PERMUTATIONS WITH FEW INVERSIONS

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Abstract. A curious generating function $S_0(x)$ for permutations of $[n]$ with exactly n inversions is presented. Moreover, $(xC(x))^i S_0(x)$ is shown to be the generating function for permutations of $[n]$ with exactly $n-i$ inversions, where $C(x)$ is the generating function for the Catalan numbers.

1. Introduction

The famous triangle of Mahonian numbers starts as follows:

1	0	0	0	0	0	0	0	0	0	⋯
1	0	0	0	0	0	0	0	0	0	⋯
1	1	0	0	0	0	0	0	0	0	⋯
1	2	2	1	0	0	0	0	0	0	⋯
1	3	5	6	5	3	1	0	0	0	⋯
1	4	9	15	20	22	20	15	9	4	⋯
1	5	14	29	49	71	90	101	101	90	⋯
1	6	20	49	98	169	259	359	455	531	⋯
1	7	27	76	174	343	602	961	1415	1940	⋯
1	8	35	111	285	628	1230	2191	3606	5545	⋯

Its n-th row records the distribution of inversions on permutations of $[n] := \{1, 2, \ldots, n\}$. The corresponding generating function is

$$(1+x)(1+x+x^2)\cdots(1+x+\cdots+x^{n-1}) = \prod_{j=1}^{n} \frac{1-x^j}{1-x}. \tag{1}$$

We shall derive generating functions for the subdiagonals on or below the main diagonal of the table above. The first three of those are

$$\begin{aligned} S_0(x) &= 1 + x^3 + 5x^4 + 22x^5 + 90x^6 + 359x^7 + 1415x^8 + \cdots \\ S_1(x) &= x + x^2 + 2x^3 + 6x^4 + 20x^5 + 71x^6 + 259x^7 + 961x^8 + \cdots \\ S_2(x) &= x^2 + 2x^3 + 5x^4 + 15x^5 + 49x^6 + 169x^7 + 602x^8 + \cdots \end{aligned}$$

In general, if i is a non-negative integer, then $S_i(x)$ is the generating function for permutations of $[n]$ with exactly $n-i$ inversions. In other words, if we let $I_n(k)$ denote the number of permutations of $[n]$ with k inversions, then

$$S_i(x) = \sum_{n\geq 0} I_n(n-i)x^n.$$

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It should be noted that there is a known closed expression for $I_n(k)$ when $k \le n$, namely the Knuth-Netto formula [5, 7]:

$$I_n(k) = \binom{n+k-1}{k} + \sum_{j=1}^{\infty}(-1)^j\binom{n+k-u_j-j-1}{k-u_j-j} + \sum_{j=1}^{\infty}(-1)^j\binom{n+k-u_j-1}{k-u_j}$$

where $u_j = j(3j-1)/2$ is the j-th pentagonal number. This formula can be proved using (1) and Euler's pentagonal number theorem [1]. For instance, $u_1 = 1$, $u_2 = 5$, and the coefficient of x^6 in $S_0(x)$ is

$$I_6(6) = \binom{11}{6} - \binom{10-u_1}{5-u_1} - \binom{11-u_1}{6-u_1} + \binom{11-u_2}{6-u_2} = 90.$$

Let $C(x) = (1-\sqrt{1-4x})/(2x)$ be the generating function for the Catalan numbers, $C_n = \binom{2n}{n}/(n+1)$. We show (Theorem 7) that, for any non-negative integer i,

$$S_i(x) = \big(xC(x)\big)^i S_0(x),$$

thus reducing the problem of determining $S_i(x)$ to that of determining $S_0(x)$.

Denote by $\sigma(n)$ the sum of divisors of n, and denote by $p(n)$ the number of integer partitions of n. We show (Theorem 8) that

$$S_0(x) = R\big(xC(x)\big),$$

where the power series $R(x)$ can be expressed in any of the following three equivalent ways

$$\begin{aligned} R(x) &= \frac{1-x}{1-2x}\prod_{k\ge1}(1-x^k); \\ \log R(x) &= \sum_{n\ge1}(2^n-\sigma(n)-1)\frac{x^n}{n}; \\ 1/R(x) &= 1-\sum_{n\ge1}\big(p(1)+p(2)+\cdots+p(n-1)-p(n)\big)x^n. \end{aligned}$$

See Equation (2), Proposition 16 and Proposition 17.

2. Factoring permutations with few inversions

Let $\mathcal{S}_n$ be the set of permutations on $[n] = \{1, 2, \ldots, n\}$. The *inversion table* of $\pi = a_1a_2\ldots a_n$ in $\mathcal{S}_n$ is defined as $b_1b_2\ldots b_n$ where b_i is the number of elements to the left of and larger than a_i; in other words, b_i is the cardinality of the set $\{j \in [i-1] : a_j > a_i\}$. For instance, the inversion table of 3152746 is 0102021. The number of inversions in π, denoted $\mathrm{inv}(\pi)$, is simply the sum of the entries in the inversion table for π. We will work with an invertible transformation of the inversion table that we call the *cumulative inversion table*. It is obtained by taking partial sums of the inversion table: b_1, b_1+b_2, $b_1+b_2+b_3$, etc. The cumulative inversion table of 3152746 is 0113356.

A *subdiagonal sequence* is a sequence of non-negative integers whose k-th entry is smaller than k. It is easy to see that the inversion table of a permutation is a subdiagonal sequence and that any such sequence is an inversion table, so the two concepts can be used interchangeably.

Lemma 1. *There are exactly* $C_n = \binom{2n}{n}/(n+1)$ *weakly increasing subdiagonal sequences of length* n*.*

Proof. Let a weakly increasing subdiagonal sequences $b_1b_2\dots b_n$ be given, and form the sequence $a_1a_2\dots a_n$ by setting $a_i = b_i + 1$. Then $a_i \leq i$ and $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$. By Exercise 6.19(s) in [9] there are exactly C_n such sequences. □

Let $\mathcal{S}_n^k = \{\pi \in \mathcal{S}_n : \mathrm{inv}(\pi) = k\}$ be the set of permutations of $[n]$ with k inversions, and let $\mathcal{C}_n$ be the subset of $\mathcal{S}_n^{n-1}$ consisting of those permutations whose every prefix of length $k \geq 1$ has fewer than k inversions. For $n = 0, 1, 2, 3, 4$ those are $\{\epsilon\}$, $\{1\}$, $\{21\}$, $\{231, 312\}$, and $\{1432, 2341, 2413, 3142, 4123\}$, where ϵ is the empty permutation.

