}$ factors in ρ. □

$d\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	2	4	8	16	32	64	128	256	512	1024	2048
1	1	1	2	5	13	35	97	275	794	2327	6905	20705	62642
2	1	1	2	5	14	41	124	384	1212	3885	12614	41400	137132
3	1	1	2	5	14	42	131	420	1375	4576	15434	52639	181230
4	1	1	2	5	14	42	132	428	1420	4796	16432	56966	199448
5	1	1	2	5	14	42	132	429	1429	4851	16718	58331	205632

Table 1: Number of 213-avoiding d-Fishburn permutations of length n.

By combining Lemma 8.3 and Proposition 8.4 we arrive at the desired generating function for 213-avoiding d-Fishburn permutations.

Theorem 8.5. *For any $d \geq 0$,*

$$\sum_{\pi \in \mathrm{F}_d(213)} x^{|\pi|} = Q_d(-1, x). \qquad \square$$

For a fixed small d one can derive an explicit expression for $Q_d(-1,x)$ by solving the corresponding system of functional equations. We have done so for $d \leq 2$:

$$\begin{aligned} Q_0(-1,x) &= \frac{1-x}{1-2x}; \\ Q_1(-1,x) &= \frac{2(1-x)}{1-2x+x^2+\sqrt{1-4x+2x^2+x^4}}; \\ Q_2(-1,x) &= \frac{2(1-x)}{1-2x+2x^2+\sqrt{1-4x+4x^3}}. \end{aligned}$$

Since $\mathrm{F}_d(213) = \mathfrak{S}(\sigma_d, 213)$ and $\#\mathfrak{S}_n(213) = C_n$, the nth Catalan number, we find that the sequence of series $\{Q_d(-1,x)\}_{d\geq 0}$ converges to the generating function for the Catalan numbers:

$$\lim_{d\to\infty} Q_d(-1,x) = \frac{2}{1+\sqrt{1-4x}}$$

The coefficient of x^n in $Q_0(-1,x)$ is 2^{n-1} for $n \geq 1$, and hence one might say that the coefficients in $Q_d(-1,x)$ "interpolate" between 2^{n-1} and C_n; in Table 1 we list the first few coefficients of $Q_d(-1,x)$ for $d \leq 5$.

The transport of patterns between Fishburn permutations and modified ascent sequences developed by the first two authors [CC23b] applies to d-Fishburn permutations and modified d-ascent sequences as well. Call two Cayley permutations α and β *equivalent* if $\mathtt{t}(\alpha) = \mathtt{t}(\beta)$, and let [Cay] denotes the set of equivalence classes over Cay defined this way. Moreover, an element $[\alpha]$ of [Cay] contains $[\rho]$ if α' contains ρ' for some $\alpha' \in [\alpha]$ and $\rho' \in [\rho]$. We denote by $[\mathrm{Cay}][\rho]$ the set of classes that avoid $[\rho]$. By the transport theorem on equivalence classes of Cayley permutations [CC23b, Theorem 4.9], the Burge transpose induces a bijection

$$\mathtt{t} : [\mathrm{Cay}][\rho] \to \mathfrak{S}\big(\mathtt{t}(\rho)\big).$$

Since each equivalence class contains at most one modified ascent sequence and $\mathtt{t}(\hat{\mathrm{A}}_0) = \mathrm{F}_0$, we obtain a size-preserving bijection

$$\mathtt{t} : \hat{\mathrm{A}}_0[\rho] \to \mathrm{F}_0\big(\mathtt{t}(\rho)\big),$$

where $\hat{\mathrm{A}}_0[\rho]$ is the set of modified ascent sequences avoiding every pattern in $[\rho]$. Equivalently [CC23b, Theorem 5.1], for every permutation τ we have a size-preserving bijection

$$\mathtt{t} : \hat{\mathrm{A}}_0(B_\tau) \to \mathrm{F}_0(\tau),$$

where $B_\tau = [\tau^{-1}]$ is the Fishburn basis of τ. A constructive procedure to compute B_τ was given in the same reference.

Now we have proved in Proposition 7.2 that the map $\mathtt{t}$ is injective on $\hat{\mathrm{A}}_d$ for every $d \geq 0$. Therefore, each equivalence class of Cayley permutations contains at most one modified d-ascent sequence. Since $\mathtt{t}(\hat{\mathrm{A}}_d) = \mathrm{F}_d$, we obtain the following transport theorem.

