Turning cycle restrictions into mesh patterns via Foata's fundamental transformation

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Abstract

An adjacent q-cycle is a natural generalization of an adjacent transposition. We show that the number of adjacent q-cycles in a permutation maps to the sum of occurrences of two mesh patterns under Foata's fundamental transformation. As a corollary we resolve Conjecture 3.14 in the paper "From Hertzprung's problem to pattern-rewriting systems" by the first author.

Let q be a positive integer. Following Brualdi and Deutsch [2], define an *adjacent q-cycle* in a permutation as a cycle of the form

$$(i, i+1, \ldots, i+q-1).$$

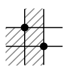
In particular, an adjacent 1-cycle is a fixed point and an adjacent 2-cycle is more commonly known as an *adjacent transposition*. Brualdi and Deutsch showed, among other things, that if $a_q(n,k)$ is the number of permutations that—when expressed as a product of disjoint cycles—have exactly k adjacent q-cycles, then

$$a_q(n,k) = \sum_{j=k}^{\lfloor n/q \rfloor} (-1)^{k+j} \binom{j}{k} \frac{(n-(q-1)j)!}{j!}.$$

Foata's fundamental transformation [4] bijectively maps a permutations π with k cycles to a permutation σ with k left-to-right minima by writing each cycle of π so that its leftmost element is the smallest, sorting the cycles in descending order with respect to their first element, and reading the resulting permutation σ as a word from left to right. For instance,

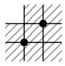
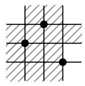
$$\pi = 213956548 = (5,6,7)(4,9,8)(3)(1,2) \quad \mapsto \quad \sigma = 567498312.$$

Consider the effect of this transformation on a fixed point (an adjacent 1-cycle) such as 3 in the permutation π above. If it is not the smallest element, then it is mapped to a left-to-right minimum in σ that is directly followed by another left-to-right minimum. In terms of mesh patterns [1] it is an occurrence of

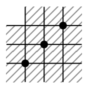
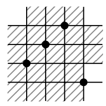
$s_1 =$ 

Indeed, the shading to the southwest of the two points guarantees that they are left-to-right minima, while the shading in the column between the points guarantees that they are adjacent. If, on the other hand, the fixed point is the smallest element, then it gets mapped to an occurrence of $r_1 =$ .

Similarly, let us consider an adjacent transposition $(i, i+1)$ in π. Depending on whether or not $(i, i+1)$ is the first cycle of π (i.e., whether or not $i = 1$) it is mapped to an occurrence of one of the patterns

$r_2 =$  or $s_2 =$ 

in σ. For instance, $(1, 2)$ in the example permutation π corresponds to an occurrence of r_2 in σ. Proceeding with adjacent 3-cycles we have

$r_3 =$  and $s_3 =$ 

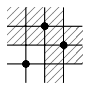
and the cycle $(5, 6, 7)$ in π corresponds to a unique occurrence of s_3 in σ.

It should now be clear how to define r_q and s_q for any $q \geq 1$, and that the following theorem is true.

Theorem 1. *Assume that $\pi \mapsto \sigma$ under Foata's fundamental transformation. For any $q \geq 1$, the number of adjacent q-cycles in π is equal to the sum of the number of occurrences of r_q and s_q in σ.*

The following corollary provides a generating function identity conjectured by the first author [3, Conjecture 3.14].

Corollary 2. *We have*

$$\sum_{n\geq 0} |S_n(p)| x^n = \sum_{m\geq 0} m! \left(\frac{x}{1+x^2}\right)^m, \quad \textit{where } p =$$ 

and $S_n(p)$ denotes the set of permutations avoiding the mesh pattern p.

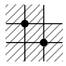
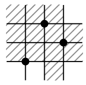
Proof. Let us denote the right-hand side of the conjectured identity by $F(x)$, and let $A(x)$ be the generating function for the number of permutations of $[n]$ whose disjoint cycle decompositions have no adjacent transpositions. In other words, the coefficient of x^n in $A(x)$ is $a_2(n, 0)$; these numbers form sequence A177249 in the OEIS [6]. Brualdi and Deutsch [2] have shown that $A(x)$ satisfies the differential equation

$$x^2(1 + x^2)A'(x) - (1 + x^2)(1 - x - x^2)A(x) + 1 - x^2 = 0, \quad A(0) = 1.$$

It is easy to verify that $F(x)/(1 + x^2)$ satisfies the same differential equation. It thus suffices to show that $\sum_{n\geq 0} |S_n(p)|x^n = (1 + x^2)A(x)$, or, equivalently,

$$|S_n(p)| = a_2(n, 0) + a_2(n - 2, 0) \quad \text{for } n \geq 2. \tag{1}$$

As a special case of Theorem 1 we find that Foata's fundamental transformation provides a one-to-one correspondence between permutations without adjacent transpositions and permutations that avoid r_2 and s_2. By symmetry (reverse followed by inverse) we may equivalently consider permutations avoiding the two patterns

$r_2' =$  and $s_2' =$ 

By the Shading lemma [5], the pattern p is *coincident* with the pattern s_2', in the sense that $S_n(p) = S_n(s_2')$ for all $n \geq 0$. By conditioning on whether a permutation avoiding s_2' also avoids r_2' or contains r_2' we find that

$$S_n(p) = S_n(r_2', s_2') \cup \big(Co_n(r_2') \cap S_n(s_2')\big),$$

where the union is disjoint and $Co_n(r_2')$ denotes the set of permutations that contain r_2'. Note that a permutation π contains r_2' and avoids s_2' precisely when it is the direct sum $\pi = 21 \oplus \sigma$ of the permutation 21 and a permutation σ that avoids r_2' and s_2'. This establishes equation (1) and concludes the proof. $\square$

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