A solution to the duality problem on modified ascent sequences

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Abstract

The in-order traversal provides a natural correspondence between binary trees with a decreasing vertex labeling and endofunctions on a finite set. By suitably restricting the vertex labeling we arrive at a class of trees that we call Fishburn trees. We give bijections between Fishburn trees and other well-known combinatorial structures that are counted by the Fishburn numbers, and by composing these new maps we obtain simplified versions of some of the known maps. With this new machinery we solve the so called flip and sum problems on modified ascent sequences.

1 Introduction

The coefficients of the elegant power series

$$\sum_{n\geq 0}\prod_{k=1}^{n}\left(1-(1-x)^k\right) \;=\; 1+x+2x^2+5x^3+15x^4+53x^5+\cdots$$

are known as the Fishburn numbers, which is sequence A022493 in the OEIS [21]. Claesson and Linusson [8] named them so in honor of Peter C. Fishburn (1936–2021), who pioneered, among other things, the study of interval orders [14, 15]. The last decade has seen a lot of interest in combinatorial structures related to this counting sequence. The starting point was the 2010 paper [2] by Bousquet-Mélou, Claesson, Dukes and Kitaev, which gave one-to-one correspondences between certain, apparently unrelated, objects: (**2**+**2**)-free posets; the set of permutations avoiding a certain bivincular pattern, now called Fishburn permutations; Stoimenow matchings [22]; and ascent sequences. They also provided an algorithm to transform an ascent sequence into its modified version, and showed that the latter is closely related to the level distribution of the corresponding (**2**+**2**)-free poset. Later, Dukes and Parviainen [12] found a bijection between ascent sequences and Fishburn matrices [15], that is, upper triangular matrices with nonnegative integer entries whose every row and column contains at least one positive entry. All these objects are enumerated by the Fishburn numbers, and for this reason we shall refer to them as *Fishburn structures*.

We will define two new structures of this kind, namely *Fishburn trees* and *Fishburn covers*. The former are decreasing binary trees satisfying some simple conditions on their labeling, while the latter encode the trees as an ordered collection of multisets. There are surprisingly straightforward bijections relating them to modified ascent sequences, Fishburn matrices and (**2**+**2**)-free posets. By composing these new maps we obtain simplified versions of those previously known in the literature. In this sense, Fishburn trees and Fishburn covers provide a transparent encoding of other Fishburn structures, and we may regard them as central objects from which the others are derived. For instance, the Dukes and Parviainen bijection [12] is obtained by composing the map between modified ascent sequences and Fishburn trees with the map between Fishburn trees and Fishburn matrices. As an application, we provide a more direct solution to the flip and sum problems (defined below).

Our work fits into an active line of research [7, 9, 10, 11, 13, 16, 17, 18, 19, 23] that explores the relations between Fishburn structures by analyzing how statistics and operations that are natural on a certain object are transported to the others. In this context, the following two problems, originally proposed by Dukes and Parviainen [12], are particularly relevant.

- *The flip problem.* Duality acts as an involution on (**2**+**2**)-free posets. On Fishburn matrices, this is equivalent to reflecting a matrix in its antidiagonal. What is the corresponding operation on ascent sequences?
- *The sum problem.* The result of adding two Fishburn matrices is another Fishburn matrix. What is the corresponding operation on ascent sequences?

Note that the Dukes and Parviainen bijection between ascent sequences and Fishburn matrices could be used to compute the flip and sum operations. For instance, if x is an ascent sequence, one could first determine the Fishburn matrix A corresponding to x, then compute $\mathrm{flip}(A)$ by reflecting A in its antidiagonal, and finally go back to the desired ascent sequence by applying the Dukes and Parviainen bijection once again. This map is, however, defined by a rather intricate recursive construction that makes this approach opaque. The goal is to find a more transparent solution. A first answer to the flip and sum problems was proposed by Ying and Yu [24]. Roughly speaking, Ying and Yu encode Fishburn matrices as, what they call, M-sequences, to then define a bijection between ascent sequences and M-sequences of Fishburn matrices. The flip and sum are computed on M-sequences, and the corresponding ascent sequences are once again obtained by composition. Unfortunately, this solution is rather cryptic, mainly due to the high amount of technicalities, and the lack of a geometric description of the construction. We believe that we have found a more transparent construction; a key in making the construction more transparent is to view it in terms of modified ascent sequences rather that plain ascent sequences.