Lemma 2. *For* $n \geq 1$ *we have* $|\mathcal{C}_n| = C_{n-1}$*.*

Proof. Clearly, the cumulative inversion table $\gamma = c_1c_2\dots c_n$ of any permutation $\pi \in \mathcal{S}_n$ is weakly increasing. Also, the last letter, c_n, of γ is the number of inversions in π. In particular, if $\pi \in \mathcal{C}_n$ then $c_n = n-1$ and π is uniquely determined by $\gamma = c_1c_2\dots c_{n-1}$. Now, any k-prefix of γ is the cumulative inversion table of a permutation with fewer than k inversions. Moreover, since the only condition on π is that each k-prefix has fewer than k inversions, any weakly increasing subdiagonal sequence of length $n-1$ is the cumulative inversion table of such a permutation. As pointed out in Lemma 1, such sequences are counted by the Catalan numbers. □

If α and β are permutations, their *direct sum*, denoted $\alpha \oplus \beta$, is the concatenation of α and β', where β' is the transformation of β that adds to each of its letters the largest letter of α. Every permutation π can be written uniquely as the direct sum of its *components*, which are the minimal segments in a direct sum decomposition of π. For example, $23145867 = 231 \oplus 1 \oplus 1 \oplus 312$ has components 231, 1, 1, and 312. A permutation consisting of a single component is *indecomposable*. Let $\mathrm{comp}(\pi)$ be the number of components in π. We will need a lemma by Claesson, Jelínek and Steingrímsson [3, Lemma 8] relating $\mathrm{comp}(\pi)$ and $\mathrm{inv}(\pi)$:

Lemma 3 ([3])**.** *For any permutation* π*,*

$$\mathrm{inv}(\pi) + \mathrm{comp}(\pi) \geq |\pi|.$$

Lemma 4. *Let* d *be a constant. If* $\mathrm{inv}(\pi) \leq |\pi| + d$ *and* π *is decomposable, say* $\pi = \alpha \oplus \beta$ *with* α *indecomposable, then* $\mathrm{inv}(\beta) \leq |\beta| + d + 1$

Proof. Since α is indecomposable we have $\mathrm{inv}(\alpha) \geq |\alpha| - 1$ by Lemma 3, and so

$$\begin{aligned} \mathrm{inv}(\beta) &\leq |\pi| + d - \mathrm{inv}(\alpha) \\ &\leq |\pi| + d - (|\alpha| - 1) \\ &\leq |\beta| + d + 1. \end{aligned}$$ □

By iterated use of Lemma 4 we arrive at the following generalisation of said lemma.

Lemma 5. *Let d be a constant. If $\mathrm{inv}(\pi) \leq |\pi|+d$ and $\pi = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_m$ with each α_i indecomposable, then $\mathrm{inv}(\alpha_m) \leq |\alpha_m| + d + m - 1$.*

Recall now that $S_i(x)$ is the generating function for permutations of length n with $n - i$ inversions:

$$S_i(x) = \sum_{n \geq 0} |\mathcal{S}_n^{n-i}| x^n.$$

Also, let $C(x) = (1 - \sqrt{1-4x})/(2x)$ be the generating function for the Catalan numbers, $C_n = \binom{2n}{n}/(n+1)$.

Theorem 6. *We have*

$$\mathcal{S}_n^{n-1} \simeq \bigcup_{k=0}^{n} \mathcal{S}_k^k \times \mathcal{C}_{n-k}$$

and thus the generating functions $S_0(x)$ and $S_1(x)$ satisfy the identity

$$S_1(x) = xC(x)S_0(x).$$

Proof. Let $\pi = a_1 a_2 \dots a_n \in \mathcal{S}_n^{n-1}$. We shall "factor" π into two parts σ and τ such that, for some k in $\{0, 1, \dots, n\}$, σ belongs to $\mathcal{S}_k^k$ and τ belongs to $\mathcal{C}_{n-k}$. Let $\sigma = a_1 a_2 \dots a_k$ be the longest prefix (possibly empty) of π with as many letters as inversions and let $\tau = a_{k+1} a_{k+2} \dots a_n$ consist of the remaining letters of π. For instance, $\pi = 4213675$ factors into $\sigma = 4213$ and $\tau = 675$. By definition, $\mathrm{inv}(\sigma) = k$. We shall prove that σ is a permutation of $[k]$, and thus τ is a permutation of $\{k+1, k+2, \dots, n\}$. Let

$$d = \#\{\, (i,j) \,:\, a_i > a_j,\ i \leq k,\ j > k \,\}.$$

That is, d is the number of inversions in π with one leg in σ ($i \leq k$) and the other leg in τ ($j > k$). Then $\mathrm{inv}(\pi) = \mathrm{inv}(\sigma) + \mathrm{inv}(\tau) + d$. We want to prove that $d = 0$. Suppose to the contrary that $d \geq 1$. Now,

$$\mathrm{inv}(\tau) = n - 1 - k - d = |\tau| - (d+1)$$

and it follows from Lemma 3 that τ has at least $d+1$ components; let us write $\tau = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_m$ with $m \geq d+1$. Using Lemma 5 we find that

$$\mathrm{inv}(\alpha_m) \leq |\alpha_m| - d + m \leq |\alpha_m| - 1.$$

Since α_m is indecomposable we also have $\mathrm{inv}(\alpha_m) \geq |\alpha_m| + 1$ by Lemma 3 and thus $\mathrm{inv}(\alpha_m) = |\alpha_m| - 1$. Let $\beta = \alpha_1 \oplus \cdots \oplus \alpha_{m-1}$. Note that no inversion can have one leg in σ and the other leg in α_m. That is, if (i,j) is an inversion with $i \leq k$ and $j > k$ then $j < n - |\alpha_m|$. This is because any such inversion would necessarily be accompanied by $m-1$ other inversions—one for each of the components $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$—contradicting $m \geq d+1$. Thus

$$\operatorname{inv}(\sigma\beta) = n - 1 - (|\alpha_m| - 1) = |\sigma\beta|$$

and we have found a prefix of π that is longer than σ with as many letters as inversions, which contradicts the definition of σ.

