Caylerian polynomials

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Abstract

The Eulerian polynomials enumerate permutations according to their number of descents. We initiate the study of descent polynomials over Cayley permutations, which we call Caylerian polynomials. Some classical results are generalized by linking Caylerian polynomials to Burge words and Burge matrices. The γ-nonegativity of the two-sided Eulerian polynomials is reformulated in terms of Burge structures. Finally, Cayley permutations with a prescribed ascent set are shown to be counted by Burge matrices with fixed row sums.

1 Introduction

Let $\mathrm{Sym}[n]$ denote the set of permutations of $[n] = \{1, 2, \dots, n\}$. A *descent* of a permutation $v = v_1 \cdots v_n \in \mathrm{Sym}[n]$ is an index $i \in [n-1]$ such that $v_i > v_{i+1}$, and we write $\mathrm{des}(v)$ to denote the total number of descents of v. The nth *Eulerian polynomial* is defined by

$$A_n(t) = \sum_{v \in \mathrm{Sym}[n]} t^{\mathrm{des}(v)} = \sum_{i=0}^{n-1} A(n,i) t^i,$$

where $A(n,i)$ is the number of permutations of $[n]$ with i descents. The coefficients $A(n,i)$ are known as the *Eulerian numbers*, and a permutation statistic that is equidistributed with des is said to be Eulerian; a well known example—due to MacMahon—is the number of exceedances. The study of Eulerian polynomials and numbers dates back to the 1755 book *Institutiones calculi differentialis* [11] by Leonard Euler. His definition was not the familiar combinatorial one given above. Rather, Euler defined his polynomials recursively by $A_0(t) = 1$ and

$$A_n(t) = \sum_{k=0}^{n-1} A_k(t)(t-1)^{n-1-k}.$$

It was MacMahon [18] who initiated the study of descent sets of permutations and later Carlitz and Riordan [7] pointed out the relation between descent sets and Eulerian numbers. Two alternative definitions of the Eulerian polynomials are given by the identities

$$\frac{tA_n(t)}{(1-t)^{n+1}} = \sum_{m \geq 1} m^n t^m \quad \text{and} \quad m^n = \sum_{i=0}^{n-1} A(n,i) \binom{m+i}{i},$$

the first due to Carlitz [6] and the second due to Worpitzky [26].

It is well known that the nth Eulerian polynomial is symmetric and unimodal [12]. A consequence is that it can be expressed as

$$A_n(t) = \sum_j \gamma_{n,j} t^j (1+t)^{n-1-2j},$$

for suitable coefficients $\gamma_{n,j}$. A beautiful combinatorial proof that the coefficients $\gamma_{n,j}$ are nonnegative was given by Foata and Strehl [13]. The proof is based on an action on $\mathrm{Sym}[n]$ known as *valley-hopping*. It induces a partition of $\mathrm{Sym}[n]$, and the contribution of each class to the Eulerian polynomial is equal to $t^j(1+t)^{n-1-2j}$, where $j-1$ is the number of valleys of any permutation in the class. As a result, the coefficient $\gamma_{n,j}$ counts the number of equivalence classes with $j-1$ valleys, and must be nonnegative. A broader introduction to this topic can be found in the book *Eulerian numbers* [22] by Petersen. The same author wrote a short survey [21] that covers the two-sided Eulerian numbers and the γ-nonnegativity of the Eulerian polynomials.

In 1978, Gessel and Stanley [15] introduced Stirling permutations; they are permutations of the multiset $\{1^2, 2^2, \dots, n^2\}$ such that all the entries between two copies of the same integer i are greater than i. Using the language of permutations patterns, this restriction translates to avoiding the pattern 212. The rationale for their name is that a formula analogous to Carlitz identity is obtained by replacing the Eulerian polynomial with the descent polynomial over Stirling permutations, and replacing m^n with the $(m+n, n)$th Stirling number of the second kind. Several variants of Stirling permutations have been studied. Quasi-Stirling permutations [2] are permutations of the multiset $\{1^2, 2^2, \dots, n^2\}$ that avoid 1212 and 2121. In a note written in 1978 and published in 2020, Gessel [14] considers Stirling permutations (in the sense of avoiding 212) of the multiset $\{1^k, \dots, n^k\}$, for any $k \geq 1$; his note also contains an up-to-date list of (some of) the literature devoted to Stirling permutations. Stirling and quasi-Stirling permutations of arbitrary multisets have been studied by Brenti [4] in the context of Hilbert polynomials, and by Kuba and Panholzer [16] in relation to increasing trees.

In this paper we explore descent polynomials over so called Cayley permutations. They are permutations of multisets (without any Stirling or quasi-Stirling restrictions), where each element has positive multiplicity. There are known links between Eulerian polynomials and Cayley permutations. For instance, it is well known (see e.g. Stanley [25]) that the nth Eulerian polynomial evaluated at 2 is equal to the number of Cayley permutations of length n, which is the nth Fubini number. Furthermore, permutations of length n whose descent set is a subset of $S = \{s_1, \dots, s_k\}$, with $1 \leq s_1 < s_2 < \cdots < s_k \leq n-1$, are enumerated by a multinomial coefficient that counts Cayley permutations containing s_1 copies of 1, $s_2 - s_1$ copies of 2, and so on. One might say (somewhat imprecisely) that descents over permutations are governed by Cayley permutations. A natural question is then what combinatorial structures govern descents over Cayley permutations. Here we give an answer in terms of Burge matrices and Burge words, providing a more general framework from which most of the classical results for permutations can be derived as special cases.

The rest of this article is structured as follows. In Section 2 we provide necessary preliminaries. For instance, we define Cayley permutations and the related symmetries. We also introduce Burge matrices and Burge words. The former are matrices with nonnegative integer entries whose every row and column contains at least one nonzero entry. The latter are biwords where the bottom row is a Cayley permutation and the top row is a weakly increasing Cayley permutation whose descent set is a subset of the descent set of the bottom row.

Section 3 is focused on binary Burge matrices and the even more restrictive subset of matrices whose every column sums to 1. These two classes of matrices will later be seen to have close connections to the Caylerian polynomials. We also make the following curious observation: By reversing the set inequality that defines Burge words we obtain a set of biwords that is equinumerous with binary Burge matrices.

In Section 4, we extend the correspondence between Burge matrices and Burge words by allowing matrices to have empty rows.

In Section 5 we study the weak and strict descent polynomials over the set of Cayley permutations, which we call Caylerian polynomials. In Theorem 5.1, we show that the weak and strict nth Caylerian polynomials evaluated at 2 count Burge matrices and binary Burge matrices, respectively.

In Section 6 we define a two-sided version of the Caylerian polynomials. The main result of this section, Theorem 6.4, highlights a surprising link with its classical counterpart. As a consequence, we are able to reformulate the γ-nonnegativity of the two-sided Eulerian polynomials in terms of Burge structures.

In Section 7 we rephrase Stanley's notion of v-compatible maps and generalize it to Cayley permutations. In Corollary 7.3, we show that the Fishburn basis [10] of a permutation v is the set of Cayley permutations that are compatible with v. Two equations relating the Caylerian polynomials to Burge matrices in which empty rows are allowed are obtained in Theorem 7.6.

In Section 8 we study Cayley permutations with a prescribed set of ascents and relate them, in Theorem 8.2, to Burge matrices with fixed row sums.

Finally, in Section 9, we suggest some directions for future work.

2 Preliminaries

A *Cayley permutation* [9, 10, 19] on $[n]$ is a mapping $v : [n] \to [n]$ such that $\mathrm{Im}(v) = [k]$ for some $k \leq n$. We often identify v with the word $v = v_1 \cdots v_n$, where $v_i = v(i)$. Let $\mathrm{Cay}[n]$ be the set of Cayley permutations on $[n]$. For instance, $\mathrm{Cay}[1] = \{1\}$, $\mathrm{Cay}[2] = \{11, 12, 21\}$ and

$$\mathrm{Cay}[3] = \{111, 112, 121, 122, 123, 132, 211, 212, 213, 221, 231, 312, 321\}.$$

A *ballot*, also called an ordered set partition, of $[n]$ is a list of disjoint blocks (nonempty sets) $B_1 B_2 \cdots B_k$ whose union is $[n]$. Let $\mathrm{Bal}[n]$ be the set of ballots of $[n]$. There is a well-known one-to-one correspondence between ballots $B_1 B_2 \cdots B_k$ and Cayley permutations v where $i \in B_{v(i)}$. For instance, $\{2,3,5\}\{6\}\{1,7\}\{4\}$ in $\mathrm{Bal}[7]$ corresponds to 3114123 in $\mathrm{Cay}[7]$. In particular, $|\mathrm{Cay}[n]|$ is equal to the nth Fubini number (A000670 in the OEIS [24]).

A bijective map $v : [n] \to [n]$ is a *permutation*. In other words, a permutation is a Cayley permutation where $\mathrm{Im}(v) = [n]$. Let $\mathrm{Sym}[n]$ be the set of permutations on $[n]$

and note that $\mathrm{Sym}[n] \subseteq \mathrm{Cay}[n]$. Denote by id_n the identity permutation $\mathrm{id}_n(i) = i$. In fact, we shall often omit the subscript in id_n and let n be inferred by context. Moreover, let $\mathrm{I}[n]$ be the subset of $\mathrm{Cay}[n]$ consisting of the weakly increasing Cayley permutations:

$$\mathrm{I}[n] = \{v \in \mathrm{Cay}[n] : v(1) \leq v(2) \leq \cdots \leq v(n)\}.$$

Note that $|\mathrm{I}[n]| = 2^{n-1}$ for each $n \geq 1$. For example,

$$\mathrm{I}[1] = \{1\},\ \mathrm{I}[2] = \{11, 12\} \ \text{ and } \ \mathrm{I}[3] = \{111, 112, 122, 123\}.$$

Given a Cayley permutation $v \in \mathrm{Cay}[n]$, define the set of *descents* and *strict descents* of v as, respectively,

$$\begin{aligned} D(v) &= \{i \in [n-1] : v(i) \geq v(i+1)\}; \\ D'(v) &= \{i \in [n-1] : v(i) > v(i+1)\}. \end{aligned}$$

Let $\mathrm{des}(v) = |D(v)|$ and $\mathrm{des}'(v) = |D'(v)|$. The sets $A(v)$ and $A'(v)$ of *ascents* and *strict ascents*, as well as their cardinalities $\mathrm{asc}(v)$ and $\mathrm{asc}'(v)$, are defined analogously. To avoid confusion, we will sometimes add the word "weak" to ascents and descents to distinguish them from their strict counterpart.

The four symmetries of a rectangle act on the plot of a Cayley permutation. They are generated by the reverse and complement operations, which we now define. For any $v \in \mathrm{Cay}[n]$ and $i \in [n]$, let the *reverse* of v be $v^r(i) = v(n+1-i)$, and let the *complement* of v be $v^c(i) = \max(v) + 1 - v(i)$, where $\max(v) = \max\{v_i : i \in [n]\}$. The following lemma is easy to prove.

Lemma 2.1. *For $v \in \mathrm{Cay}[n]$,*

$$\begin{aligned} D(v^c) &= A(v), & D'(v^c) &= A'(v), \\ A(v^c) &= D(v), & A'(v^c) &= D'(v), \\ D(v^r) &= n - A(v), & D'(v^r) &= n - A'(v), \\ A(v^r) &= n - D(v), & A'(v^r) &= n - D'(v), \end{aligned}$$

where $n - X = \{n - x : x \in X\}$ for any set of numbers X.

A *Burge matrix* [10] is a matrix with nonnegative integer entries whose every row and column has at least one nonzero entry. The size of a Burge matrix is the sum of its entries and we let $\mathrm{Mat}[n]$ denote the set of Burge matrices of size n. For instance, $\mathrm{Mat}[2]$ contains the five matrices

$$\begin{bmatrix} 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1}$$

With each matrix $A = (a_{ij})$ in $\mathrm{Mat}[n]$ we associate a biword $\binom{u}{v}$ of length n as follows. Any column $\binom{i}{j}$ appears a_{ij} times and the columns are sorted in ascending order with respect to the top entry, breaking ties by sorting in descending order with respect to the bottom entry. For instance, the biword corresponding to the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{is} \quad \begin{pmatrix} 1\,1\,1\,2\,2\,2\,3\,3 \\ 3\,3\,1\,3\,2\,2\,2\,1 \end{pmatrix}.$$

To ease notation, we will often write the biword $\binom{u}{v}$ as a pair (u, v).