In Section 2 we introduce a family of decreasing binary trees called *endotrees*. We show that endotrees bijectively map to endofunctions via the in-order traversal of the tree. To describe the inverse of this bijection, we define the *max-decomposition* of an endofunction and use it to recursively build an endotree. Similarly, Cayley permutations are in one-to-one correspondence with endotrees whose labels form an interval, we call them *regular endotrees*.

In Section 3 we introduce Fishburn trees as the set of regular endotrees satisfying an additional property. By decomposing a Fishburn tree in maximal right paths, each one labeled with a unique integer, we are able to encode it as an ordered collection of multisets, the *Fishburn cover*. The main result of this section, Theorem 3.9, is a bijection from Fishburn covers to Fishburn trees.

In Section 4, we obtain a bijection between Fishburn trees and modified ascent sequences by restricting the in-order sequence and the max-decomposition.

In Section 5 we define a bijection from Fishburn trees to Fishburn matrices by simply mapping each maximal right path to a specific row of the matrix. More specifically, we set the (i, j)-th entry of the matrix equal to the number of nodes with label j contained in the i-th path. On the other hand, a Fishburn matrix naturally induces a Fishburn cover, and the corresponding Fishburn tree is determined by Theorem 3.9.

The bijection from Fishburn trees to $(\mathbf{2}+\mathbf{2})$-free posets has a similar flavour, and is illustrated in Section 6. Nodes with label i are mapped to the i-th level of the poset, and the j-th strict down-set contains those that belong to a maximal right path with index strictly less than j. Conversely, we show that Fishburn covers naturally define a canonical labeling of $(\mathbf{2}+\mathbf{2})$-free posets.

In Section 7 we use Fishburn covers as stepping stones to provide a solution to the flip and sum problems. We end this section with two concrete examples.

In Section 8 we provide a high-level description of the framework introduced in this paper and leave some open problems and suggestions for future work.

2 Endofunctions and decreasing binary trees

For any natural number n, let End_n be the set of *endofunctions*, $x : [n] \to [n]$, where $[n] = \{1, 2, \ldots, n\}$. We often identify an endofunction x with the word $x = x_1 \ldots x_n$, where $x_i = x(i)$ for each $i \in [n]$. Let $\mathrm{End} = \cup_{n\geq 0}\mathrm{End}_n$. In general, given a set A whose elements are equipped with a notion of size, we will denote by A_n the set of elements in A that have size n. Or, conversely, given a definition of A_n (of elements of size n) we let $A = \cup_{n\geq 0} A_n$. If $\mathrm{Im}(x) = [k]$, for some $k \leq n$, then x is a *Cayley permutation* [4, 20]. The set of Cayley permutations is denoted by Cay. In other words, x is a Cayley permutation if it contains at least one copy of each integer between 1 and its maximum element. For example, $\mathrm{Cay}_1 = \{1\}$, $\mathrm{Cay}_2 = \{11, 12, 21\}$ and

$$\mathrm{Cay}_3 = \{111, 112, 121, 122, 123, 132, 211, 212, 213, 221, 231, 312, 321\}.$$

There is a well known bijection between Cayley permutations and ballots (ordered set partitions) of $[n]$. Indeed, a Cayley permutation x encodes the ballot $B_1 \ldots B_k$, where $i \in B_{x(i)}$. In particular, $|\mathrm{Cay}_n|$ is the n-th Fubini number, which is sequence A000670 in the OEIS [21].

A *binary tree* is either the empty tree or a triple

$$T = (L, r, R),$$

where r is a node called the *root* of T and L and R are binary trees called the *left subtree* and the *right subtree* of T, respectively. Equivalently, a binary tree is a rooted plane tree where each node has either 0 children; 1 child, which can be either a left or right child; or 2 children, namely a left child and a right child.