Having proved that $\sigma \in \mathcal{S}_k^k$ it immediately follows that $\operatorname{inv}(\tau) = n - k - 1$. It remains to prove that τ has no nonempty prefix with as many inversions as letters, but that is trivially true as σ together with any such prefix would, again, be a longer prefix than σ, with as many inversions as letters. □

The proof above can be generalised to prove the following.

Theorem 7. *For $i \geq 0$ we have*

$$\mathcal{S}_n^{n-i-1} \simeq \bigcup_{k=0}^{n} \mathcal{S}_k^{k-i} \times \mathcal{C}_{n-k}$$

and thus the generating functions $S_{i+1}(x)$ and $S_i(x)$ satisfy the identity

$$S_{i+1}(x) = xC(x)S_i(x),$$

Equivalently,

$$S_i(x) = (xC(x))^i S_0(x).$$

While the above theorems represent some progress in understanding permutations with few inversions one crucial piece of the puzzle is missing. Theorem 7 relates all the $S_i(x)$'s to $S_0(x)$, but we need a formula for $S_0(x)$, which is what we shall offer in the next section.

3. A formula for $S_0(x)$

Let us write $\lambda \vdash n$ to indicate that λ is an integer partition of n, and $\mu \vDash n$ to indicate that μ is an integer composition of n. Further, let

$$\operatorname{Par}(x) = \prod_{k \geq 1} \frac{1}{1 - x^k} \quad \text{and} \quad \operatorname{Comp}(x) = \frac{1-x}{1-2x}$$

be the generating functions for integer partitions and compositions. With $\operatorname{Par}_+(x) = \operatorname{Par}(x) - 1$ denoting the generating function for nonempty integer partitions we have

$$\operatorname{Par}(x)^{-1} = \frac{1}{1 + \operatorname{Par}_+(x)} = \sum_{k \geq 0} (-1)^k (\operatorname{Par}_+(x))^k.$$

Thus $\operatorname{Par}(x)^{-1}$ counts signed tuples of nonempty integer partitions, where the sign of such a tuple $(\lambda^1, \ldots, \lambda^k)$ is $(-1)^k$. Define

$$\begin{aligned} R(x) &= \operatorname{Comp}(x)\operatorname{Par}(x)^{-1} \qquad (2) \\ &= 1 + x^3 + 2x^4 + 5x^5 + 9x^6 + 19x^7 + 37x^8 + \cdots \end{aligned}$$

Then $R(x)$ counts elements of the set

$$\mathcal{R}_n = \{ (\lambda^1, \ldots, \lambda^k; \mu) : \lambda^i \vdash n_i,\ \mu \vDash m,\ n_1 + n_2 + \cdots + n_k + m = n \},$$

where the sign of the tuple $(\lambda^1, \dots, \lambda^k; \mu)$ is $(-1)^k$. Writing $(\lambda^1, \dots, \lambda^k; \mu) \vdash n$ when $(\lambda^1, \dots, \lambda^k; \mu)$ is in $\mathcal{R}_n$ we then have, by definition,

$$R(x) = \sum_{n \geq 0} \left(\sum_{(\lambda^1, \dots, \lambda^k;\, \mu) \vdash n} (-1)^k \right) x^n.$$

For illustration we list the elements of $\mathcal{R}_3$ below. Negative elements are found in the left column and positive elements in the right column:

$$\begin{array}{rc}
(1\,;11) & (\emptyset\,;111) \\
(1\,;2) & (\emptyset\,;12) \\
(1,1,1\,;\epsilon) & (\emptyset\,;21) \\
(11\,;1) & (\emptyset\,;3) \\
(111\,;\epsilon) & (1,1\,;1) \\
(21\,;\epsilon) & (1,11\,;\epsilon) \\
(2\,;1) & (1,2\,;\epsilon) \\
(3\,;\epsilon) & (11,1\,;\epsilon) \\
 & (2,1\,;\epsilon)
\end{array}$$

The sequence $1, 0, 0, 1, 2, 5, 9, 19, 37, 74, \dots$ of coefficients of $R(x)$ is recorded in entry A178841 of the OEIS [8]. There it is said to count the number of *pure inverting compositions* of n; see Propositions 2 and 3 in [6].

We are now in position to state our main result regarding $S_0(x)$.

Theorem 8. *We have* $S_0(x) = R(xC(x))$, *or, equivalently,* $S_0(x(1-x)) = R(x)$, *which, by Theorem 7, implies that* $S_i(x) = \big(xC(x)\big)^i R(xC(x))$.

Before proving this we need to better understand what combinatorial structures $R(x)$ enumerates, so we shall define a sign-reversing involution ϕ on $\mathcal{R}$ that singles out a positive subset $\mathrm{Fix}(\phi)$ of $\mathcal{R}$ for which

$$R(x) = \sum_{n \geq 0} |\,\mathrm{Fix}(\phi) \cap \mathcal{R}_n|\, x^n.$$

First, however, we define the auxiliary function

$$\mathrm{split} : \{\mu : \mu \vDash n\} \to \bigcup_{i=0}^{n} \{\lambda : \lambda \vdash i\} \times \{\mu : \mu \vDash n - i\}$$

by $\mathrm{split}(\mu) = (\lambda, \mu')$ where $\mu = \lambda\mu'$ and λ is the longest prefix of μ that is weakly decreasing, and thus defines a partition. For instance, $\mathrm{split}(311212) = (311, 212)$, $\mathrm{split}(21) = (21, \epsilon)$, $\mathrm{split}(12) = (1, 2)$ and $\mathrm{split}(\epsilon) = (\epsilon, \epsilon)$. Let $\mathrm{lir}(\mu)$ be the length of the longest strictly increasing prefix (also called leftmost increasing run) of μ. For instance, $\mathrm{lir}(121) = 2$, $\mathrm{lir}(213) = \mathrm{lir}(1122) = 1$ and $\mathrm{lir}(\epsilon) = 0$.