Let φ be the map associating each Burge matrix A with the corresponding biword (u, v), as described above, and let $\mathrm{Bur}[n] = \varphi(\mathrm{Mat}[n])$. Note that the number of rows of A is equal to $\max(u)$; similarly, the number of columns of A is equal to $\max(v)$. Furthermore, the requirement that each row and column of A is not null guarantees that both u and v are Cayley permutations. In particular, u is a weakly increasing Cayley permutation. Finally, we have $D(u) \subseteq D(v)$ due to the procedure adopted to sort the columns of (u, v). Indeed, we have $D(u) = \{i : u(i) = u(i+1)\}$ and entries of v that have the same top entry are sorted in weakly decreasing order. Conversely, any such biword is associated with a unique Burge matrix; more explicitly, the biword (u, v) is associated with the matrix $A = (a_{ij})$, of size $\max(u) \times \max(v)$, where a_{ij} is equal to the number of columns $\binom{i}{j}$ in (u, v). In fact, the set $\mathrm{Bur}[n]$ can be alternatively defined as

$$\mathrm{Bur}[n] = \{(u, v) \in \mathrm{I}[n] \times \mathrm{Cay}[n] : D(u) \subseteq D(v)\},$$

and the map φ is a bijection between $\mathrm{Mat}[n]$ and $\mathrm{Bur}[n]$. Elements of $\mathrm{Bur}[n]$ are called *Burge words*. This terminology is due to Alexandersson and Uhlin [1] and the connection to Burge is with his variant of the RSK correspondence [5]. The sequence of cardinalities $|\mathrm{Bur}[n]|$ is A120733 in the OEIS [24].

Given $v \in \mathrm{Cay}[n]$, we define

$$\mathrm{I}(v) = \{u \in \mathrm{I}[n] : D(u) \subseteq D(v)\}$$

so that

$$\mathrm{Bur}[n] = \bigcup_{v \in \mathrm{Cay}[n]} \mathrm{I}(v) \times \{v\},$$

where the union is disjoint.

Let $A = (a_{ij}) \in \mathrm{Mat}[n]$ and let $x = \varphi(A)$ be its corresponding biword in $\mathrm{Bur}[n]$. It is straightforward to compute the biword x^T corresponding to the transpose $A^T = (a_{ji})$ of A: turn each column of x upside down and then sort the columns of the resulting biword as described previously. Following [10], we shall write

$$\begin{pmatrix} u \\ v \end{pmatrix}^T = \begin{pmatrix} \mathrm{sort}(v) \\ \Gamma(u, v) \end{pmatrix},$$

where $\mathrm{sort}(v)$ is obtained by sorting v in weakly increasing order and

$$\Gamma : \mathrm{Bur}[n] \to \mathrm{Cay}[n]$$

is the map associating (u, v) with the bottom row of $(u, v)^T$. As a notable instance of this construction, if p is a permutation, then

$$\begin{pmatrix} \mathrm{id} \\ p \end{pmatrix}^T = \begin{pmatrix} \mathrm{id} \\ p^{-1} \end{pmatrix} \quad \text{and} \quad \Gamma(\mathrm{id}, p) = p^{-1}.$$

Now, it is clear that transposition acts as an involution on the set of Burge matrices, and thus $\mathrm{Bur}[n]$ is closed under transpose too. Indeed, we [10] showed that the transpose operation gives an alternative characterization of $\mathrm{Bur}[n]$: For any biword $(u, v) \in \mathrm{I}[n] \times \mathrm{Cay}[n]$,

$$D(u) \subseteq D(v) \quad \text{if and only if} \quad ((u, v)^T)^T = (u, v).$$

Note that the Burge transpose extends naturally to any biword (u, v) where u and v are maps $[n] \to \{1, 2, 3, \dots\}$ such that u is weakly increasing and $D(u) \subseteq D(v)$. To compute $(u, v)^T$ we flip (u, v) upside down and sort the columns of the resulting biword according to the same procedure as before. It is easy to see that the equality $((u, v)^T)^T = (u, v)$ still holds. Indeed it depends solely on the procedure adopted to sort the columns and on the initial requirement that $D(u) \subseteq D(v)$. The map Γ extends in a similar fashion. For instance,

$$\left(\begin{pmatrix} 1\,1\,2\,4\,4\,7\,8\,8 \\ 3\,3\,1\,5\,1\,5\,6\,1 \end{pmatrix}^T\right)^T = \begin{pmatrix} 1\,1\,1\,3\,3\,5\,5\,6 \\ 8\,4\,2\,1\,1\,7\,4\,8 \end{pmatrix}^T = \begin{pmatrix} 1\,1\,2\,4\,4\,7\,8\,8 \\ 3\,3\,1\,5\,1\,5\,6\,1 \end{pmatrix}$$

and $\Gamma(11244788, 33151561) = 84211748$. Note that $\Gamma(u_1, v) \neq \Gamma(u_2, v)$ if $u_1 \neq u_2$. In other words, for any fixed $v \in \mathrm{Cay}[n]$, the map $u \mapsto \Gamma(u, v)$ is injective on $\mathrm{I}(v) \times \{v\}$.

3 Binary Burge matrices

We shall now define two subsets of $\mathrm{Mat}[n]$ that will be particularly relevant in this paper. Let $\mathrm{Mat}^{01}[n]$ be the set of *binary Burge matrices* of size n, that is, matrices in $\mathrm{Mat}[n]$ with coefficients in $\{0, 1\}$. The cardinality of $\mathrm{Mat}^{01}[n]$ is given by A101370 in the OEIS [24]. The four biwords corresponding to matrices in $\mathrm{Mat}^{01}[2]$ are

$$\begin{pmatrix} 11 \\ 21 \end{pmatrix}, \begin{pmatrix} 12 \\ 12 \end{pmatrix}, \begin{pmatrix} 12 \\ 21 \end{pmatrix}, \begin{pmatrix} 12 \\ 11 \end{pmatrix}.$$

Let $\mathrm{Mat}^![n]$ be the set of Burge matrices whose every column sums to 1; equivalently, whose every column contains precisely one nonzero entry, which is equal to 1. Clearly,

$$\mathrm{Mat}^![n] \subseteq \mathrm{Mat}^{01}[n] \subseteq \mathrm{Mat}[n].$$

For instance, among the five Burge matrices of size two listed in (1), three belong to $\mathrm{Mat}^![2]$, namely

$$\begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

whereas the matrix $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ does not, but it is binary.

Let us define the Burge words corresponding to $\mathrm{Mat}^{01}[n]$ and $\mathrm{Mat}^![n]$:

$$\begin{aligned} \mathrm{Bur}^{01}[n] &= \big\{(u, v) \in \mathrm{Bur}[n] : D(u) \subseteq D'(v)\big\}; \\ \mathrm{Bur}^![n] &= \big\{(u, v) \in \mathrm{Bur}[n] : v \in \mathrm{Sym}[n]\big\}. \end{aligned}$$

Let $v \in \mathrm{Cay}[n]$. In analogy with $\mathrm{I}(v)$, we also define

$$\mathrm{I}^{01}(v) = \{u \in \mathrm{I}[n] : D(u) \subseteq D'(v)\}$$

so that

$$\mathrm{Bur}^{01}[n] = \bigcup_{v \in \mathrm{Cay}[n]} \mathrm{I}^{01}(v) \times \{v\},$$

where the union is disjoint. Note that

$$\mathrm{Bur}^![n] \subseteq \mathrm{Bur}^{01}[n] \subseteq \mathrm{Bur}[n].$$

Indeed, $\mathrm{Bur}^{01}[n] \subseteq \mathrm{Bur}[n]$ since $D'(v) \subseteq D(v)$ and $\mathrm{I}^{01}(v) \subseteq \mathrm{I}(v)$. Furthermore, if $v \in \mathrm{Sym}[n]$, then $D(v) = D'(v)$, from which $\mathrm{Bur}^![n] \subseteq \mathrm{Bur}^{01}[n]$ immediately follows.

Proposition 3.1. *For $n \geq 0$,*

$$\varphi(\mathrm{Mat}^{01}[n]) = \mathrm{Bur}^{01}[n] \quad \textit{and} \quad \varphi(\mathrm{Mat}^{!}[n]) = \mathrm{Bur}^{!}[n].$$

In particular,

$$|\mathrm{Mat}^{01}[n]| = |\mathrm{Bur}^{01}[n]| \quad \textit{and} \quad |\mathrm{Mat}^{!}[n]| = |\mathrm{Bur}^{!}[n]|.$$

Proof. Let $A \in \mathrm{Mat}[n]$ and $\varphi(A) = (u, v) \in \mathrm{Bur}[n]$. We shall start by proving that $\varphi(\mathrm{Mat}^{01}[n]) = \mathrm{Bur}^{01}[n]$. If A is binary, then (u, v) has no repeated columns. In particular, if $u(i) = u(i+1)$, then $v(i) \neq v(i+1)$. Moreover, $v(i) \geq v(i+1)$ due to the inequality $D(u) \subseteq D(v)$ defining biwords in $\mathrm{Bur}[n]$. Thus $u(i) = u(i+1)$ implies $v(i) > v(i+1)$, and so $D(u) \subseteq D'(v)$ and $(u, v) \in \mathrm{Bur}^{01}[n]$. Similarly, if $(u, v) \in \mathrm{Bur}^{01}[n]$, i.e. $D(u) \subseteq D'(v)$, then there are no repeated columns in (u, v) and $A \in \mathrm{Mat}^{01}[n]$. The equality $\varphi(\mathrm{Mat}^{!}[n]) = \mathrm{Bur}^{!}[n]$ follows from the following simple observation: Each column of A sums to 1 if and only if v contains exactly one copy of each integer, that is, $v \in \mathrm{Sym}[n]$ is a permutation. □

We shall shortly define a natural set of biwords (denoted $\Omega[n]$ below) that is equinumerous with binary Burge words. While we find this connection interesting, we should point out that the machinery we wish to develop surrounding Cayley polynomials does not depend on it, and in that sense the reminder of this section can be skipped.

Recall that

$$\mathrm{Bur}[n] = \{(u, v) \in \mathrm{I}[n] \times \mathrm{Cay}[n] : D(u) \subseteq D(v)\}.$$

What happens if we reverse the set inequality and define

$$\Omega[n] = \{(u, v) \in \mathrm{I}[n] \times \mathrm{Cay}[n] : D(u) \supseteq D(v)\}?$$

Curiously, it turns out that $\Omega[n]$ and $\mathrm{Bur}^{01}[n]$ are equinumerous. This is due to a symmetry on $\mathrm{I}[n]$ that we now define. Any $u \in \mathrm{I}[n]$ is determined by its descent set $D(u)$. Indeed, if we know that $D(u) = S \subseteq [n-1]$ and $u \in \mathrm{I}[n]$, with $n \geq 1$, then u is given by $u(1) = 1$ and $u(i+1) = u(i) + [i \notin S]$. Here we are using the Iverson bracket, so that $[i \notin S]$ is 1 if $i \notin S$ and 0 if $i \in S$. We define $u^\star$ as the unique weakly increasing Cayley permutation with descent set $[n-1] \setminus D(u)$, and we call $u^\star$ the *conjugate* of u. For instance, if $u = 12223445555$, then $u^\star = 11233344567$, where $D(u^\star) = \{1, 2, \ldots, 10\} \setminus D(u) = \{1, 4, 5, 7\}$. Note that if we add the ith letter of u with the ith letter of $u^\star$, for $i = 1$ through $i = 11$, we get 2,3,…,12. This is not a coincidence. In fact, we have the following lemma, which could serve as the definition of $u^\star$. We give two proofs, one using induction, the other more direct.

Lemma 3.2. *Let $u \in \mathrm{I}[n]$. For each $i \in [n]$ we have $u^\star(i) = i + 1 - u(i)$.*

First proof. Since $u(1) = u^\star(1) = 1$, the statement trivially holds for $i = 1$. Let $S = D(u)$ and $T = D(u^\star)$, so that $T = [n-1] \setminus S$. For $i \geq 1$ we have

$$\begin{aligned} u^\star(i+1) = u^\star(i) + [i \notin T] &= i + 1 - u(i) + [i \notin T] \\ &= i + 1 - u(i) + (1 - [i \in T]) \\ &= i + 2 - (u(i) + [i \notin S]) = i + 2 - u(i+1), \end{aligned}$$

which—by induction—concludes the proof. □

Second proof. Reading from left to right, the ith letter of u is simply one more than the number of strict ascents seen up to that point, where a strict ascent is an element not in $D(u)$; in other words, a strict ascent is an element in $D(u^\star)$. Thus

$$u(i) = 1 + \mathrm{des}(u^\star \circ \iota_{i,n}),$$

where $\iota_{i,n} : [i] \to [n]$ denotes the inclusion map. Since conjugation is an involution we also have $u^\star(i) = 1 + \mathrm{des}(u \circ \iota_{i,n})$. Finally, $\mathrm{des}(u \circ \iota_{i,n}) + \mathrm{des}(u^\star \circ \iota_{i,n}) = i - 1$ and the result follows. $\square$

The following lemma is easy to prove.