Let T be a binary tree. We denote by $\mathrm{V}(T)$ the set of nodes of T and by $\mathrm{r}(T)$ its root. The *size* of T is the cardinality of $\mathrm{V}(T)$. Now, suppose that T is equipped with

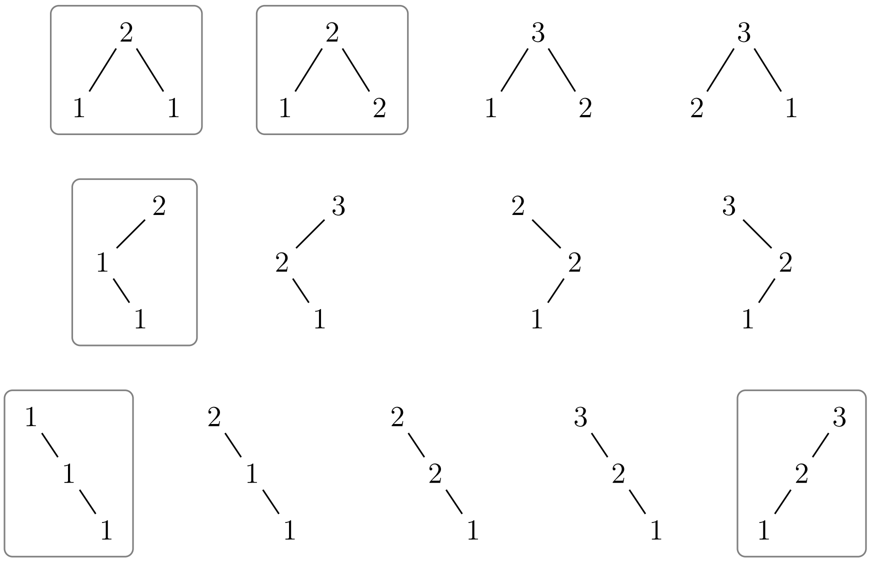


Figure 1: Regular endotrees of size 3. Fishburn trees have been highlighted.

a *vertex labeling* $\mathfrak{l} : \mathrm{V}(T) \to \{1, 2, \dots\}$ assigning to each node $v \in \mathrm{V}(T)$ a positive integer label $\mathfrak{l}(v)$. Then, assuming that T is nonempty, we let

$$\max(T) = \max\bigl\{\mathfrak{l}(v) : v \in \mathrm{V}(T)\bigr\}$$

denote the largest value among the labels of T. For convenience we also let $\max(T) = 0$ when $T = \emptyset$ is the empty tree.

A *decreasing binary tree* is a vertex-labeled binary tree T such that either T is empty or $T = (L, r, R)$, where L and R are decreasing binary trees and

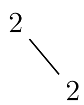
$$\mathfrak{l}(r) \geq \max\bigl\{\max(L), \max(R)\bigr\}.$$

We say that T is *strictly decreasing to the left* if it is empty or $\mathfrak{l}(r) > \max(L)$ and L and R are strictly decreasing to the left. Less formally, a vertex-labeled binary tree T is decreasing if on any path from the root to a leaf we encounter the labels in weakly decreasing order. It is strictly decreasing to the left if on any such path when we take a left turn we encounter a smaller label.

Definition 2.1. A decreasing binary tree T of size n is said to be an *endotree* if it is strictly decreasing to the left and $\mathfrak{l}(v) \in [n]$ for each $v \in \mathrm{V}(T)$. The last condition may be more compactly written $\mathrm{Im}(\mathfrak{l}) \subseteq [n]$. If, in addition, $\mathrm{Im}(\mathfrak{l}) = [k]$, for some $k \leq n$, then T is said to be *regular*. We denote by Tree the set of endotrees and by $\mathrm{Tree}^{\star}$ the set of regular endotrees.

Of the four endotrees of size 2 there is exactly one which is not regular, namely

$$\bigl(\emptyset, 2, (\emptyset, 2, \emptyset)\bigr) \;=$$



Of size 3 there are 13 regular endotrees and they are illustrated in Figure 1.

The *in-order traversal* of a binary tree $T = (L, r, R)$ is performed as follows: recursively traverse the left subtree L, visit the root r, and recursively traverse the right subtree R. For the rest of this paper, we will denote by v_i the i-th visited node in the in-order traversal of T. The *in-order sequence* of a vertex-labeled binary tree T is defined by $\alpha(T) = x_1 \dots x_n$, in which $x_i = \mathfrak{l}(v_i)$. We can alternatively define $\alpha(T)$ recursively as follows. If T is the empty tree, then $\alpha(T)$ is the empty string. Otherwise, T is nonempty and we can write $T = (L, r, R)$. Then

$$\alpha(T) = \alpha(L)\mathfrak{l}(r)\alpha(R).$$

It is easy to see that if T is an endotree (of size n and maximum k), then $\alpha(T)$ is an endofunction (of size n and maximum k). That is, we have a map