Lemma 9. *Let* λ *be a nonempty partition and* μ *a composition such that* $\mathrm{lir}(\mu)$ *is even. Then* $\mathrm{lir}(\lambda\mu)$ *is odd. Moreover, if* a *is the last element of* λ, *then*

$$\mathrm{split}(\lambda\mu) = \begin{cases} (\lambda, \epsilon) & \text{if } \mu = \epsilon \text{ is empty} \\ (\lambda, \mu) & \text{if } (b, \mu') = \mathrm{split}(\mu) \text{ and } a < b; \\ (\lambda b, \mu') & \text{if } (b, \mu') = \mathrm{split}(\mu) \text{ and } a \geq b. \end{cases}$$

Note that if μ is nonempty and $\mathrm{lir}(\mu)$ is even, then the first element of μ must be smaller than the second, and hence the longest weakly decreasing prefix of μ is a singleton (the first letter of μ). Thus the lemma above covers all cases. We now define the promised involution ϕ on $\mathcal{R}_n$.

Definition 10. Let $(\lambda^1, \dots, \lambda^k; \mu) \vdash n$. If $\mathrm{lir}(\mu)$ is even then

$$\phi(\lambda^1, \dots, \lambda^k; \mu) = \begin{cases} (\emptyset; \mu) & \text{if } k = 0; \\ (\lambda^1, \dots, \lambda^{k-1}; \lambda^k \mu) & \text{if } k > 0. \end{cases}$$

If $\mathrm{lir}(\mu)$ is odd and $(\rho x, \mu') = \mathrm{split}(\mu)$ then

$$\phi(\lambda^1, \dots, \lambda^k; \mu) = \begin{cases} (\lambda^1, \dots, \lambda^k, \rho x; \mu') & \text{if } \mathrm{lir}(\mu') \text{ is even;} \\ (\lambda^1, \dots, \lambda^k, \rho; x\mu') & \text{if } \mathrm{lir}(\mu') \text{ is odd.} \end{cases}$$

The idea behind the map is that we can create an involution by moving a partition λ back and forth between being considered as part of the list of partitions or as a prefix of our composition μ. The parity of $\mathrm{lir}(\mu)$ allows us to know if we, so to speak, have already prepended a λ or not; indeed $\mathrm{lir}(\lambda\mu)$ is odd if $\mathrm{lir}(\mu)$ is even.

Let us look at a few cases illustrating Definition 10. A simple case is that of a fixed point: $\mathrm{lir}(3644) = 2$ is even and

$$\phi(\emptyset; 3644) = (\emptyset; 3644).$$

Consider $(\lambda^1; \mu) = (6211; \epsilon) \vdash 10$. Then $\mathrm{lir}(\mu) = 0$ is even, $k = 1$ and

$$\phi(6211; \epsilon) = (\emptyset; 6211).$$

Another example of when $\mathrm{lir}(\mu)$ is even is

$$\phi(11, 62; 243352) = (11; 62243352).$$

Finally, two cases when $\mathrm{lir}(\mu)$ is odd are

$$\phi(11, 62; 643452) = (11, 62, 643; 452);$$
$$\phi(11, 62; 643425) = (11, 62, 64; 3425).$$

Theorem 11. *The map ϕ is a sign-reversing involution on $\mathcal{R}_n$ whose fixed points are of the form $(\emptyset; \mu)$ with $\mu \vDash n$ and $\mathrm{lir}(\mu)$ even.*

Proof. Let $w = (\lambda^1, \dots, \lambda^k; \mu) \vdash n$ be given. It is clear that $\phi(w) \vdash n$ and that the first case of the definition, namely $\mathrm{lir}(\mu)$ is even and $k = 0$, covers all fixed points. Further, the second case shortens the list of partitions by one while the third and fourth cases lengthen the same list by one. In all three cases the sign of w is thus reversed. It remains to show that $\phi(\phi(w)) = w$ and we consider each of the last three cases of the definition of ϕ separately.

If $\mathrm{lir}(\mu)$ is even and $k > 0$, then $\phi(w) = (\lambda^1, \dots, \lambda^{k-1}; \lambda^k \mu)$. To show that $\phi(\phi(w)) = w$ we consider the three cases of Lemma 9. If μ is empty then $\mathrm{split}(\lambda^k \mu) = (\lambda^k, \epsilon)$, $\mathrm{lir}(\epsilon) = 0$ is even and

$$\phi(\lambda^1, \dots, \lambda^{k-1}; \lambda^k \mu) = (\lambda^1, \dots, \lambda^{k-1}, \lambda^k; \mu) = w.$$

If μ is nonempty then let $(b, \mu') = \mathrm{split}(\mu)$. Also, let a be the last element of λ^k. If $a < b$ then $\mathrm{split}(\lambda^k\mu) = (\lambda^k, \mu)$, $\mathrm{lir}(\mu)$ is even (by assumption) and

$$\phi(\lambda^1, \dots, \lambda^{k-1}; \lambda^k\mu) = (\lambda^1, \dots, \lambda^{k-1}, \lambda^k; \mu) = w.$$

If $a \geq b$ then $\mathrm{split}(\lambda^k\mu) = (\lambda^k b, \mu')$, $\mathrm{lir}(\mu') = \mathrm{lir}(\mu) - 1$ is odd and

$$\phi(\lambda^1, \dots, \lambda^{k-1}; \lambda^k\mu) = (\lambda^1, \dots, \lambda^{k-1}, \lambda^k; b\mu') = (\lambda^1, \dots, \lambda^k; \mu) = w.$$

If $\mathrm{lir}(\mu)$ is odd then let $(\rho x, \mu') = \mathrm{split}(\mu)$. If, in addition, $\mathrm{lir}(\mu')$ is even, then

$$\phi(\phi(w)) = \phi(\lambda^1, \dots, \lambda^k, \rho x; \mu') = (\lambda^1, \dots, \lambda^k; \rho x\mu') = w.$$

If $\mathrm{lir}(\mu')$ is odd, then $\phi(w) = (\lambda^1, \dots, \lambda^k, \rho; x\mu')$ and, since $\mathrm{lir}(x\mu')$ is even,

$$\phi(\phi(w)) = \phi(\lambda^1, \dots, \lambda^k, \rho; x\mu') = (\lambda^1, \dots, \lambda^k; \rho x\mu') = w,$$

which concludes the last case and thus also the proof. □

Next we aim at proving Theorem 8. That is, we wish to prove that

$$S_0(x(1-x)) = R(x) \tag{3}$$

and we start by giving a combinatorial interpretation of $S_0(x(1-x))$.