Lemma 3.3. *For $u \in \mathrm{I}[n]$ we have $D(u^\star) = A'(u)$ and $A'(u^\star) = D(u)$.*

Proposition 3.4. *Define the mapping $\theta : \mathrm{Bur}^{01}[n] \to \Omega[n]$ by $\theta(u,v) = \big((u^{rc})^\star, v^r\big)$. Then θ is a bijection.*

Proof. Note that if $u \in \mathrm{I}[n]$, then $u^{rc} \in \mathrm{I}[n]$, and thus $(u^{rc})^\star$ is well defined. Since reverse, complement and conjugation are involutions, it is clear that θ is injective. It only remains to show that the image of $\mathrm{Bur}^{01}[n]$ under θ is $\Omega[n]$. By Lemma 2.1 and Lemma 3.3 we have

$$D\big((u^{rc})^\star\big) = A'(u^{rc}) = D'(u^r) = n - A'(u) \quad \text{and} \quad D(v^r) = n - A(v).$$

Since $A'(u) \cup D(u) = [n-1] = A(v) \cup D'(v)$ it follows that

$$\begin{aligned} \theta(u,v) \in \Omega[n] \iff & A'(u) \supseteq A(v) \\ \iff & D(u) \subseteq D'(v) \iff (u,v) \in \mathrm{Bur}^{01}[n], \end{aligned}$$

which concludes the proof. $\square$

Example 3.5. We have

$$\mathrm{Bur}^{01}[2] = \left\{ \begin{pmatrix} 1\,1 \\ 2\,1 \end{pmatrix}, \begin{pmatrix} 1\,2 \\ 1\,1 \end{pmatrix}, \begin{pmatrix} 1\,2 \\ 1\,2 \end{pmatrix}, \begin{pmatrix} 1\,2 \\ 2\,1 \end{pmatrix} \right\};$$

$$\Omega[2] = \left\{ \begin{pmatrix} 1\,2 \\ 1\,2 \end{pmatrix}, \begin{pmatrix} 1\,1 \\ 1\,1 \end{pmatrix}, \begin{pmatrix} 1\,1 \\ 2\,1 \end{pmatrix}, \begin{pmatrix} 1\,1 \\ 1\,2 \end{pmatrix} \right\}.$$

With the elements in the order they are listed above, θ maps the ith element of $\mathrm{Bur}^{01}[2]$ to the ith element of $\Omega[2]$.

4 Allowing empty rows

We wish to generalize our setting by allowing Burge matrices to have empty rows, the reason for which will become clear in the coming sections. Let $\mathfrak{Mat}$ consist of all matrices with nonnegative integer entries whose every column has at least one nonzero entry, and let $\mathfrak{Mat}_m$ be the set of matrices in $\mathfrak{Mat}$ with m rows. Also, denote by $\mathfrak{Mat}[n]$ and $\mathfrak{Mat}_m[n]$ the corresponding sets of matrices whose entries sum to n. Clearly, $\mathrm{Mat}[n] \subseteq \mathfrak{Mat}[n]$. Matrices in $\mathfrak{Mat}_m[n]$ have been studied by Munarini, Poneti and Rinaldi [20] under the name of *m-compositions* of n. It is easy to see that the procedure defining the map φ applies to matrices in $\mathfrak{Mat}$ as well. However, since empty rows are now admitted, the top row of the resulting biwords is a weakly increasing map, but not necessarily a Cayley permutation. For instance, the biword associated with

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{is} \quad \begin{pmatrix} 1\,1\,1\,3\,3 \\ 3\,3\,1\,2\,1 \end{pmatrix}.$$

Another consequence of allowing empty rows is that the correspondence between matrices and biwords is not bijective anymore. Returning to the matrix of the previous example, all the matrices obtained by inserting any number of empty rows at the bottom give rise to the same biword:

$$\varphi^{-1}\begin{pmatrix} 1\,1\,1\,3\,3 \\ 3\,3\,1\,2\,1 \end{pmatrix} = \left\{ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \dots \right\}.$$

Here the ambiguity lies in that we did not specify the codomain of top row u of the biword $(u, v) = (11133, 33121)$. It is clear that, if we regard u as a mapping $u : [n] \to [m]$, for a fixed value of m, then there is a unique matrix A with m rows associated with (u, v). Note also that $m \geq \operatorname{asc}(v) + 1$ because of the inclusion $D(u) \subseteq D(v)$. Define the set

$$\mathfrak{I}_m[n] = \{u : [n] \to [m] \ \& \ u(1) \leq u(2) \leq \cdots \leq u(n)\}$$

of weakly increasing maps from $[n]$ to $[m]$, and let

$$\mathfrak{Bur}_m[n] = \{(u, v) \in \mathfrak{I}_m[n] \times \mathrm{Cay}[n] : D(u) \subseteq D(v)\}.$$

To ease notation, let $\mathrm{Bur} = \cup_{n\geq 0}\mathrm{Bur}[n]$, $\mathfrak{Bur} = \cup_{n\geq 0}\mathfrak{Bur}[n]$, $\mathfrak{Bur}_m = \cup_{n\geq 0}\mathfrak{Bur}_m[n]$, etc. We may also write $(u, v) \in \mathrm{I} \times \mathrm{Cay}$ instead of $(u, v) \in \mathrm{I}[n] \times \mathrm{Cay}[n]$, and $(u, v) \in \mathfrak{I}_m \times \mathrm{Cay}$ instead of $(u, v) \in \mathfrak{I}_m[n] \times \mathrm{Cay}[n]$. This should not lead to any confusion since n is determined by the length of u, or v, and in all biwords (u, v) considered in this paper u and v will be of equal length. The correspondence between matrices and biwords described above induces a family of bijections

$$\varphi_m : \mathfrak{Mat}_m \to \mathfrak{Bur}_m.$$

For instance, if we regard $u = 11133$ as a map $u : [5] \to [3]$, then

$$\begin{pmatrix} 1\,1\,1\,3\,3 \\ 3\,3\,1\,2\,1 \end{pmatrix} \in \mathfrak{Bur}_3 \quad \text{and} \quad \varphi_3^{-1}\begin{pmatrix} 1\,1\,1\,3\,3 \\ 3\,3\,1\,2\,1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \in \mathfrak{Mat}_3.$$

On the other hand, if we regard u as a map $u : [5] \to [4]$, then

$$\begin{pmatrix} 1\,1\,1\,3\,3 \\ 3\,3\,1\,2\,1 \end{pmatrix} \in \mathfrak{Bur}_4 \quad \text{and} \quad \varphi_4^{-1}\begin{pmatrix} 1\,1\,1\,3\,3 \\ 3\,3\,1\,2\,1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{Mat}_4.$$

As a further illustration, the biwords $(u, v) \in \mathfrak{Bur}_3[2]$ in which $v = 11$, and the corresponding matrices in $\mathfrak{Mat}_3[2]$, are listed below:

$$\binom{11}{11} \leftrightarrow \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \qquad \binom{12}{11} \leftrightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \binom{13}{11} \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$\binom{22}{11} \leftrightarrow \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \qquad \binom{23}{11} \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \binom{33}{11} \leftrightarrow \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

We define subsets of $\mathfrak{Mat}$ and $\mathfrak{Bur}$ that are analogous to Mat^{01}, $\mathrm{Mat}^{!}$, Bur^{01} and $\mathrm{Bur}^{!}$ as follows. Let $\mathfrak{Mat}^{01}_m$ be the set of binary matrices in $\mathfrak{Mat}_m$ and let $\mathfrak{Mat}^{!}_m$ be the set of matrices in $\mathfrak{Mat}_m$ where each column sums to 1. Furthermore, let

$$\mathfrak{Bur}^{01}_m = \{(u, v) \in \mathfrak{Bur}_m : D(u) \subseteq D'(v)\}$$

and

$$\mathfrak{Bur}^{!}_m = \{(u, v) \in \mathfrak{Bur}_m : v \in \mathrm{Sym}\}.$$

Clearly, $\mathfrak{Mat}^{!}_m \subseteq \mathfrak{Mat}^{01}_m \subseteq \mathfrak{Mat}_m$ and $\mathfrak{Bur}^{!}_m \subseteq \mathfrak{Bur}^{01}_m \subseteq \mathfrak{Bur}_m$. Given $v \in \mathrm{Cay}[n]$, let

$$\mathfrak{I}_m(v) = \{u \in \mathfrak{I}_m[n] : D(u) \subseteq D(v)\} \quad \text{and} \quad \mathfrak{I}^{01}_m(v) = \{u \in \mathfrak{I}_m[n] : D(u) \subseteq D'(v)\}$$

so that we have the disjoint unions

$$\mathfrak{Bur}_m = \bigcup_{v \in \mathrm{Cay}} \mathfrak{I}_m(v) \times \{v\} \quad \text{and} \quad \mathfrak{Bur}^{01}_m = \bigcup_{v \in \mathrm{Cay}} \mathfrak{I}^{01}_m(v) \times \{v\}.$$

The proof of the following result is akin to the proof of Proposition 3.1 and is omitted.

Proposition 4.1. *For each $m \geq 0$,*

$$\varphi_m(\mathfrak{Mat}^{01}_m) = \mathfrak{Bur}^{01}_m \quad \textit{and} \quad \varphi_m(\mathfrak{Mat}^{!}_m) = \mathfrak{Bur}^{!}_m.$$

In particular, for each $n \geq 0$,

$$|\mathfrak{Mat}^{01}_m[n]| = |\mathfrak{Bur}^{01}_m[n]| \quad \textit{and} \quad |\mathfrak{Mat}^{!}_m[n]| = |\mathfrak{Bur}^{!}_m[n]|.$$

5 Caylerian polynomials

Let $n \geq 0$. The *nth Eulerian polynomial* is

$$A_n(t) = \sum_{v \in \mathrm{Sym}[n]} t^{\mathrm{des}(v)}.$$

We shall study the corresponding descent polynomials over Cayley permutations. There are two reasonable definitions: the *nth (weak) Caylerian polynomial* and the *nth strict Caylerian polynomial* are defined, respectively, as

$$C_n(t) = \sum_{v \in \mathrm{Cay}[n]} t^{\mathrm{des}(v)} \quad \text{and} \quad C'_n(t) = \sum_{v \in \mathrm{Cay}[n]} t^{\mathrm{des}'(v)}.$$

It is clear (e.g. by Lemma 2.1) that in the above definitions we can replace $\operatorname{des}(v)$ with $\operatorname{asc}(v)$ and $\operatorname{des}'(v)$ with $\operatorname{asc}'(v)$, respectively. In other words, we have

$$A_n(t) = \sum_{v \in \mathrm{Sym}[n]} t^{\operatorname{asc}(v)}, \quad C_n(t) = \sum_{v \in \mathrm{Cay}[n]} t^{\operatorname{asc}(v)}, \quad C'_n(t) = \sum_{v \in \mathrm{Cay}[n]} t^{\operatorname{asc}'(v)}.$$

Furthermore, since $\operatorname{des}'(v) = n - 1 - \operatorname{asc}(v)$ for each $v \in \mathrm{Cay}_n$, the coefficients of the strict Caylerian polynomial $C'_n(t)$ are simply the reverse of the coefficients of $C_n(t)$:

$$C'_n(t) = t^{n-1} C_n\big(1/t\big).$$

The first Caylerian polynomials are

$$\begin{aligned}
C_0(t) &= 1;\\
C_1(t) &= 1;\\
C_2(t) &= 1 + 2t;\\
C_3(t) &= 1 + 8t + 4t^2;\\
C_4(t) &= 1 + 24t + 42t^2 + 8t^3;\\
C_5(t) &= 1 + 64t + 276t^2 + 184t^3 + 16t^4.
\end{aligned}$$

At the time of writing, the resulting triangle of coefficients is not present in the OEIS [24].