$$\alpha : \mathrm{Tree}_n \to \mathrm{End}_n.$$

We wish to define the inverse map $\bar{\alpha}$. The *max-decomposition* of a nonempty endofunction $x = x_1 \dots x_n$ is

$$x \;=\; \mathrm{pref}(x)\, x_m\, \mathrm{suff}(x),$$

where $\mathrm{pref}(x) = x_1 \dots x_{m-1}$, $\mathrm{suff}(x) = x_{m+1} \dots x_n$ and $m = \min\bigl(x^{-1}(\max(x))\bigr)$ is the index of the leftmost occurrence of $\max(x) = \max\{x_i : i \in [n]\}$ in x. The tree $\bar{\alpha}(x)$ is then defined using recursion: If x is the empty word, then $\bar{\alpha}(x)$ is the empty tree. Otherwise, x is nonempty and using the max-decomposition we can write $x = \mathrm{pref}(x)x_m\mathrm{suff}(x)$. Now, let

$$\bar{\alpha}(x) = \bigl(L, r, R\bigr)$$

be the tree with root r labeled $\mathfrak{l}(r) = x_m$, left subtree $L = \bar{\alpha}\bigl(\mathrm{pref}(x)\bigr)$ and right subtree $R = \bar{\alpha}\bigl(\mathrm{suff}(x)\bigr)$.

Proposition 2.2. *If $x \in \mathrm{End}_n$, then $\bar{\alpha}(x) \in \mathrm{Tree}_n$.*

Proof. Let $T = \bar{\alpha}(x)$. If x is empty, then there is nothing to prove. Assume that x is nonempty and let $x = \mathrm{pref}(x)x_m\mathrm{suff}(x)$ be its max-decomposition. By definition of the map $\bar{\alpha}$ we may write $T = (L, r, R)$, where $\mathfrak{l}(r) = x_m$, $L = \bar{\alpha}\bigl(\mathrm{pref}(x)\bigr)$ and $R = \bar{\alpha}\bigl(\mathrm{suff}(x)\bigr)$. Since x_m is the leftmost occurrence of $\max(x)$ in x, each label in L is strictly smaller than the label x_m of the root r. Similarly, T is weakly decreasing to the right since each label in R is at most equal to x_m. The result follows from applying the induction hypothesis to L and R. □

Proposition 2.3. *The inverse map of α is $\bar{\alpha}$.*

Proof. Using induction we shall show that $\alpha \circ \bar{\alpha}$ is the identity function on End_n, and that $\bar{\alpha} \circ \alpha$ is the identity function on Tree_n. The base cases are trivial and omitted. Assume $n \geq 1$. Let $x \in \mathrm{End}_n$. Applying the induction hypothesis to $\mathrm{pref}(x)$ and $\mathrm{suff}(x)$ we find that

$$\alpha\bigl(\bar{\alpha}(x)\bigr) = \alpha\bigl(\bar{\alpha}(\mathrm{pref}(x)\bigr)x_m\alpha\bigl(\bar{\alpha}(\mathrm{suff}(x)\bigr) = \mathrm{pref}(x)x_m\mathrm{suff}(x) = x.$$

Let $T = (L, r, R) \in \mathrm{Tree}_n$. By definition of α, we have

$$\alpha(T) = \alpha(L)\mathfrak{l}(r)\alpha(R). \tag{1}$$

Since T is strictly decreasing to the left, we have $\mathfrak{l}(r) > \mathfrak{l}(u)$ for each node u in L. Thus, Equation 1 is the max-decomposition of $\alpha(T)$ and, by induction,

$$\bar{\alpha}\bigl(\alpha(T)\bigr) = \bigl(\bar{\alpha}(\alpha(L)), r, \bar{\alpha}(\alpha(R))\bigr) = (L, r, R) = T.$$ □

A corollary of the previous result is that $\alpha : \text{Tree} \to \text{End}$ is a size-preserving bijection with inverse $\bar{\alpha}$. Furthermore, it is easy to see that $T \in \text{Tree}^\star_n$ if and only if $\alpha(T) \in \text{Cay}_n$. That is, the (restricted) map $\alpha : \text{Tree}^\star \to \text{Cay}$ is a size-preserving bijections between $\text{Tree}^\star$ and Cay.

Corollary 2.4. *For each $n \geq 1$ we have*

$$|\text{Tree}_n| = |\text{End}_n| \quad \textit{and} \quad |\text{Tree}^\star_n| = |\text{Cay}_n|.$$

See Figure 2 for a concrete endotree and its corresponding endofunction. In Figure 3 (on the left) an example of a decreasing binary tree that fails to be an endotree is given.