Let $T_{n,k}$ be the set of pairs (S, β), where $S \subseteq [n-k]$, $|S| = k$, and

$$\beta = (\beta_1, \beta_2, \dots, \beta_{n-k})$$

is a subdiagonal sequence with sum $\beta_1 + \beta_2 + \cdots + \beta_{n-k} = n - k$. Also, let

$$T_n = \bigcup_{k=0}^{n} T_{n,k}.$$

For instance, $T_0 = \{(\emptyset, \epsilon)\}$, T_1 and T_2 are empty, $T_3 = \{(\emptyset, 012)\}$, and T_4 consists of the following 8 elements:

$$(\emptyset, 0121),\ (\emptyset, 0112),\ (\emptyset, 0103),\ (\emptyset, 0022),\ (\emptyset, 0013),$$
$$(\{1\}, 012),\ (\{2\}, 012),\ (\{3\}, 012).$$

Lemma 12. *We have*

$$S_0(x(1-x)) = \sum_{n\geq 0} \Bigg(\sum_{(S,\beta)\,\in\, T_n} (-1)^{|S|} \Bigg) x^n.$$

Proof. Let $(S, \beta) \in T_{n,k}$. View β as an inversion table and let π be the corresponding permutation on $[n-k]$. Note that π has exactly $n-k$ inversions and thus the cardinality of $T_{n,k}$ is $\left|\mathcal{S}_{n-k}^{n-k}\right| \binom{n-k}{k}$. The result now follows from a direct calculation:

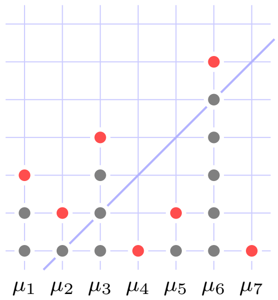
$$\begin{aligned}
S_0(x(1-x)) &= \sum_{n\geq 0} |\mathcal{S}_n^n|\, x^n (1-x)^n \\
&= \sum_{n\geq 0} |\mathcal{S}_n^n|\, x^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k \\
&= \sum_{n\geq 0} \left(\sum_{k=0}^{n} \left|\mathcal{S}_{n-k}^{n-k}\right| \binom{n-k}{k} (-1)^k \right) x^n \\
&= \sum_{n\geq 0} \sum_{k=0}^{n} \left(\sum_{(S,\beta)\,\in\, T_{n,k}} (-1)^{|S|} \right) x^n \\
&= \sum_{n\geq 0} \left(\sum_{(S,\beta)\,\in\, T_n} (-1)^{|S|} \right) x^n. \qquad \square
\end{aligned}$$

We shall show that the set T_n in the inner summation in Lemma 12 can be replaced with a smaller set, but first a few definitions.

For a composition $\mu = (\mu_1, \ldots, \mu_k)$ define $\mathrm{dmax}(\mu)$ as 0 if $k \leq 1$ and

$$\mathrm{dmax}(\mu) = \max\{\, \mu_j - j + 1 : 2 \leq j \leq k \,\}$$

otherwise. If we plot μ_j against j this is the largest distance it goes over the line $y = x - 1$, excluding μ_1 for technical reasons. For instance, if $\mu = 3241261$ then $\mathrm{dmax}(\mu) = \mu_3 - 3 + 1 = 2$ as depicted below:



Up until this point we have listed the parts of a partition λ in weakly decreasing order. In what follows, it will be convenient to instead list them in weakly increasing order. For instance, we may write $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (1, 3, 4) \vdash 8$.

Definition 13. Let λ be an integer partition and μ an integer composition. Let their total sum be n and let $d = \mathrm{dmax}(\mu)$. We shall write

$$(\lambda, \mu) \Vdash n$$

if the following three conditions hold:

- λ has distinct parts (and is hence strictly increasing);
- $\lambda \neq \epsilon \implies \lambda_{|\lambda|} < d$,
- $\mu \neq \epsilon \implies \mu_1 \leq d$.

For instance, (λ, μ) with $\mu = 3241261$ as in the example above does not satisfy Definition 13 regardless of what the partition λ is; the reason being that $3 = \mu_1 > \mathrm{dmax}(\mu) = 2$. Let us consider the sets of pairs $(\lambda, \mu) \Vdash n$ for small n. For $n = 0$ there is a single pair, (ϵ, ϵ); for $n = 1, 2$ there are none; for $n = 3$ there is a single pair, $(\epsilon, 12)$; for $n = 4$ there are two, $(\epsilon, 121)$ and $(\epsilon, 13)$; and for $n = 5$ there are seven:

$$(\epsilon, 113),\ (\epsilon, 1211),\ (\epsilon, 122),\ (\epsilon, 131),\ (\epsilon, 14),\ (\epsilon, 23),\ (1, 13).$$

As a larger example we offer $(134, 161121) \Vdash 20$.

Theorem 14. *We have*

$$\sum_{(S,\beta)\,\in\, T_n} (-1)^{|S|} = \sum_{(\lambda,\mu)\,\Vdash\, n} (-1)^{|\lambda|}.$$

Proof. We shall give a sign-reversing involution on T_n whose fixed points can be bijectively mapped to pairs $(\lambda, \mu) \Vdash n$.

Let $(S, \beta) \in T_{n,n-r}$ with $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$. We will say that β_i is *marked* if $i \in S$. An index i such that $\beta_i = 0$ and $\beta_{i+1} > 0$ will be called a *0-ascent*. If $\beta_i = i - 1$, then we call i a *diagonal index* and β_i a *diagonal entry*. We shall now define an endofunction

$$\psi : T_n \to T_n$$

which we will later prove is a sign-reversing involution. Consider the entries β_i in descending order by index and define $\psi(S, \beta)$ according to which of the following four mutually exclusive cases is encountered first:

1. If β_i is marked and there is no 0-ascent $j > i$, then we replace β_i with an unmarked bigram xy whose first letter is zero, $x = 0$, and whose last letter is $y = \beta_i + 1$. In particular, S is mapped to $S \setminus \{i\}$.
2. If β_i is marked, i is not a diagonal index and there is a 0-ascent $j > i$, then we replace β_i with an unmarked $\beta_i + 1$ and append an unmarked zero to the end of β. Again, S is mapped to $S \setminus \{i\}$.
3. If β_i and β_{i+1} are both unmarked, i is a 0-ascent and there is no diagonal index $j > i + 1$, then we replace the bigram $\beta_i\beta_{i+1}$ by a single marked $\beta_{i+1} - 1$. Here, S is mapped to $S \cup \{i\}$.
4. If $\beta_i \neq 0$ is unmarked, $\beta_r = 0$ and there is some 0-ascent $j > i$, then we replace β_i by a marked $\beta_i - 1$ and remove β_r. Here, S is mapped to $S \cup \{i\}$.