It is well known [10, 25] that the nth Eulerian polynomial evaluated at 2 equals the number of Cayley permutations of $[n]$. A simple proof in terms of the Burge transpose goes as follows. For any permutation v, we have $|\mathrm{I}(v)| = 2^{\operatorname{des}(v)}$. In other words, there are $2^{\operatorname{des}(v)}$ weakly increasing Cayley permutations u such that $(u, v) \in \mathrm{Bur}^!$. Indeed, in order to satisfy the inclusion $D(u) \subseteq D(v)$, if $v(i) < v(i+1)$ is an ascent, then we must have $u(i+1) = u(i)$. On the other hand, if $v(i) > v(i+1)$ is a descent, then both $u(i+1) = u(i)$ and $u(i+1) = u(i) + 1$ are admitted. Thus,

$$\begin{aligned}
A_n(2) \;&=\; \sum_{v \in \mathrm{Sym}[n]} 2^{\operatorname{des}(v)}\\
&=\; |\{(u, v) \in \mathrm{I}[n] \times \mathrm{Sym}[n] : u \in \mathrm{I}(v)\}| \;=\; |\mathrm{Bur}^![n]|.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathrm{Bur}^![n]^T &= \{(u, v) \in \mathrm{I}[n] \times \mathrm{Sym}[n] : u \in \mathrm{I}(v)\}^T\\
&= \{(\mathrm{id}, x) : x \in \mathrm{Cay}[n]\},
\end{aligned}$$

and clearly the cardinality of the latter set is $|\mathrm{Cay}[n]|$. The main advantage of this approach is that it works for Cayley permutations as well. For $v \in \mathrm{Cay}$ there are $2^{\operatorname{des}(v)}$ weakly increasing Cayley permutations $u \in \mathrm{I}(v)$ such that $(u, v) \in \mathrm{Bur}$. Therefore,

$$\begin{aligned}
C_n(2) \;&=\; \sum_{v \in \mathrm{Cay}[n]} 2^{\operatorname{des}(v)}\\
&=\; \sum_{v \in \mathrm{Cay}[n]} |\mathrm{I}(v)| \;=\; |\mathrm{Bur}[n]|.
\end{aligned}$$

Similarly, there are $2^{\mathrm{des}'(v)}$ weakly increasing Cayley permutations $u \in \mathrm{I}^{01}(v)$ such that $(u,v) \in \mathrm{Bur}^{01}$. Therefore,

$$\begin{aligned} C'_n(2) &= \sum_{v \in \mathrm{Cay}[n]} 2^{\mathrm{des}'(v)} \\ &= \sum_{v \in \mathrm{Cay}[n]} |\mathrm{I}^{01}(v)| \;=\; |\mathrm{Bur}^{01}[n]|. \end{aligned}$$

Let us summarize these results in the following theorem.

Theorem 5.1. *For $n \geq 0$,*

(i) $C_n(2) = \big|\mathrm{Bur}[n]\big| = \big|\mathrm{Mat}[n]\big|$;

(ii) $C'_n(2) = \big|\mathrm{Bur}^{01}[n]\big| = \big|\mathrm{Mat}^{01}[n]\big|$;

(iii) $A_n(2) = \big|\mathrm{Bur}^{!}[n]\big| = \big|\mathrm{Mat}^{!}[n]\big|$.

Let us push this approach a bit further. Recall from [10] that the *Fishburn basis* of $v \in \mathrm{Sym}$ is defined by

$$B(v) = \Gamma\big(\mathrm{I}(v) \times \{v\}\big),$$

or, equivalently,

$$B(v) = \big\{x \in \mathrm{Cay} : (\mathrm{id}, x)^T = (\mathrm{sort}(x), v)\big\}\,.$$

Once again, it is easy to see that $|B(v)| = 2^{\mathrm{des}(v)}$. Now, for each $x \in \mathrm{Cay}$ there is exactly one permutation $v \in \mathrm{Sym}$ such that $x \in B(v)$. Indeed, we have

$$x \in B(v) \iff \begin{pmatrix}\mathrm{id} \\ x\end{pmatrix}^T = \begin{pmatrix}\mathrm{sort}(x) \\ v\end{pmatrix} \tag{2}$$

and thus $v = \Gamma(\mathrm{id}, x)$. As a result, the disjoint union

$$\mathrm{Cay} = \bigcup_{v \in \mathrm{Sym}} B(v)$$

holds. In particular, since $(\mathrm{id}, v) \in \mathrm{Bur}$ and $(\mathrm{id}, v)^T = (\mathrm{id}, v^{-1})$, the inverse permutation v^{-1} belongs to the Fishburn basis of v. Next we show that every Cayley permutation in $B(v)$ has the same weak descent set (and thus also strict ascent set) as v^{-1}.

Lemma 5.2. *Let v be a permutation. For each $x \in B(v)$, we have $D(x) = D(v^{-1})$ and $A'(x) = A(v^{-1})$. Furthermore,*

$$\sum_{x \in B(v)} t^{\mathrm{des}(x)} = 2^{\mathrm{des}(v)} t^{\mathrm{des}(v^{-1})} \quad \text{and} \quad \sum_{x \in B(v)} t^{\mathrm{asc}'(x)} = 2^{\mathrm{des}(v)} t^{\mathrm{asc}(v^{-1})}.$$

Proof. Let $v \in \mathrm{Sym}[n]$, $x \in B(v)$ and $u = \mathrm{sort}(x)$, so that $(\mathrm{id}, x)^T = (u, v)$. For $i \in [n-1]$, the columns

$$\begin{pmatrix} i \\ x_i \end{pmatrix} \text{ and } \begin{pmatrix} i+1 \\ x_{i+1} \end{pmatrix} \text{ in } \begin{pmatrix} \mathrm{id} \\ x \end{pmatrix}$$

are mapped via the Burge transpose to the columns

$$\binom{x_i}{i} \quad \text{and} \quad \binom{x_{i+1}}{i+1} \quad \text{in} \quad \binom{u}{v}.$$

Because of how the columns of $(\mathrm{id}, x)^T$ are sorted via the Burge transpose, $i \in D(x)$ if and only if $i+1$ precedes i in v. In particular, if $i \in D(x)$ and $x_i = x_{i+1}$, then $i+1$ precedes i since the corresponding top entries x_{i+1} and x_i are tied and $i+1 > i$. We leave the remaining details to the reader. Furthermore, $i+1$ precedes i in v if and only if i is a descent in v^{-1}, from which $D(x) = D(v^{-1})$ follows. Thus,

$$\begin{aligned} A'(x) &= [n-1] \setminus D(x) \\ &= [n-1] \setminus D(v^{-1}) = A(v^{-1}). \end{aligned}$$

To prove the remaining identities, recall that $v^{-1} \in B(v)$ and $|B(v)| = 2^{\mathrm{des}(v)}$. Thus,

$$\sum_{x \in B(v)} t^{\mathrm{des}(x)} = \sum_{x \in B(v)} t^{\mathrm{des}(v^{-1})} = 2^{\mathrm{des}(v)} t^{\mathrm{des}(v^{-1})}.$$

The second identity follows in a similar manner. □

Recall that $\mathrm{Cay}[n]$ is the disjoint union $\mathrm{Cay}[n] = \bigcup_{v \in \mathrm{Sym}[n]} B(v)$. By Lemma 5.2 we now have

$$C_n(t) = \sum_{v \in \mathrm{Sym}[n]} \sum_{x \in B(v)} t^{\mathrm{des}(x)} = \sum_{v \in \mathrm{Sym}[n]} 2^{\mathrm{des}(v)} t^{\mathrm{des}(v^{-1})}$$

and a similar formula holds for the strict Caylerian polynomials. Let us summarize this in a theorem:

Theorem 5.3. *For $n \geq 0$,*

$$C_n(t) = \sum_{v \in \mathrm{Sym}[n]} 2^{\mathrm{des}(v)} t^{\mathrm{des}(v^{-1})} \quad \text{and} \quad C'_n(t) = \sum_{v \in \mathrm{Sym}[n]} 2^{\mathrm{des}(v)} t^{\mathrm{asc}(v^{-1})}.$$

6 Two-sided Caylerian polynomials

The nth *two-sided Eulerian polynomial* [8, 21] is

$$A_n(s,t) = \sum_{v \in \mathrm{Sym}[n]} s^{\mathrm{des}(v)} t^{\mathrm{des}(v^{-1})}.$$

Note that the identity for the weak Caylerian polynomial $C_n(t)$ in Theorem 5.3 can be expressed in terms of these polynomials as $C_n(t) = A_n(2,t)$. We define a strict version of the two-sided Eulerian polynomials by

$$A'_n(s,t) = \sum_{v \in \mathrm{Sym}[n]} s^{\mathrm{des}(v)} t^{\mathrm{asc}(v^{-1})},$$

so that $C'_n(t) = A'_n(2,t)$. The (single-sided) Eulerian polynomials are related to Cayley permutations by

$$A_n(1+t) = \sum_{v \in \mathrm{Cay}[n]} t^{n-\max(v)} \tag{3}$$

and

$$\sum_{n\geq 0} A_n(1+t)\frac{x^n}{n!} = \frac{t}{1+t-\exp(tx)}. \tag{4}$$

We wish to derive analogous results for descent polynomials over Cayley permutations and Burge structures. To this end, we define the *weak* and *strict two-sided Caylerian polynomials* by

$$\begin{aligned} B_n(s,t) &= \sum_{(u,v)\in \mathrm{Bur}[n]} s^{n-\max(u)} t^{n-\max(v)};\\ B'_n(s,t) &= \sum_{(u,v)\in \mathrm{Bur}^{01}[n]} s^{n-\max(u)} t^{n-\max(v)}. \end{aligned}$$

Equivalently—in terms of Burge matrices—we have

$$\begin{aligned} B_n(s,t) &= \sum_{A\in \mathrm{Mat}[n]} s^{n-\mathrm{row}(u)} t^{n-\mathrm{col}(v)};\\ B'_n(s,t) &= \sum_{A\in \mathrm{Mat}^{01}[n]} s^{n-\mathrm{row}(u)} t^{n-\mathrm{col}(v)}, \end{aligned}$$

where $\mathrm{row}(A)$ and $\mathrm{col}(A)$ denote the number of rows and columns of A, respectively. The polynomial $B'_n(s,t)$ has been studied by Riordan and Stein [23] in terms of arrangements on chessboards (where each line contains at least one piece).

Lemma 6.1. *For $v \in \mathrm{Cay}[n]$,*

$$(1+t)^{\mathrm{des}(v)} = \sum_{u\in \mathrm{I}(v)} t^{n-\max(u)}$$

and

$$(1+t)^{\mathrm{des}'(v)} = \sum_{u\in \mathrm{I}^{01}(v)} t^{n-\max(u)}.$$

Proof. Let $v \in \mathrm{Cay}[n]$. Also, let $u \in \mathrm{I}(v)$ and recall that by definition we then have $D(u) \subseteq D(v)$. If $v(i) < v(i+1)$ is a strict ascent, then $u(i+1) = u(i)+1$. Otherwise, if $v(i) \geq v(i+1)$ is a weak descent, then we have either $u(i+1) = u(i)$ or $u(i+1) = u(i)+1$. Let us mark every such entry $u(i+1)$ with t if $u(i+1) = u(i)$, and with 1 if $u(i+1) = u(i)+1$. From

$$n = \max(u) + \{i \in [n-1] : u(i) = u(i+1)\}$$

it follows that $n - \max(u)$ is equal to the number of entries of u marked with t. By summing the contribution $t^{n-\max(u)}$ of each weakly increasing Cayley permutation $u \in \mathrm{I}(v)$ we obtain $(1+t)^{\mathrm{des}(v)}$. The second identity is proved in a similar manner. □

Lemma 6.2. *For $v \in \mathrm{Sym}[n]$,*

$$\sum_{x\in B(v)} t^{n-\max(x)} = \sum_{u\in \mathrm{I}(v)} t^{n-\max(u)}.$$

Proof. Let $v \in \mathrm{Sym}[n]$. Recall the characterization of the Fishburn basis given in equation (2): $x \in B(v) \iff (\mathrm{id}, x)^T = (\mathrm{sort}(x), v)$. In particular, the Burge transpose bijectively maps

$$\left\{ \binom{\mathrm{id}}{x} : x \in B(v) \right\} \quad \text{to} \quad \left\{ \binom{u}{v} : u \in \mathrm{I}(v) \right\}.$$

Since $\max(x) = \max\bigl(\mathrm{sort}(x)\bigr)$, the claimed identity immediately follows. □

Let us show how equations (3) and (4) can be derived from Lemmas 6.1 and 6.2. First, equation (3):

$$\begin{aligned} A_n(1+t) &= \sum_{v \in \mathrm{Sym}[n]} (1+t)^{\mathrm{des}(v)} \\ &= \sum_{v \in \mathrm{Sym}[n]} \sum_{u \in \mathrm{I}(v)} t^{n-\max(u)} && \text{(by Lemma 6.1)} \\ &= \sum_{v \in \mathrm{Sym}[n]} \sum_{x \in B(v)} t^{n-\max(x)} && \text{(by Lemma 6.2)} \\ &= \sum_{x \in \mathrm{Cay}[n]} t^{n-\max(x)}, \end{aligned}$$

where the last equality follows from $\mathrm{Cay}[n] = \bigcup_{v \in \mathrm{Sym}[n]} B(v)$. Second, equation (4):

$$\begin{aligned} \sum_{n \geq 0} A_n(1+t) \frac{x^n}{n!} &= \sum_{n \geq 0} \sum_{v \in \mathrm{Sym}[n]} (1+t)^{\mathrm{des}(p)} \frac{x^n}{n!} \\ &= \sum_{n \geq 0} \sum_{x \in \mathrm{Cay}[n]} t^{n-\max(x)} \frac{x^n}{n!} \\ &= \frac{t}{1+t-\exp(tx)}, \end{aligned}$$

where the last expression is the exponential generating function of weighted ballots (i.e. Cayley permutations) with weight n minus the number of blocks (i.e. the maximum value). For an elegant proof of this using weighted $\mathbb{L}$-species see Exercise (1a) of Section 5.1 in the book by Bergeron, Labelle and Leroux [3].