3 Fishburn trees and Fishburn covers

Throughout the preceding section we have denoted by $T = (L, r, T)$ a binary tree with root r, left subtree L and right subtree R. In the same vein, given a node $v \in \text{V}(T)$, let $T(v)$ denote the subtree of T consisting of v together with all the descendants of v, and let $L(v)$ and $R(v)$ denote the left and right subtrees of $T(v)$, so that $T(v) = (L(v), v, R(v))$. Recall that v_i denotes the i-th visited node in the in-order traversal of T; in particular, v_1 is the first visited node and $L(v_1) = \emptyset$. Assuming that T is and endotree and that $x = \alpha(T)$ is the corresponding endofunction, then $x_{i-1} < x_i$ if and only if $L(v_i)$ is nonempty. Such an x_i is called an ascent top; by convention and convenience we will also include x_1 among the ascent tops. This justifies us defining

$$\text{asctops}(T) = \{v_1\} \cup \{v_i : L(v_i) \neq \emptyset\}$$

as the set consisting of v_1 and nodes that have a left child. We also define

$$\text{nub}(T) = \{v_j : \mathfrak{l}(v_i) \neq \mathfrak{l}(v_j) \text{ for each } i < j\}$$

as the set of nodes v_j whose label $\ell = \mathfrak{l}(v_j)$ is the first occurrence of ℓ in the in-order sequence of T. As illustrated in Figure 2, we can represent an endotree so that labels of nodes in $\text{nub}(T)$ are the "leftmost" occurrences among the labels of T. The in-order sequence $\alpha(T)$ is then obtained by simply reading the labels of T from left to right. With this in mind, we say that $v_j \in \text{nub}(T)$ is the leftmost occurrence of $\mathfrak{l}(v_j)$ in T.

We are now ready to give the definition of Fishburn tree.

Definition 3.1. A *Fishburn tree* is a regular endotree T in which $\text{nub}(T) = \text{asctops}(T)$, and we denote by $\mathcal{T}$ the set of Fishburn trees.

An example of a Fishburn tree is given in Figure 2; an example of a non-Fishburn tree is given in Figure 3 (on the right). Five of the 13 endotrees of size 3 are Fishburn trees; they are highlighted in Figure 1. We continue this section with a couple of simple lemmas concerning Fishburn trees.

Lemma 3.2. *If $T \in \mathcal{T}$, then $\mathfrak{l}(v_1) = 1$.*

Proof. Let $v_j \in \text{nub}(T)$ be the leftmost occurrence of 1 in T. By definition of Fishburn tree we have $\text{nub}(T) = \text{asctops}(T)$ and thus $v_j \in \text{asctops}(T)$. Hence the disjunction $j = 1$ or $L(v_j) \neq \emptyset$ holds true. The latter disjunct is, however, false since $\mathfrak{l}(v_j) = 1$ and T is strictly decreasing to the left. □

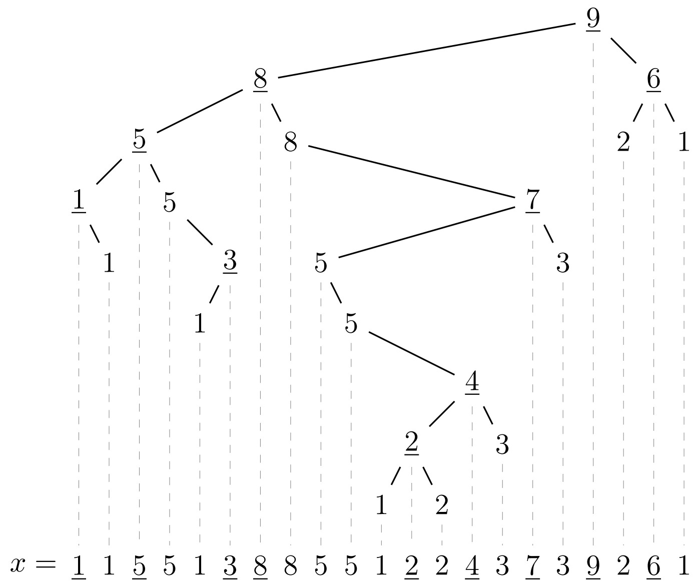


Figure 2: A Fishburn tree T and the in-order sequence $x = \alpha(T)$. Nodes are spaced so that the in-order sequence is obtained by reading the labels of the nodes from left to right. Finally, labels of nodes in $\operatorname{asctops}(T) = \operatorname{nub}(T)$, as well as the corresponding entries in $\operatorname{asctops}(x) = \operatorname{nub}(x)$, are underlined.