If none of these cases are encountered we let $\psi(S, \beta) = (S, \beta)$ be a fixed point. It is easy to see that each case preserves subdiagonality. Cases 1 and 2 remove a mark, increase the sum by one and add an element; consequently the image $\psi(S, \beta)$ is in $T_{n,n-r-1}$. Cases 3 and 4 add a mark, decrease the sum by one and remove an element, so in these two cases $\psi(S, \beta)$ is in $T_{n,n-r+1}$. Thus ψ is well-defined and sign-reversing. Let us consider some examples:

- Case 1 at $i = 3$: $\psi(\{1, 3\}, 0103) = (\{1\}, 01013)$
- Case 2 at $i = 3$: $\psi(\{2, 3\}, 0010150) = (\{2\}, 00201500)$
- Case 3 at $i = 4$: $\psi(\{1\}, 002040) = (\{1, 4\}, 00230)$
- Case 4 at $i = 2$: $\psi(\{3\}, 0120250000) = (\{2, 3\}, 002025000)$
- A fixed point: $\psi(\{1, 3\}, 0020152000) = (\{1, 3\}, 0020152000)$.

Next we shall prove that ψ is an involution; that is, $\psi(\psi(S,\beta)) = (S,\beta)$. If (S,β) is a fixed point, then the claim is trivially true, so we can assume that we encounter one of the four cases above. Suppose $\psi(S,\beta) = (T,\gamma)$ after falling into one of the cases at β_i. We want to show that $\psi(T,\gamma) = (S,\beta)$. The map ψ leaves the suffix $\beta_{i+2}\beta_{i+3}\dots\beta_r$ of β unchanged, aside from possibly appending an unmarked trailing zero; hence this suffix, with possibly an appended zero, will also be present in γ. If no zero was appended then clearly none of the cases apply to (T,γ) at $j > i$, or else that case would have applied to (S,β) as well. Suppose a zero was appended. Since this trailing zero is unmarked (T,γ) cannot fall into case 1 or 2 for any $j > i$. Adding a zero at the end cannot introduce a 0-ascent, so (T,γ) cannot fall into case 3 for $j > i$. Case 4 is also easy to exclude, so we conclude that (T,γ) cannot fall into any of the four cases at an index $j > i$.

Going through each of the cases at index i, we see that if β_i falls into case 1, then γ_i must fall into case 3, which undoes what case 1 just did. Similarly, case 2 is cancelled by case 4, 3 by 1, and 4 by 2; thus, ψ is an involution.

We now consider the fixed points of ψ. We wish to show that any nonempty fixed point (S,β) of ψ, when considered as a marked sequence, can be written

$$\sigma\tau\zeta$$

where each letter of σ is either a marked diagonal entry or an unmarked zero, ending in an unmarked zero; τ consists of unmarked positive entries at least one of which is a diagonal entry; and ζ is a (possibly empty) sequence of zeros. One instance of a fixed point is $(\emptyset, 0121)$ in which $\sigma = 0$, $\tau = 121$ and $\zeta = \epsilon$. Another instance is $(\{1,3\}, 0020152000)$ in which $\sigma = 0020$, $\tau = 152$ and $\zeta = 000$.

Since β is subdiagonal it starts with a zero. Moreover, its sum r is positive, and hence it must contain a 0-ascent. Let σ be the prefix of β consisting of every letter of β up to and including the rightmost 0-ascent. Define τ as the subsequent contiguous run of positive entries in β and let the remaining suffix be ζ. In particular, τ is nonempty. Also, ζ must consist entirely of zeros; otherwise, it would contain a 0-ascent, contradicting that ζ is right of the rightmost 0-ascent in β. Now, if β_i is marked and there is no 0-ascent $j > i$, then case 1 would apply at β_i. Thus every entry of $\sigma_{|\sigma|}\tau\zeta$ must be unmarked. There also has to be a diagonal entry in τ, otherwise the bigram $\sigma_{|\sigma|}\tau_1$ would make us fall into case 3. Thus, there is an $\ell > 1$ such that $\tau_\ell = \ell - 1 + |\sigma|$. If σ_i is marked, then i is a diagonal index, else we would fall into case 2 at σ_i because it is to the left of a 0-ascent. To show that σ is of the desired form we shall consider two cases.

Suppose ζ is empty. Every element to the right of τ_ℓ is ≥ 1, and $\tau_1 > 0$, so

$$\tau_1 + \tau_\ell + \cdots + \tau_{r-|\sigma|} \geq 1 + (\ell - 1 + |\sigma|) + (r - |\sigma| - \ell) \geq r.$$

The sum of entries in β is r (by definition of $T_{n,n-r}$) and consequently the sum above is exactly r. Thus, every entry of σ is zero. Aside from the first one, none of those zeros can be marked, or else we would have a contradiction with the earlier result that any marked element of σ is a diagonal entry. Thus σ is of the desired form.

Suppose ζ is nonempty. There cannot be any positive unmarked σ_i since then we would fall into case 4 at σ_i. Thus, σ is of the desired form by the same argument as above.

Let us now define a function θ mapping fixed points of ψ to pairs $(\lambda, \mu) \Vdash n$. Given a fixed point (S, β) factored as $\sigma\tau\zeta$ we let $\theta(S, \beta) = (\lambda, \mu)$, where λ consists of the marked indices of σ written in increasing order and $\mu = \tau$. In other words, the entries of λ are the elements of S, the reason being that τ and ζ contain only unmarked elements. For example $\theta(\emptyset, 0121) = (\epsilon, 121)$ and $\theta(\{1, 3\}, 0020152000) = (13, 152)$. It is clear that λ has distinct parts and that μ defines a composition. Furthermore, the sum of values in λ and μ is the sum of elements in β plus the number of marked elements, which is $r + n - r = n$. Note that the sign simply is $(-1)^{|\lambda|}$.