Proposition 6.3. *For* $p \in \mathrm{Sym}[n]$,

$$(1+s)^{\mathrm{des}(p)}(1+t)^{\mathrm{des}(p^{-1})} = \sum_{x \in B(p^{-1})} \sum_{u \in \mathrm{I}(x)} s^{n-\max(u)} t^{n-\max(x)}.$$

Proof. Recall that $p \in B(p^{-1})$. By Lemma 5.2, each Cayley permutation $x \in B(p^{-1})$ has the same weak descent set $D(x) = D(p)$ as p. In particular, $\mathrm{I}(x) = \mathrm{I}(p)$ for each

$x \in B(p^{-1})$. Thus

$$\begin{aligned}
\sum_{x\in B(p^{-1})}\sum_{u\in \mathrm{I}(x)} s^{n-\max(u)}t^{n-\max(x)} &= \sum_{x\in B(p^{-1})} t^{n-\max(x)}\left(\sum_{u\in \mathrm{I}(p)} s^{n-\max(u)}\right)\\
&= \sum_{x\in B(p^{-1})} t^{n-\max(x)}(1+s)^{\mathrm{des}(p)}\\
&= (1+s)^{\mathrm{des}(p)}\sum_{v\in \mathrm{I}(p^{-1})} t^{n-\max(v)}\\
&= (1+s)^{\mathrm{des}(p)}(1+t)^{\mathrm{des}(p^{-1})},
\end{aligned}$$

where the second and the last equalities follow by Lemma 6.1 and the penultimate equality follows from Lemma 6.2. □

To illustrate the previous result, let $p = 2413$ and $q = p^{-1} = 3142$. Then $\mathrm{des}(p) = 1$, $\mathrm{des}(q) = 2$, and

$$\begin{aligned}
\mathrm{I}(q) &= \{1122, 1233, 1123, 1234\};\\
B(q) &= \Gamma\left(\mathrm{I}(q)\times\{q\}\right) = \{1212, 1312, 2313, 2413\}.
\end{aligned}$$

Note that $p \in B(q)$ and $D(x) = D(p)$ for each $x \in B(q)$; in particular, $\mathrm{I}(x) = \mathrm{I}(p)$ for each $x \in B(q)$. Now, we have $\mathrm{I}(p) = \{1223, 1234\}$ and

$$\sum_{u\in \mathrm{I}(p)} s^{n-\max(u)} = s + 1 = (1+s)^{\mathrm{des}(p)},$$

as claimed in Lemma 6.1. Referring to the same lemma, $u = 1223$ is obtained by letting $u(3) = u(2)$, and it thus contributes with s to the summand. On the other hand, the contribution of $u = 1234$ is 1 since u is obtained by letting $u(3) = u(2)+1$. Finally,

$$\begin{aligned}
\sum_{x\in B(q)}\sum_{u\in \mathrm{I}(x)} s^{n-\max(u)}t^{n-\max(x)} &= (1+s)^{\mathrm{des}(p)}\sum_{x\in B(q)} t^{n-\max(x)}\\
&= (1+s)^{\mathrm{des}(p)}(t^2+2t+1)\\
&= (1+s)^{\mathrm{des}(p)}(1+t)^2\\
&= (1+s)^{\mathrm{des}(p)}(1+t)^{\mathrm{des}(q)},
\end{aligned}$$

as claimed in Proposition 6.3.

Theorem 6.4. *For $n \geq 0$,*

$$A_n(1+s, 1+t) = B_n(s,t) \quad \textit{and} \quad A'_n(1+s, 1+t) = B'_n(s,t).$$

Proof. Recall that $\mathrm{Cay}[n]$ is the disjoint union $\mathrm{Cay}[n] = \bigcup_{q\in \mathrm{Sym}[n]} B(q)$. By letting $q = p^{-1}$, we have

$$\begin{aligned}
\mathrm{Bur}[n] &= \bigcup_{v\in \mathrm{Cay}[n]}\bigcup_{u\in \mathrm{I}(v)}\{(u,v)\}\\
&= \bigcup_{p\in \mathrm{Sym}[n]}\bigcup_{x\in B(p^{-1})}\bigcup_{u\in \mathrm{I}(x)}\{(u,x)\},
\end{aligned}$$

where once again all the unions are disjoint. The desired equation for $A(1+s,1+t)$ follows by Proposition 6.3:

$$\begin{aligned}
B_n(s,t) &= \sum_{(u,v)\in \mathrm{Bur}[n]} s^{n-\max(u)} t^{n-\max(v)} \\
&= \sum_{p\in \mathrm{Sym}[n]} \sum_{x\in B(p^{-1})} \sum_{u\in \mathrm{I}(x)} s^{n-\max(u)} t^{n-\max(x)} \\
&= \sum_{p\in \mathrm{Sym}[n]} (1+s)^{\mathrm{des}(p)} (1+t)^{\mathrm{des}(p)^{-1}} \\
&= A_n(1+s,1+t).
\end{aligned}$$

A more involved computation is required for the strict case:

$$\begin{aligned}
B'_n(s,t) &= \sum_{(u,v)\in \mathrm{Bur}^{01}[n]} s^{n-\max(u)} t^{n-\max(v)} \\
&= \sum_{v\in \mathrm{Cay}[n]} \sum_{u\in \mathrm{I}^{01}(v)} s^{n-\max(u)} t^{n-\max(v)} \\
&= \sum_{v\in \mathrm{Cay}[n]} t^{n-\max(v)} (1+s)^{\mathrm{des}'(v)} && \text{(by Lemma 6.1)} \\
&= \sum_{w\in \mathrm{Cay}[n]} t^{n-\max(w)} (1+s)^{\mathrm{asc}'(w)} && (w = v^r) \\
&= \sum_{q\in \mathrm{Sym}[n]} \sum_{w\in B(q^{-1})} t^{n-\max(w)} (1+s)^{\mathrm{asc}'(w)} \\
&= \sum_{q\in \mathrm{Sym}[n]} \sum_{w\in B(q^{-1})} t^{n-\max(w)} (1+s)^{\mathrm{asc}(q)} && \text{(by Lemma 5.2)} \\
&= \sum_{q\in \mathrm{Sym}[n]} (1+s)^{\mathrm{asc}(q)} \left(\sum_{w\in \mathrm{I}(q^{-1})} t^{n-\max(w)} \right) && \text{(by Lemma 6.2)} \\
&= \sum_{q\in \mathrm{Sym}[n]} (1+s)^{\mathrm{asc}(q)} (1+t)^{\mathrm{des}(q^{-1})} && \text{(by Lemma 6.1)} \\
&= \sum_{p\in \mathrm{Sym}[n]} (1+s)^{\mathrm{des}(p)} (1+t)^{\mathrm{asc}(p^{-1})} && (p = q^r). \qquad \square
\end{aligned}$$

As discussed at the beginning of this section we have $C_n(t) = A_n(2,t)$ and $C'_n(t) = A'_n(2,t)$. Now, by Theorem 6.4, we obtain the following equations for the Caylerian polynomials:

$$C_n(1+t) = B_n(1,t) \quad \text{and} \quad C'_n(1+t) = B'_n(1,t).$$

The above equations can be seen as analogs of equation (3). The problem of finding an exponential generating function for $C_n(1+t)$ and $C'_n(1+t)$, akin to equation (4), remains open.

We shall reformulate Theorem 6.4 in terms of the joint distribution of $\max(u)$ and $\max(v)$ on Burge words (u,v), or, equivalently, the joint distribution of $\mathrm{row}(A)$ and $\mathrm{col}(A)$ on Burge matrices A. Let

$$\hat{B}_n(s,t) = \sum_{(u,v)\in \mathrm{Bur}[n]} s^{\max(u)} t^{\max(v)} = \sum_{A\in \mathrm{Mat}[n]} s^{\mathrm{row}(A)} t^{\mathrm{col}(A)}.$$

Then

$$\hat{B}_n(s,t) = (st)^n B_n\left(\frac{1}{s}, \frac{1}{t}\right) = (st)^n A_n\left(\frac{s+1}{s}, \frac{t+1}{t}\right);$$

alternatively, by setting $x = (s+1)/s$ and $y = (t+1)/t$, we get

$$\frac{A_n(x,y)}{(x-1)^n(y-1)^n} = \hat{B}_n\left(\frac{1}{x-1}, \frac{1}{y-1}\right).$$

Similarly, for the strict case we let

$$\hat{B}'_n(s,t) = \sum_{(u,v)\in \mathrm{Bur}^{01}[n]} s^{\max(u)} t^{\max(v)} = \sum_{A\in \mathrm{Mat}^{01}[n]} s^{\mathrm{row}(A)} t^{\mathrm{col}(A)}.$$

Then

$$\hat{B}'_n(s,t) = (st)^n A'_n\left(\frac{s+1}{s}, \frac{t+1}{t}\right)$$

and

$$\frac{A'_n(x,y)}{(x-1)^n(y-1)^n} = \hat{B}'_n\left(\frac{1}{x-1}, \frac{1}{y-1}\right).$$

In terms of the Caylerian polynomials, we have

$$C_n(t) = B_n(1, t-1) = (t-1)^n \hat{B}_n\left(1, \frac{1}{t-1}\right)$$

and

$$C'_n(t) = B'_n(1, t-1) = (t-1)^n \hat{B}'_n\left(1, \frac{1}{t-1}\right).$$

We have thus proved the following result.

Corollary 6.5. *For $n \geq 0$,*

$$C_n(t) = (t-1)^n \hat{B}_n\left(1, \frac{1}{t-1}\right) \quad \textit{and} \quad C'_n(t) = (t-1)^n \hat{B}'_n\left(1, \frac{1}{t-1}\right).$$

Furthermore,

$$\hat{B}_n(s,t) = (st)^n A_n\left(\frac{s+1}{s}, \frac{t+1}{t}\right) \quad \textit{and} \quad \hat{B}'_n(s,t) = (st)^n A'_n\left(\frac{s+1}{s}, \frac{t+1}{t}\right).$$

Remark 6.6. We can use Theorem 6.4 and Corollary 6.5 to reinterpret the symmetry $C'_n(t) = t^{n-1}C_n(1/t)$ in terms of the two-sided Caylerian polynomials. A simple computation leads to

$$B'_n(1, t-1) = t^{n-1} B_n\left(1, \frac{1-t}{t}\right),$$

or, alternatively,

$$\hat{B}'_n\left(1, \frac{1}{t-1}\right) = \frac{(-1)^n}{t} \hat{B}_n\left(1, \frac{t}{1-t}\right).$$

What is the combinatorial meaning of this equality in terms of Burge structures?