Theorem 8.6. *For any $d \geq 0$ and permutation τ,*

$$\mathtt{t} : \hat{\mathrm{A}}_d(B_\tau) \longrightarrow \mathrm{F}_d(\tau)$$

is a size-preserving bijection, where B_τ is the Fishburn basis of τ, $\hat{\mathrm{A}}_d(B_\tau)$ is the set of modified d-ascent sequences avoiding every pattern in B_τ, and $F_d(\tau)$ is the set of d-Fishburn permutations avoiding τ. In particular,

$$\#\,\mathrm{F}_{d,n}(\tau) = \#\,\hat{\mathrm{A}}_{d,n}(B_\tau). \qquad \square$$

For instance, $B_{213} = \{112, 213\}$ and by combining Theorems 8.5 and 8.6 we get the following result.

Corollary 8.7. *For any $d \geq 0$,*

$$\sum_{\alpha \in \hat{\mathrm{A}}_d(112,213)} x^{|\alpha|} \;=\; Q_d(-1, x). \qquad \square$$

It would be interesting to make a deeper study of pattern avoidance in d-Fishburn permutations and (modified) d-ascent sequences.

9 Final remarks

It would be desirable to have a better understanding of $\hat{\mathrm{I}}$. Computer calculations show that the first few terms of the sequence $|\,\hat{\mathrm{I}}_n\,|$, starting from $n = 0$, are

$$1, 1, 3, 10, 43, 224, 1396, 10136, 84057.$$

We also recall the open problem from Section 5.

Problem 9.1. *Find a characterization of which Cayley permutations lie in $\hat{\mathrm{I}}$, perhaps similar to that of A_0 in equation (1).*

There are many properties of the bijection $\mathrm{hat}_{\max}$ which remain to be investigated. In Section 6, we characterized the image of A_0 under this map. It is natural to ask which sets of permutations are obtained by restricting $\mathrm{hat}_{\max}$ to the set $\mathrm{A}_0(p)$ of ascent sequences which avoid a pattern p. In this regard, we have several conjectures.

Conjecture 9.2. *The map* $\mathrm{hat}_{\max}$ *restricts to the following bijections.*

(a) $\mathrm{A}_0(123) \longrightarrow \mathfrak{S}(123, 213)$,

(b) $\mathrm{A}_0(112) \longrightarrow \mathfrak{S}(213, 312)$,

(c) $\mathrm{A}_0(121) \longrightarrow \mathfrak{S}(213, 231)$,

(d) $\mathrm{A}_0(213) \longrightarrow \mathfrak{S}(213, 45123)$.

We note that the enumeration of $\mathrm{A}_0(p)$, for $p \in \{111, 211, 221, 231, 312\}$, is currently open.

One could also hope to find analogues of the characterization of $\mathrm{hat}_{\max}(\mathrm{A}_0)$ in terms of ir-subdiagonal permutations for larger d.

Question 9.3. *What can we say about* $\mathrm{hat}_{\max}(\mathrm{A}_d)$*, for* $d > 0$*? Since* $\mathrm{A}_0 \subseteq \mathrm{A}_d$*, can we describe* $\mathrm{hat}_{\max}(\mathrm{A}_d)$ *by a similar notion of subdiagonality?*

The approach adopted in Section 6 can be generalized as follows. Let $U \subseteq \mathrm{I}$ be any subset of I. Given any $\alpha \in U$, choose uniquely a nonnegative integer d_α, with $d_\alpha \geq \operatorname{dmin} \alpha$. By Proposition 4.6, we obtain an injection

$$\begin{aligned} \{(\alpha, d_\alpha)\}_{\alpha \in U} &\longrightarrow \hat{\mathrm{I}} \\ (\alpha, d_\alpha) &\longmapsto \mathrm{hat}_{d_\alpha}(\alpha). \end{aligned}$$

What other choices of U and d_α give interesting examples? A natural choice consists in using $d_\alpha = \operatorname{dmin} \alpha$. Can we describe the corresponding subset of $\hat{\mathrm{I}}$? Conversely, what sets of permutations $T \subseteq \mathfrak{S}$ can be pulled back to interesting sets of pairs $\{(\alpha, d_\alpha)\}_{\alpha \in U}$?

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