We wish to show that $(\lambda, \mu) \Vdash n$. Our diagonal index ℓ gives us

$$\begin{aligned} \mathrm{dmax}(\mu) &\geq \mu_\ell - \ell + 1 \\ &= \tau_\ell - \ell + 1 \\ &= (\ell - 1 + |\sigma|) - \ell + 1 = |\sigma| \,. \end{aligned}$$

Suppose $\mathrm{dmax}(\mu) = \mu_j - j + 1$. Then

$$\begin{aligned} \mathrm{dmax}(\mu) &= \beta_{j+|\sigma|} - j + 1 \\ &\leq j + |\sigma| - 1 - j + 1 = |\sigma| \,. \end{aligned}$$

in which the inequality is a consequence of subdiagonality. Thus $\mathrm{dmax}(\mu) = |\sigma|$. If λ is nonempty, then λ_1 corresponds to a marked diagonal index, which must then be in σ. Thus $\lambda < |\sigma| = \mathrm{dmax}(\mu)$ since σ ends on a zero. If μ is nonempty, then $\mu_1 = \tau_1 \leq |\sigma|$ by subdiagonality, and hence $\mu_1 \leq \mathrm{dmax}(\mu)$. Thus $(\lambda, \mu) \Vdash n$.

To complete our proof we have to show that θ is bijective, which we do by constructing its inverse. Assume that $(\lambda, \mu) \Vdash n$ and $k = \mathrm{dmax}(\mu)$. Let $\sigma = \sigma_1\sigma_2 \dots \sigma_k$, where $\sigma_i = i - 1$ is a marked diagonal entry if $i = \lambda_j$ for some $j \in [|\lambda|]$, and $\sigma_i = 0$ is an unmarked zero otherwise. Also, let $\tau = \mu$ and let ζ be a segment consisting of $n - |\sigma| - |\tau|$ unmarked zeros. By the same argument as above we have $|\lambda|$ marked elements, and the sum of all elements in λ and μ is $n - |\lambda|$, so $r = |\lambda|$. Since $\lambda \neq \epsilon \Rightarrow \lambda_1 < \mathrm{dmax}(\mu)$ we have $\sigma_{|\sigma|} = 0$. Thus the image is in $T_{n,n-r}$ and θ maps $\sigma\tau\zeta$ back to (λ, μ), completing our proof. □

Theorem 15. *We have*

$$\sum_{(\lambda,\mu) \Vdash n} (-1)^{|\lambda|} \;=\; \big|\{\mu \vDash n : \mathrm{lir}(\mu) \text{ even}\}\big|.$$

Proof. Let $\sim$ be the equivalence relation generated by postulating that

$$(\lambda a, \mu) \sim (\lambda, a\mu)$$

whenever both $(\lambda a, \mu) \Vdash n$ and $(\lambda, a\mu) \Vdash n$ hold. For example, when $n = 5$ the equivalence classes are all singletons except the class $\{(\epsilon, 113), (1, 13)\}$. For $n = 6$ there are three non-singleton classes, namely

$$\{(\epsilon, 1131), (1, 131)\},\ \{(\epsilon, 114), (1, 14)\} \text{ and } \{(\epsilon, 123), (1, 23)\}.$$

We wish to show that the inner sum in the expression for $R(x)$ above when restricted to a single equivalence class is 0 or 1. In other words, if C is an equivalence class, then

$$\sum_{(\lambda,\mu)\in C} (-1)^{|\lambda|} \in \{0,1\}.$$

Assume $(\lambda a,\mu) \Vdash n$. Then $a < \mathrm{dmax}(\mu)$, but $\mathrm{dmax}(a\mu) \geq \mathrm{dmax}(\mu) - 1$, so $a \leq \mathrm{dmax}(a\mu)$. Furthermore, if λ is nonempty, then $\lambda_{|\lambda|} < a$ because λa is strictly increasing. Thus $\lambda_{|\lambda|} < a \leq \mathrm{dmax}(a\mu)$, and so $(\lambda, a\mu) \Vdash n$. By induction on the number of elements moved we see that (λ,μ) is in the same equivalence class as $(\epsilon, \lambda\mu)$. Clearly we cannot have two pairs of the form (ϵ,μ) in the same equivalence class, so we make them our representatives.

Let (ϵ,μ) be such a representative. We will call k *valid* if

$$(\mu_1 \dots \mu_k,\, \mu_{k+1} \dots \mu_{|\mu|}) \Vdash n.$$

Let $\lambda = \mu_1 \dots \mu_k$ and $\nu = \mu_{k+1} \dots \mu_{|\mu|}$. By the argument above, if some k is valid, then all smaller k are valid too. We want to find the largest valid k. The sign of $(\mu_1 \dots \mu_k, \mu_{k+1} \dots \mu_{|\mu|})$ is $(-1)^k$, so if the largest valid value is ℓ, then the sum of the equivalence class of (ϵ,μ) is $(-1)^0 + (-1)^1 + \cdots + (-1)^\ell$, which is zero if ℓ is odd and 1 if ℓ is even.

Let $s = \mathrm{lir}(\mu)$. We wish to show that ℓ and s have the same parity. Clearly, $\ell \leq s$ since otherwise $\lambda = \mu_1 \dots \mu_\ell$ would not be strictly increasing. If $\ell = s$ we have nothing left to prove, so we can assume that $\ell < s$. Then ν is nonempty and $\nu_1 \leq \mathrm{dmax}(\nu)$, so $|\nu| \geq 2$ and $\ell \leq s-2$. Let $k = s-2$. Then $\lambda = \mu_1 \dots \mu_k$ is strictly increasing and $\nu_1 < \nu_2$. Thus $\mathrm{dmax}(\nu) \geq \nu_2 - 1 \geq \nu_1$. If λ is nonempty, then $\lambda_k < \nu_1 \leq \mathrm{dmax}(\nu)$. Thus $k = s-2$ is valid, so $\ell = s-2$, which has the same parity as s. The representatives (ϵ,μ) whose equivalence classes have sum one are hence exactly those where $\mathrm{lir}(\mu)$ is even. ☐

Theorem 8 follows directly from Lemma 12 and Theorems 11, 14 and 15.