Figure 1 shows the polynomials $\hat{B}_n(s,t)$ and $\hat{B}'_n(s,t)$ up to $n=5$. Below we illustrate the equalities of Corollary 6.5 for $n=2$. The five matrices in Mat[2], and the corresponding contribution $s^{\mathrm{row}(A)}t^{\mathrm{col}(A)}$ to $\hat{B}_2(s,t)$, are

$$\begin{bmatrix}2\end{bmatrix} = st, \quad \begin{bmatrix}1 & 1\end{bmatrix} = st^2, \quad \begin{bmatrix}1\\1\end{bmatrix} = s^2t, \quad \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = s^2t^2, \quad \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = s^2t^2,$$

which gives

$$\hat{B}_2(s,t) = st + st^2 + s^2t + 2s^2t^2.$$

On the other hand, we have

$$\begin{aligned}
&(st)^2 A_2\left(\frac{s+1}{s}, \frac{t+1}{t}\right)\\
&\quad = (st)^2\left[\left(\frac{s+1}{s}\right)^{\mathrm{des}(12)}\left(\frac{t+1}{t}\right)^{\mathrm{des}(12)} + \left(\frac{s+1}{s}\right)^{\mathrm{des}(21)}\left(\frac{t+1}{t}\right)^{\mathrm{des}(21)}\right]\\
&\quad = (st)^2\left[1 + \left(\frac{s+1}{s}\right)\left(\frac{t+1}{t}\right)\right]\\
&\quad = st\left[st + st + s + t + 1\right]\\
&\quad = 2s^2t^2 + s^2t + st^2 + st\\
&\quad = \hat{B}_2(s,t),
\end{aligned}$$

as expected. The analogous computation for binary matrices and the strict two-sided Caylerian polynomial gives

$$\hat{B}'_2(s,t) = st^2 + s^2t + 2s^2t^2$$

and

$$\begin{aligned}
&(st)^2 A'_2\left(\frac{s+1}{s}, \frac{t+1}{t}\right)\\
&\quad = (st)^2\left[\left(\frac{s+1}{s}\right)^{\mathrm{des}(12)}\left(\frac{t+1}{t}\right)^{\mathrm{asc}(12)} + \left(\frac{s+1}{s}\right)^{\mathrm{des}(21)}\left(\frac{t+1}{t}\right)^{\mathrm{asc}(21)}\right]\\
&\quad = (st)^2\left[\frac{t+1}{t} + \frac{s+1}{s}\right]\\
&\quad = s^2t^2 + s^2t + s^2t^2 + st^2\\
&\quad = \hat{B}'_2(s,t).
\end{aligned}$$

Finally, it is easy to check that

$$\begin{aligned}
(t-1)^2\hat{B}_2\left(1, \frac{1}{t-1}\right) &= (t-1)^2\left[\frac{1}{t-1}\left(1 + 1 + \frac{1}{t-1} + \frac{2}{t-1}\right)\right]\\
&= (t-1)^2\left[\frac{2t-2+3}{(t-1)^2}\right]\\
&= 2t+1\\
&= C_2(t),
\end{aligned}$$

and, similarly,

$$(t-1)^2\hat{B}'_2\left(1, \frac{1}{t-1}\right) = t + 2 = C'_2(t).$$

$$[1], \quad \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}, \quad \begin{bmatrix}1 & 2 & 1\\ 2 & 8 & 6\\ 1 & 6 & 6\end{bmatrix}, \quad \begin{bmatrix}1 & 3 & 3 & 1\\ 3 & 19 & 30 & 14\\ 3 & 30 & 63 & 36\\ 1 & 14 & 36 & 24\end{bmatrix}, \quad \begin{bmatrix}1 & 4 & 6 & 4 & 1\\ 4 & 36 & 90 & 88 & 30\\ 6 & 90 & 306 & 372 & 150\\ 4 & 88 & 372 & 528 & 240\\ 1 & 30 & 150 & 240 & 120\end{bmatrix}$$

$$[1], \quad \begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 1\\ 0 & 4 & 6\\ 1 & 6 & 6\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 1 & 12 & 14\\ 0 & 12 & 45 & 36\\ 1 & 14 & 36 & 24\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 0 & 0 & 1\\ 0 & 0 & 6 & 32 & 30\\ 0 & 6 & 90 & 228 & 150\\ 0 & 32 & 228 & 432 & 240\\ 1 & 30 & 150 & 240 & 120\end{bmatrix}.$$

Figure 1: Coefficient matrices of $\hat{B}_n(s,t)$, in the top row, and $\hat{B}'_n(s,t)$, in the bottom row, for $n = 1, 2, 3, 4, 5$. The (i, j)-th entry is the coefficient of $s^i t^j$.

Remark 6.7. A well known conjecture by Gessel asserts the γ-nonnegativity of the two-sided Eulerian polynomial. That is, for $n \geq 0$ there exist nonnegative integers $\gamma_{n,i,j}$, with $0 \leq i-1, j, j+2i \leq n-1$, such that

$$A_n(s,t) = \sum_{i,j} \gamma_{n,i,j} (st)^i (s+t)^j (1+st)^{n-1-j-2i}.$$

For example, for $n = 5$ we have

$$A_5(s,t) = (1+st)^4 + 16(st)(1+st)^2 + 16(st)^2 + 6(st)(s+t)(1+st).$$

This conjecture has recently been proved by Lin [17] via algebraic means. However, the γ-nonnegativity of the classical Eulerian polynomial $A_n(t)$ has a beautiful combinatorial proof by *valley-hopping* [21] and it would be interesting to find a similar combinatorial proof for the two-sided case. In Theorem 6.4, we showed that

$$\begin{aligned} B_n(s,t) &= A_n(1+s, 1+t) \\ &= \sum_{i,j} \gamma_{n,i,j} (st+s+t+1)^i (s+t+2)^j (st+s+t+2)^{n-1-j-2i}. \end{aligned}$$

Therefore, a combinatorial proof of the γ-nonnegativity could potentially be obtained by working with $B_n(s,t)$, in terms of Burge words or Burge matrices. For instance, one could try to partition the set of Burge words (matrices) in classes whose contribution to $B_n(s,t)$ is $(st+s+t+1)^i(s+t+2)^j(st+s+t+2)^{n-1-j-2i}$, where possibly i and j are some suitable parameters on $\mathrm{Bur}[n]$ (or $\mathrm{Mat}[n]$). This way, the coefficient $\gamma_{n,i,j}$ would count the number of such classes (with parameters i and j), and thus would be nonnegative. In fact, it turns out that looking at binary matrices (or at the corresponding set of biwords) would be enough to prove the nonnegativity of the coefficients $\gamma_{n,i,j}$. Indeed, by Theorem 6.4,

$$B'_n(s,t) = A'_n(1+s, 1+t).$$

Let $\gamma'_{n,i,j}$ be such that

$$A'_n(s,t) = \sum_{i,j} \gamma'_{n,i,j} (st)^i (s+t)^j (1+st)^{n-1-j-2i}.$$

Then

$$\begin{aligned}
A'_n(s,t) &= \sum_{p\in\mathrm{Sym}[n]} s^{\mathrm{des}(p)} t^{\mathrm{asc}(p)} \\
&= \sum_{p\in\mathrm{Sym}[n]} s^{\mathrm{des}(p)} t^{n-1-\mathrm{des}(p)} \\
&= t^{n-1} A_n\Big(s, \frac{1}{t}\Big) \\
&= t^{n-1} \sum_{i,j} \gamma_{n,i,j} \Big(\frac{s}{t}\Big)^i \Big(s+\frac{1}{t}\Big)^j \Big(1+\frac{s}{t}\Big)^{n-1-j-2i} \\
&= \sum_{i,j} \gamma_{n,i,j} s^i t^{n-1-i-j-n+1+j+2i} (st+1)^j (t+s)^{n-1-j-2i} \\
&= \sum_{i,j} \gamma_{n,i,j} (st)^i (s+t)^{n-1-j-2i} (1+st)^j.
\end{aligned}$$

Finally,

$$\sum_{i,j} \gamma'_{n,i,j} (st)^i (s+t)^j (1+st)^{n-1-j-2i} = \sum_{i,j} \gamma_{n,i,j} (st)^i (s+t)^{n-1-j-2i} (1+st)^j$$

from which

$$\gamma'_{n,i,j} = \gamma_{n,i,n-1-j-2i}.$$

follows. Ultimately, the nonnegativity of the coefficients $\gamma_{n,i,j}$ is expressed in terms of the strict two-sided Eulerian polynomial as

$$B'_n(s,t) = \sum_{i,\ell} \gamma_{n,i,\ell} (st+s+t+1)^i (s+t+2)^{n-1-\ell-2i} (st+s+t+2)^{\ell},$$

where $\ell = n-1-j-2i$.

7 Stanley's v-compatible maps

Let us now return to the Eulerian polynomial $A_n(t)$. Recall Carlitz identity:

$$\frac{tA_n(t)}{(1-t)^{n+1}} = \sum_{m\geq 1} m^n t^m. \tag{5}$$

We wish to derive an analogous result for Cayley permutations. Let us first sketch the proof presented by Stanley [25, Proposition 1.4.4] for the classical case.

Definition 7.1 (Stanley [25])**.** Let $v \in \mathrm{Sym}[n]$. A map $x : [n] \to \mathbb{N}$ is *v-compatible* if it satisfies the following two conditions:

(i) $x(v_1) \leq x(v_2) \leq \cdots \leq x(v_n)$;

(ii) $x(v_i) < x(v_{i+1})$ for each $i \in A(v)$.

We remark that to better fit our presentation the roles of ascents and descents, as well as the inequalities in (i) and (ii), are reversed with respect to Stanley's exposition.

For $m \geq 0$, denote by $\mathrm{Comp}_m(v)$ the set of v-compatible maps $x : [n] \to [m]$. Using a direct combinatorial argument, Stanley proves that

$$|\mathrm{Comp}_m(v)| = \left(\!\!\binom{m - \mathrm{asc}(v)}{n}\!\!\right),$$

where $\left(\!\binom{a}{b}\!\right) = \binom{a+b-1}{b}$ is the number of multisets of cardinality b over a set of size a. Stanley proceeds by showing that for each map $x : [n] \to [m]$ there is a unique permutation v of $[n]$ such that x is v-compatible. As a result, the disjoint union

$$[m]^{[n]} = \bigcup_{v \in \mathrm{Sym}[n]} \mathrm{Comp}_m(v),$$

holds, which leads to the equation

$$m^n = \sum_{v \in \mathrm{Sym}[n]} |\mathrm{Comp}_m(v)| = \sum_{v \in \mathrm{Sym}[n]} \left(\!\!\binom{m - \mathrm{asc}(v)}{n}\!\!\right). \tag{6}$$

Now, the generating function for multisets over $[n]$, according to size, is

$$\sum_{k \geq 0} \left(\!\!\binom{n}{k}\!\!\right) t^k = (1 + t + t^2 + \cdots)^n = \frac{1}{(1-t)^n}. \tag{7}$$

Moreover,

$$\begin{aligned}
\frac{t^{1+\mathrm{asc}(v)}}{(1-t)^{n+1}} &= \sum_{k \geq 0} \left(\!\!\binom{n+1}{k}\!\!\right) t^{1+\mathrm{asc}(v)+k} \\
&= \sum_{k \geq 0} \binom{n+k}{k} t^{1+\mathrm{asc}(v)+k} \\
&= \sum_{k \geq 0} \binom{n+k}{n} t^{1+\mathrm{asc}(v)+k} \\
&= \sum_{m \geq 1} \binom{n + m - \mathrm{asc}(v) - 1}{n} t^m \qquad (m = 1 + \mathrm{asc}(v) + k) \\
&= \sum_{m \geq 1} \left(\!\!\binom{m - \mathrm{asc}(v)}{n}\!\!\right) t^m \\
&= \sum_{m \geq 1} |\mathrm{Comp}_m(v)| t^m
\end{aligned}$$

and Carlitz identity follows:

$$\begin{aligned}
\frac{tA_n(t)}{(1-t)^{n+1}} &= \sum_{v \in \mathrm{Sym}[n]} \frac{t^{1+\mathrm{asc}(v)}}{(1-t)^{n+1}} \\
&= \sum_{m \geq 1} \sum_{v \in \mathrm{Sym}[n]} |\mathrm{Comp}_m(v)| t^m \\
&= \sum_{m \geq 1} m^n t^m.
\end{aligned}$$

We wish to use the Burge transpose to generalize the notion of v-compatibility and obtain similar equations for the Caylerian polynomials. The relation between v-compatible maps and the Burge transpose is highlighted in the next proposition.