4. Structure of $R(x)$

By Theorem 11 the elements of $\mathrm{Fix}(\phi) \cap \mathcal{R}_n$ are of the form $(\emptyset;\mu)$ with $\mu \vDash n$ and $\mathrm{lir}(\mu)$ even. With this in mind let

$$\mathrm{Fix}_n(\phi) = \{\mu \vDash n : \mathrm{lir}(\mu) \text{ even}\}.$$

Let $\mathcal{M}_n$ be the set of compositions of n that start with an ascent and are weakly decreasing after the initial ascent and let $M(x)$ be the corresponding generating function. That is, $(\mu_1,\dots,\mu_k) \in \mathcal{M}_n$ if and only if $k \geq 2$, $\mu_1 < \mu_2 \geq \mu_3 \geq \cdots \geq \mu_k$ and $\mu_1 + \cdots + \mu_k = n$. For instance, $\mathcal{M}_n = \emptyset$ for $n \leq 2$, $\mathcal{M}_3 = \{12\}$, $\mathcal{M}_4 = \{121, 13\}$, $\mathcal{M}_5 = \{1211, 122, 131, 14, 23\}$ and the first few terms of the power series $M(x)$ are

$$M(x) = x^3 + 2x^4 + 5x^5 + 8x^6 + 15x^7 + 23x^8 + 37x^9 + \cdots$$

Let $\mu^1, \mu^2, \dots, \mu^k$ be compositions with $\mu^i \in \mathcal{M}_{n_i}$. Their concatenation

$$\mu = \mu^1 \mu^2 \cdots \mu^k$$

is a composition of $n = n_1 + \cdots + n_k$ with $\mathrm{lir}(\mu)$ even, and so $\mu \in \mathrm{Fix}_n(\phi)$.

Conversely, given a composition $\mu \in \mathrm{Fix}_n(\phi)$, let μ^1 be the longest prefix of μ that belongs to $\mathcal{M}_{n_1}$, where n_1 is the length of μ^1. Writing $\mu = \mu^1\nu$ we can recursively do the same with ν, stopping if ν is empty. This way we arrive at a factorisation $\mu = \mu^1\mu^2 \cdots \mu^k$ with $\mu^i \in \mathcal{M}_{n_i}$ and $n = n_1 + \cdots + n_k$. For instance, the factors of $123511211 \in \mathrm{Fix}_{17}(\phi)$ are 12, 351 and 1211.

In terms of generating functions the factorisation we have established translates to the functional equation $R(x) = (1 - M(x))^{-1}$. Now, by (2),

$$M(x) = 1 + \left(\frac{x}{1-x} - 1\right)\mathrm{Par}(x).$$

Thus, aside from the constant term, the coefficient of x^n in $M(x)$ equals

$$p(0) + p(1) + \cdots + p(n-1) - p(n) \tag{4}$$

and coincides with sequence A058884 in the OEIS [8]. Moreover, (4) is the number of compositions with exactly one inversion according to Theorem 4.1 of [4]. To summarise we have established the following proposition.

Proposition 16. *With $p(n)$ denoting the number of partitions of n,*

$$R(x) = \Big(1 - \sum_{n\geq 1}(p(1) + p(2) + \cdots + p(n-1) - p(n))x^n\Big)^{-1}.$$

An alternative formula can be obtained from considering the logarithmic derivative of $R(x)$:

Proposition 17. *With $\sigma(n)$ denoting the sum of the divisors of n,*

$$R(x) = \exp\left(\sum_{n\geq 1}(2^n - \sigma(n) - 1)\,\frac{x^n}{n}\right).$$

Proof. Taking the logarithmic derivative of (2) we get

$$\begin{aligned} x\big(\log R(x)\big)' &= \frac{x\mathrm{Comp}'(x)}{\mathrm{Comp}(x)} - \frac{x\mathrm{Par}'(x)}{\mathrm{Par}(x)} \\ &= \frac{x}{(1-x)(1-2x)} - \sum_{k\geq 1}\frac{kx^k}{1-x^k} \end{aligned}$$

An expression of the form $F(x) = \sum_{k\geq 1} a_k x^k/(1-x^k)$ is called a Lambert series, and it is well known, and easy to see, that

$$F(x) = \sum_{n\geq 1} b_n x^n, \text{ where } b_n = \sum_{k|n} a_k.$$

In our case $a_k = k$ and hence $b_n = \sigma(n)$. Consequently,

$$x\big(\log R(x)\big)' = \sum_{n\geq 1}(2^n - 1 - \sigma(n))x^n \tag{5}$$

and it follows that

$$\log R(x) = \int_0^x \sum_{n\geq 1} (2^n - 1 - \sigma(n)) t^{n-1} dt$$
$$= \sum_{n\geq 1} (2^n - 1 - \sigma(n)) \frac{x^n}{n},$$

which proves the claim. □

Corollary 18. *The cardinalities $r_n = |\mathcal{R}_n|$ can be computed recursively by $r_0 = 1$ and, for $n \geq 1$,*

$$r_n = \frac{1}{n} \sum_{k=1}^{n} r_{n-k}(2^k - \sigma(k) - 1).$$

Moreover, we have the closed formula

$$r_n = \frac{1}{n!} \sum_{\pi \in \mathrm{Sym}(n)} \prod_{\ell \in C(\pi)} (2^\ell - \sigma(\ell) - 1),$$

where $\mathrm{Sym}(n)$ is the symmetric group of degree n and $C(\pi)$ is a multiset that encodes the cycle type of π, that is, there is an $\ell \in C(\pi)$ for each ℓ-cycle of π.

Proof. Since $\left(\log R(x)\right)' = R'(x)/R(x)$ it follows from (5) that

$$xR'(x) = R(x) \sum_{n\geq 1} (2^n - 1 - \sigma(n)) x^n$$

and on identifying coefficients we get the claimed recursion. For the closed formula we refer to Equation (8) in [2] and the paragraph preceding that formula. □

It easy to see that the coefficient of x^n in $xM'(x)/(1-M(x))$ is $2^n - \sigma(n) - 1$. Thus, if we consider the factorisation $\mu = \mu^1\mu^2 \cdots \mu^k$ of elements in $\mathcal{R}$, as above, together with a distinguished site of μ^1, then such structures should be counted by $2^n - \sigma(n) - 1$. Finding a bijective proof of this remains an open problem.

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