Proposition 7.2. *For $v \in \mathrm{Sym}[n]$ and $x : [n] \to [m]$,*

$$x \in \mathrm{Comp}_m(v) \iff x \circ v \in \mathfrak{I}_m(v).$$

In particular,

$$\mathrm{Comp}_m(v) = \Gamma\big(\mathfrak{I}_m(v) \times \{v\}\big) \quad \textit{and} \quad |\mathrm{Comp}_m(v)| = |\mathfrak{I}_m(v)|.$$

Proof. It is easy to see that $x(v_i) < x(v_{i+1})$ for each $i \in A(v)$ if and only if $D\big(x(v_1)\cdots x(v_n)\big) \subseteq D(v)$, i.e. $x(v_1)\cdots x(v_n) \in \mathfrak{I}_m(v)$. This proves the first statement. Let us now prove the equality $\mathrm{Comp}_m(v) = \Gamma\left(\mathfrak{I}_m(v) \times \{v\}\right)$. If $x \in \mathrm{Comp}_m(v)$, then $\big(x(v_1)\cdots x(v_n), v\big) \in \mathfrak{Bur}^!_m$ and

$$\begin{pmatrix} x(v_1) & \dots & x(v_n) \\ v_1 & \dots & v_n \end{pmatrix}^T = \begin{pmatrix} 1 & \dots & n \\ x(1) & \dots & x(n) \end{pmatrix} = \begin{pmatrix} \mathrm{id} \\ x \end{pmatrix}.$$

In particular, $x = \Gamma\big(x(v_1)\cdots x(v_n), v\big)$, as desired. Conversely, let $x = \Gamma(u, v)$ for some $u \in \mathfrak{I}_m(v)$. Then

$$\begin{pmatrix} u_1 & \dots & u_n \\ v_1 & \dots & v_n \end{pmatrix}^T = \begin{pmatrix} 1 & \dots & n \\ x(1) & \dots & x(n) \end{pmatrix}.$$

Since the columns of the biword on the left-hand side are obtained by flipping the right-hand side upside down we have

$$\begin{pmatrix} x(i) \\ i \end{pmatrix} = \begin{pmatrix} x(v_j) \\ v_j \end{pmatrix} = \begin{pmatrix} u_j \\ v_j \end{pmatrix} \quad \text{and} \quad u_j = x(v_j),$$

for each $i \in [n]$, where $j = v^{-1}(i)$. In particular, $\big(x(v_1)\cdots x(v_n), v\big) \in \mathrm{Bur}^!_m$ and $x \in \mathrm{Comp}_m(v)$, which gives the desired equality. Finally, $|\mathrm{Comp}_m(v)| = |\mathfrak{I}_m(v)|$ is an immediate consequence of Γ being injective on $\mathfrak{I}_m(v) \times \{v\}$. □

Stanley showed that for each map $x : [n] \to [m]$ there is a unique permutation $v \in \mathrm{Sym}[n]$ such that x is v-compatible. Let us rederive this result using our machinery. By Proposition 7.2, x is v-compatible if and only if $x = \Gamma(u, v)$, where $u = \mathrm{sort}(x)$ and $(u, v) \in \mathfrak{Bur}_m[n]$. Thus, if x is v-compatible, we have $(u, v)^T = (\mathrm{id}, x)$, or, equivalently, $(\mathrm{id}, x)^T = (\mathrm{sort}(x), v)$ and v is uniquely determined as

$$v = \Gamma(\mathrm{id}, x).$$

Let $\mathrm{Comp}(v) = \bigcup_{m \geq 0} \mathrm{Comp}_m(v)$.

Corollary 7.3. *The Fishburn basis of a permutation v is the set of v-compatible maps that are Cayley permutations:*

$$B(v) = \mathrm{Cay} \cap \mathrm{Comp}(v).$$

Proof. Assume $v \in \mathrm{Sym}$ and let $\mathfrak{I}(v) = \bigcup_{m\geq 1} \mathfrak{I}_m(v)$. By definition,

$$\mathrm{Comp}(v) = \Gamma\big(\mathfrak{I}(v) \times \{v\}\big) \quad \text{and} \quad B(v) = \Gamma\big(\mathrm{I}(v) \times \{v\}\big).$$

Clearly, $\mathrm{I}(v) = \mathfrak{I}(v) \cap \mathrm{Cay}$ and thus

$$\begin{aligned}
\mathrm{Comp}(v) \cap \mathrm{Cay} &= \Gamma\Big(\mathfrak{I}(v) \times \{v\}\Big) \cap \mathrm{Cay} \\
&= \Gamma\Big(\big(\mathfrak{I}(v) \cap \mathrm{Cay}\big) \times \{v\}\Big) \\
&= \Gamma\Big(\mathrm{I}(v) \times \{v\}\Big) \\
&= B(v).
\end{aligned}$$

□

Now, it is not immediately clear how to extend Stanley's notion of v-compatibilty to Cayley permutations. In fact, if $v \in \mathrm{Cay}[n]$ and $x : [n] \to [m]$, then the presence of repeated entries in v makes the inequalities (i) and (ii) in Definition 7.1 meaningless. On the other hand, we showed that the set of v-compatible maps is characterized by $\mathrm{Comp}_m(v) = \Gamma\left(\mathfrak{I}_m(v) \times \{v\}\right)$, and $\mathfrak{I}_m(v)$ is defined for any Cayley permutation v. We shall generalize the definition of *v-compatible maps* accordingly by setting, for any $v \in \mathrm{Cay}$ and $m \geq 0$,

$$\mathrm{Comp}_m(v) = \Gamma\big(\mathfrak{I}_m(v) \times \{v\}\big).$$

Furthermore, we define the set of *strictly v-compatible maps* by

$$\mathrm{Comp}^{01}_m(v) = \Gamma\big(\mathfrak{I}^{01}_m(v) \times \{v\}\big).$$

By Proposition 7.2, our notion of v-compatibilty matches Stanley's definition when $v \in \mathrm{Sym}$. The same goes for strict v-compatibility: if $v \in \mathrm{Sym}$, then $D(v) = D'(v)$ and $u \in \mathfrak{I}_m$ if and only if $u \in \mathfrak{I}^{01}_m$.

Example 7.4. Let us give an example to illustrate the notion of v-compatibility. Recall that $\mathfrak{I}_m(v) = \{u \in \mathfrak{I}_m[n] : D(u) \subseteq D(v)\}$, where n is the length of v. To compute $\mathfrak{I}_m(v)$, consider a biword

$$\binom{u}{v} = \begin{pmatrix} u(1) & \dots & u(n) \\ v(1) & \dots & v(n) \end{pmatrix},$$

where $u : [n] \to [m]$ is weakly increasing. Note that $D(u) \subseteq D(v)$ if and only if $A'(v) \subseteq A'(u)$; that is, among all the weakly increasing maps, to obtain $\mathfrak{I}_m(v)$ we will pick only those u where $u(i+1) > u(i)$ for each $i \in A'(v)$. Similarly, the set $\mathfrak{I}^{01}_m(v)$ consists of the weakly increasing maps $u : [n] \to [m]$ that satisfy the stricter requirement that $u(i+1) > u(i)$ for each $i \in A(v)$. For instance, let $v = 331412$ and $m = 4$. Here $n = 6$, $D(v) = \{1, 2, 4\}$ and $D'(v) = \{2, 4\}$. Thus,

$$\mathfrak{I}_4(v) = \{111223, 111224, 111234, 111334, 112334, 122334, 222334\}$$

and $\mathfrak{I}^{01}_4(v) = \{122334\}$. The sets of v-compatible and strictly v-compatible maps are obtained by applying Γ to the biwords in $\mathrm{Comp}_m(v) \times \{v\}$ and $\mathrm{Comp}^{01}_m(v) \times \{v\}$, respectively. We get

$$\mathrm{Comp}_4(v) = \{213112, 214112, 314112, 314113, 324113, 324213, 324223\},$$

where for instance $214112 = \Gamma(111224, 331412)$ since $\begin{pmatrix} 111224 \\ 331412 \end{pmatrix}^T = \begin{pmatrix} 112334 \\ 214112 \end{pmatrix}$. Also,

$$\mathrm{Comp}^{01}_4(v) = \{\Gamma(122334, v)\} = \{324213\}.$$

Next we wish to count compatible and strictly compatible maps.

Proposition 7.5. *Let $v \in \mathrm{Cay}[n]$ and $m \geq 0$. Then*

$$|\mathrm{Comp}_m(v)| = \left(\!\!\binom{m - \mathrm{asc}'(v)}{n}\!\!\right) \quad \text{and} \quad |\mathrm{Comp}^{01}_m(v)| = \left(\!\!\binom{m - \mathrm{asc}(v)}{n}\!\!\right).$$

Proof. We shall focus on the first identity. Let $v \in \mathrm{Cay}[n]$ and $a = \mathrm{asc}'(v)$. Recall that $\mathrm{Comp}_m(v) = \Gamma\big(\mathfrak{I}_m(v) \times \{v\}\big)$. Since Γ is an injective map on $\mathfrak{I}_m(v) \times \{v\}$, it suffices to show that $|\mathfrak{I}_m(v)| = \left(\!\binom{m-a}{n}\!\right)$. Let $Y = Y_m(v)$ be the set of strings with letters "$\star$" and "$|$" that contain exactly n stars and $m - a - 1$ bars. Clearly,

$$|Y| = \binom{n + m - a - 1}{n} = \left(\!\!\binom{m - a}{n}\!\!\right).$$

We shall construct a bijection between $\mathfrak{I}_m(v)$ and Y. To each string $y \in Y$, we associate a weakly increasing map u of length n by setting, for $i \in [n]$,

$$u(i) = a_i + b_i + 1,$$

where $a_i = |\{j \in [i-1] : v(j) < v(j+1)\}|$ is the number of strict ascents preceding the ith entry of v, and b_i is the number of bars preceding the ith star of y.

By definition, u is weakly increasing and

$$\begin{aligned} \max(u) = u(n) &= a_n + b_n + 1 \\ &\leq a + (m - a - 1) + 1 = m. \end{aligned}$$

Thus $u \in \mathfrak{I}_m$. Furthermore, if $i \in A'(v)$, then $a_{i+1} = a_i + 1$ and hence

$$\begin{aligned} u(i+1) &= a_{i+1} + b_{i+1} + 1 \\ &= a_i + 1 + b_{i+1} + 1 \\ &\geq a_i + 1 + b_i + 1 = u(i) + 1. \end{aligned}$$

In other words, $A'(v) \subseteq A'(u)$. This is of course equivalent to $D(u) \subseteq D(v)$, i.e. $u \in \mathfrak{I}_m(v)$, as wanted. On the other hand, every $u \in \mathfrak{I}_m(v)$ gives rise to a string $y \in Y$, where the ith star is preceded by $u(i) - a_i - 1$ bars. Therefore, the correspondence between Y and $\mathrm{Comp}_m(v)$ is bijective, which completes the proof of the first equality. We omit the proof of the second equality since it is obtained by a straightforward modification of the proof just given. □

Let us give an example illustrating the bijection between $Y_m(v)$ and $\mathrm{Comp}_m(v)$ in the proof of Proposition 7.5. Let $m = 9$, $n = 6$ and $v = 221312 \in \mathrm{Cay}[n]$. Note that $D(v) = \{1, 2, 4\}$, $A'(v) = \{3, 5\}$ and $a_1 a_2 \dots a_n = 000112$. Consider the string

$$y \;=\; |\star\star|\star||\star\star|\star|.$$

We have $b_1 b_2 \dots b_n = 112445$ and calculating $u(i) = a_i + b_i + 1$, for each $i \in [n]$, we find that the map $u : [n] \to [m]$ associated with y is $u = 223668$. Note that $D(u) = \{1, 4\} \subseteq D(v) = \{1, 2, 4\}$ and hence $u \in \mathfrak{I}_m(v)$.

Theorem 7.6. *For each $n \geq 0$,*

1. $\dfrac{tC_n(t)}{(1-t)^{n+1}} = \sum_{m\geq 1} |\mathfrak{Bur}_m[n]| t^m$;

2. $\dfrac{tC'_n(t)}{(1-t)^{n+1}} = \sum_{m\geq 1} |\mathfrak{Bur}^{01}_m[n]| t^m$;

3. $\dfrac{tA_n(t)}{(1-t)^{n+1}} = \sum_{m\geq 1} |\mathfrak{Bur}^{!}_m[n]| t^m$.

Proof. Let $a \in \mathbb{N}$. Using equation (7) we obtain

$$\frac{t^{a+1}}{(1-t)^{n+1}} = \sum_{k\geq 0} \left(\!\!\binom{n+1}{k}\!\!\right) t^k t^{a+1} = \sum_{m\geq 1} \left(\!\!\binom{m-a}{n}\!\!\right) t^m,$$

where the last equality follows by setting $m = k+a+1$ and noting that $\left(\!\binom{n+1}{k}\!\right) = \binom{n+k}{n} = \left(\!\binom{m-a}{n}\!\right)$. Now, by using Proposition 7.5 and setting $a = \mathrm{asc}'(v)$ in the above equation, we get

$$\begin{aligned}
\frac{tC_n(t)}{(1-t)^{n+1}} &= \sum_{v\in \mathrm{Cay}[n]} \frac{t^{\mathrm{asc}'(v)+1}}{(1-t)^{n+1}} \\
&= \sum_{v\in \mathrm{Cay}[n]} \sum_{m\geq 1} \left(\!\!\binom{m-\mathrm{asc}'(v)}{n}\!\!\right) t^m \\
&= \sum_{m\geq 1} \left(\sum_{v\in \mathrm{Cay}[n]} |\mathrm{Comp}_m(v)| \right) t^m \\
&= \sum_{m\geq 1} \left(\sum_{v\in \mathrm{Cay}[n]} |\mathfrak{I}_m(v)| \right) t^m \\
&= \sum_{m\geq 1} |\mathfrak{Bur}_m[n]| t^m,
\end{aligned}$$

where the last equality follows from the disjoint union

$$\mathfrak{Bur}_m[n] = \bigcup_{v\in \mathrm{Cay}[n]} \mathfrak{I}_m(v) \times \{v\}.$$

The equation for $C'_n(t)$ can analogously be obtained by setting $a = \mathrm{asc}(v)$ instead of $a = \mathrm{asc}'(v)$, and the equation for $A_n(t)$ follows in a similar manner. □

8 Cayley permutations with a prescribed ascent set

Let $n \geq 1$ and consider a subset $S \subseteq [n-1]$ of size r, say $S = \{s_1, \ldots, s_r\}$, with $s_1 < s_2 < \cdots < s_r$. Define

$$\alpha_n(S) = |\{v \in \mathrm{Sym}[n] : A(v) \subseteq S\}|.$$

It is well known (see e.g. Stanley [25], Proposition 1.4.1) that $\alpha_n(S)$ is a polynomial in n given by the multinomial coefficient

$$\begin{aligned}\alpha_n(S) &= \binom{n}{s_1, s_2 - s_1, s_3 - s_2, \ldots, n - s_r} \\ &= \frac{n!}{s_1!(s_2 - s_1)!(s_3 - s_2)! \cdots (n - s_r)!}.\end{aligned}$$

Indeed, any permutation $v \in \mathrm{Sym}[n]$ such that $A(v) \subseteq S$ is obtained by first choosing s_1 elements $v_1 > v_2 > \cdots > v_{s_1}$, then $s_2 - s_1$ elements $v_{s_1+1} > v_{s_1+2} > \cdots > v_{s_2}$, and so on. In other words, $\alpha_n(S)$ equals the number of ballots on $[n]$ whose block sizes are given by s_1, $s_2 - s_1$, $\ldots$, $n - s_r$.

Transitioning from permutations to Cayley permutations—the theme of this article—we define

$$\begin{aligned}\kappa_n(S) &= |\{v \in \mathrm{Cay}[n] : A(v) \subseteq S\}|; \\ \kappa'_n(S) &= |\{v \in \mathrm{Cay}[n] : A'(v) \subseteq S\}|.\end{aligned}$$

We wish to show that Burge matrices and Burge words provide an answer to the following natural question:

What is the combinatorial structure we should replace ballots with, so that $\kappa_n(S)$ and $\kappa'_n(S)$ equals the number of structures on $[n]$ with block sizes s_1, $s_2 - s_1$, $\ldots$, $n - s_r$?

Given $v \in \mathrm{Cay}[n]$, consider the biword

$$\binom{\eta(S)}{v} = \begin{pmatrix} 1 \ldots 1 & 2 \ldots 2 & \ldots & r+1 \ldots r+1 \\ v_1 \ldots v_{s_1} & v_{s_1+1} \ldots v_{s_2} & \ldots & v_{s_r+1} \ldots v_n \end{pmatrix},$$

where

$$\eta(S) = 1^{s_1} 2^{s_2 - s_1} 3^{s_3 - s_2} \cdots r^{s_r - s_{r-1}} (r+1)^{n - s_r}$$

is the unique weakly increasing Cayley permutation of length n whose descent set is $D\big(\eta(S)\big) = [n-1] \setminus S$. We will refer to the entries in v that lie below the integer i as the ith *block* of v in the ballot induced by S.

Lemma 8.1. *For any $v \in \mathrm{Cay}[n]$,*

$$A'(v) \subseteq S \iff \binom{\eta(S)}{v} \in \mathrm{Bur} \quad \textit{and} \quad A(v) \subseteq S \iff \binom{\eta(S)}{v} \in \mathrm{Bur}^{01}.$$

Proof. To prove the first statement, observe that $A'(v) \subseteq S$ if and only if $[n-1] \setminus S \subseteq D(v)$, which is equivalent to $(\eta(S), v) \in \mathrm{Bur}$ since $D(\eta(S)) = [n-1] \setminus S$. In other words, we have just showed that $A'(v) \subseteq S$ if and only if each block of v in the ballot induced by S is weakly decreasing, which in turn is the same as $(\eta(S), v) \in \mathrm{Bur}$. The second statement can be proved in a similar fashion. In this case, $A(v) \subseteq S$ if and only if each block of v is strictly decreasing, that is, $D(\eta(S)) \subseteq D'(v)$ and $(\eta(S), v) \in \mathrm{Bur}^{01}$. □

Now, define

$$\mathrm{Bur}(S) = \left\{ \binom{\eta(S)}{v} : v \in \mathrm{Cay} \right\} \cap \mathrm{Bur}.$$

Similarly, define $\mathrm{Bur}^{01}(S) = \mathrm{Bur}(S) \cap \mathrm{Bur}^{01}$ and $\mathrm{Bur}^{!}(S) = \mathrm{Bur}(S) \cap \mathrm{Bur}^{!}$. It is straightforward to describe the sets of Burge matrices corresponding to $\mathrm{Bur}(S)$, $\mathrm{Bur}^{01}(S)$ and $\mathrm{Bur}^{!}(S)$: Recall that, if $A = \varphi^{-1}(u, v)$ is the Burge matrix corresponding to $(u, v) \in \mathrm{Bur}$, then A is the $\max(u) \times \max(v)$ matrix whose entry a_{ij} is equal to the number of columns $\binom{i}{j}$ contained in (u, v). Now, if $u = \eta(S)$, then the first row of A sums to s_1, the second row sums to $s_2 - s_1$, and so on up the $(r+1)$th (and last) row, whose sum is $n - s_r$. To express this in a more compact form, let $\mathbf{1}$ denote the all ones vector so that $A \cdot \mathbf{1}$ is the *row sum vector* of A, i.e. the vector whose ith entry is equal to the sum of the ith row of A. We then define

$$\mathrm{Mat}(S) = \{A \in \mathrm{Mat} : A \cdot \mathbf{1} = \Delta(S)\},$$

where $\Delta(S) = (s_1, s_2 - s_1, \ldots, s_r - s_{r-1}, n - s_r)$. We further define

$$\mathrm{Mat}^{01}(S) = \mathrm{Mat}(S) \cap \mathrm{Bur}^{01}(S);$$
$$\mathrm{Mat}^{!}(S) = \mathrm{Mat}(S) \cap \mathrm{Bur}^{!}(S).$$

In analogy with Proposition 3.1, we have

$$\varphi\big(\mathrm{Mat}(S)\big) = \mathrm{Bur}(S); \quad \varphi\big(\mathrm{Mat}^{01}(S)\big) = \mathrm{Bur}^{01}(S); \quad \varphi\big(\mathrm{Mat}^{!}(S)\big) = \mathrm{Bur}^{!}(S).$$

Let $\mathrm{Bur}(S)[n] = \mathrm{Bur}(S) \cap \mathrm{Bur}[n]$, $\mathrm{Mat}(S)[n] = \mathrm{Mat}(S) \cap \mathrm{Mat}[n]$, etc. The following proposition is an immediate consequence of Lemma 8.1.

Theorem 8.2. *For each $n \geq 1$ and $S \subseteq [n-1]$,*

1. $\kappa'_n(S) = |\mathrm{Bur}(S)[n]| = |\mathrm{Mat}(S)[n]|$;
2. $\kappa_n(S) = |\mathrm{Bur}^{01}(S)[n]| = |\mathrm{Mat}^{01}(S)[n]|$;
3. $\alpha_n(S) = |\mathrm{Bur}^{!}(S)[n]| = |\mathrm{Mat}^{!}(S)[n]|$.

Arguably, we have now established that Burge matrices of size n with fixed row sums, i.e. $\mathrm{Mat}(S)[n]$ and $\mathrm{Mat}^{01}(S)[n]$, and Burge words of size n whose top row is equal to $\eta(S)$, i.e. $\mathrm{Bur}(S)[n]$ and $\mathrm{Bur}^{01}(S)[n]$, are combinatorial structures that generalize ballots so that $\kappa'_n(S)$ and $\kappa_n(S)$ count the number of structures on $[n]$ whose "block sizes" are given by $S \subseteq [n-1]$.

Two expressions relating $\kappa'_n(S)$ and $\kappa_n(S)$ to permutations follow from Lemma 5.2:

$$\kappa'_n(S) = \sum_{\substack{v \in \mathrm{Sym}[n] \\ A(v) \subseteq S}} 2^{\mathrm{des}(v^{-1})};$$

$$\kappa_n(S) = \sum_{\substack{v \in \mathrm{Sym}[n] \\ D(v) \subseteq S}} 2^{\mathrm{des}(v^{-1})}.$$

Indeed, for any $w \in \mathrm{Sym}$ and Cayley permutation $y \in B(w)$ we have $D(y) = D(w^{-1})$ and $A'(y) = A(w^{-1})$. Furthermore, $|B(w)| = 2^{\mathrm{des}(w)}$, $\mathrm{Cay}[n] = \bigcup_{w \in \mathrm{Sym}[n]} B(w)$ and the desired identities follow by setting $v = w^{-1}$.

9 Future work

In Theorem 6.4, we showed that $A_n(1+s, 1+t) = B_n(s,t)$. In other words, by marking descents and inverse descents of permutations with $1+s$ and $1+t$, respectively, we obtain Burge matrices A with weight $s^{n-\mathrm{row}(A)}t^{n-\mathrm{col}(A)}$; or, equivalently, Burge words (u,v) with weight $s^{n-\max(u)}t^{n-\max(v)}$. Going one step further, what structure arises from $B_n(1+s, 1+t)$?

Working with generating functions it is easy to show that

$$A_n(-1) = \begin{cases} (-1)^{(n-1)/2}E_n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

where E_n is the nth Euler number. Can something interesting be said about the Caylerian polynomials evaluated at -1? Due to the symmetry $C'_n(t) = t^{n-1}C_n(1/t)$, we have $C'_n(-1) = (-1)^{n-1}C_n(-1)$. The sequence $C_n(-1)$, $n \geq 0$, begins

$$1, 1, -1, -3, 11, 45, -301, -1475, 14755, 85629.$$

As mentioned in Section 8, the number $\alpha_n(S)$ of permutations of size n whose ascent set is contained in S is given by a polynomial in n. The corresponding property is false for $\kappa'_n(S)$. In particular, $\kappa'_n(\emptyset) = 2^{n-1}$ for $n \geq 1$. We have, however, gathered some numerical evidence that suggests that $\kappa_n(S)$ is a polynomial and we make the following conjecture (in which $\max(\emptyset) = 0$ by convention).

Conjecture 9.1. *Let S be any finite subset of positive integers and let $k = \max(S)$. Then the sequence $n \mapsto \kappa_{n+k}(S)$ is a a polynomial in n of degree k.*

For instance, the polynomials for $S \subseteq \{1,2,3\}$ are conjectured to be

$$\kappa_n(\emptyset) = 1; \quad \kappa_{n+1}(\{1\}) = 1 + 2n; \quad \kappa_{n+2}(\{2\}) = 1 + 2n + 2n^2;$$

$$\kappa_{n+2}(\{3\}) = 1 + \frac{8}{3}n + 2n^2 + \frac{4}{3}n^3; \quad \kappa_{n+2}(\{1,2\}) = 3 + 6n + 4n^2;$$

$$\kappa_{n+3}(\{1,3\}) = \kappa_{n+3}(\{2,3\}) = 5 + 12n + 10n^2 + 4n^3;$$

$$\kappa_{n+3}(\{1,2,3\}) = 13 + 30n + 24n^2 + 8n^3.$$

There is a nice determinant formula for the number,

$$\beta_n(S) = |\{v \in \mathrm{Sym}[n] : A(v) = S\}|,$$

of permutations of $[n]$ whose ascent set is equal to S, namely

$$\beta_n(S) = n! \det\big[1/(s_j - s_{i-1})\big],$$

where $S = \{s_1, s_2, \dots, s_k\}$ and $(i,j) \in [k+1] \times [k+1]$. See Stanley [25, p. 229] for further details. Is there a nice formula for the number of Cayley permutations whose ascent set is S?

One way to generalize the number $\alpha_n(S)$ to a polynomial in t is to let

$$\alpha_n(S;t) = \sum_{\substack{v \in \mathrm{Sym}[n] \\ A(v) \subseteq S}} t^{\mathrm{des}(v^{-1})}.$$

for $S \subseteq [n-1]$. Then $\alpha_n(S;1) = \alpha_n(S)$ and $\kappa_n(S) = \alpha_n(S;2)$. What can we more generally say about $\alpha_n(S;t)$?

In this paper, we have obtained a variety of results linking the (two-sided) Caylerian polynomials, as well as the coefficients $\kappa_n(S)$ and $\kappa'_n(S)$, to certain sets of Burge words and Burge matrices. An in-depth study of the enumerative properties of the latter structures will be carried out in a forthcoming paper.

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