Lemma 3.3. *Let $T \in \mathcal{T}$ and let $k = \max(T)$. Then*

$$|\operatorname{asctops}(T)| = k \quad \textit{and} \quad \mathfrak{l}\big(\operatorname{asctops}(T)\big) = [k].$$

Proof. We have $\operatorname{Im}(\mathfrak{l}) = [k]$. In particular, $\operatorname{nub}(T)$ contains exactly one node with label i, for each $i \in [k]$. The claim then immediately follows from $\operatorname{nub}(T) = \operatorname{asctops}(T)$. ☐

A *maximal right path* of a binary tree T is a nonempty sequence of nodes $W = (w_1, w_2, \dots, w_k)$ such that w_{i+1} is the right child of w_i, for each $i = 1, \dots, k-1$; and W is maximal in the sense that the first node w_1 is not the right child of any node and the last node w_k has no right child. *Maximal left path* is defined analogously. It is easy to see that for any node $v \in V(T)$ there is a unique maximal right path to which v belongs. Similarly, there is a unique maximal left path to which v belongs. We shall denote those by $\operatorname{rpath}(v)$ and $\operatorname{lpath}(v)$, respectively. Furthermore, we define the *diagonal* of a nonempty binary tree $T = (L, r, R)$ by

$$\operatorname{diag}(T) = \operatorname{lpath}(r).$$

Note that $\operatorname{diag}(T) \subseteq \operatorname{asctops}(T)$. We shall partition $\operatorname{asctops}(T)$ accordingly as

$$\operatorname{asctops}(T) = \operatorname{diag}(T) \cup \overline{\operatorname{diag}}(T),$$

where $\overline{\operatorname{diag}}(T) = \{v \in \operatorname{asctops}(T) : v \notin \operatorname{diag}(T)\}$. We shall also say that a node $v \in \operatorname{asctops}(T)$ is *diagonal* if $v \in \operatorname{diag}(T)$ and that it is *non-diagonal* if $v \in \overline{\operatorname{diag}}(T)$. Note that v_1 is always diagonal.

Next we show that in a Fishburn tree the first node of a maximal right path is either diagonal or the left child of a non-diagonal node.

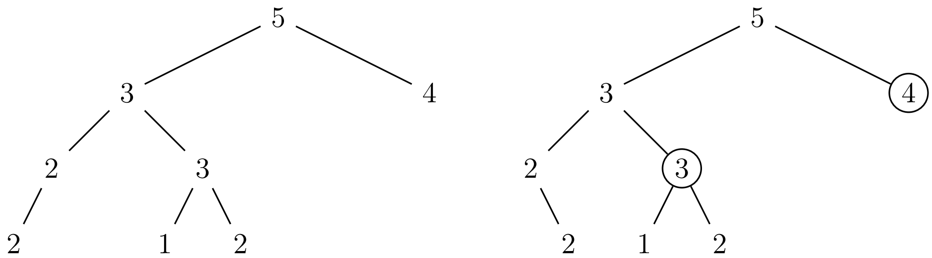


Figure 3: The tree on the left is not an endotree since it is not strictly decreasing to the left. The tree on the right is a regular endotree, but not a Fishburn tree since $\mathrm{nub}(T) \neq \mathrm{asctops}(T)$. Indeed, if u and v are the distinguished nodes with $\ell(u) = 3$ and $\ell(v) = 4$, respectively, then $u \in \mathrm{asctops}(T) \setminus \mathrm{nub}(T)$ and $v \in \mathrm{nub}(T) \setminus \mathrm{asctops}(T)$. Note that the two trees have the same in-order sequence $x = 22313254$.

Lemma 3.4. *Let W be a maximal right path of a Fishburn tree T and let w be the first node of W. Then either $w \in \mathrm{diag}(T)$ or w is the left child of some $v \in \overline{\mathrm{diag}}(T)$.*

Proof. Let r be the root of T. Since W is maximal, w is not a right child. If w is not a left child, then $w = r$ and thus $w \in \mathrm{diag}(T)$. Otherwise, w is the left child of some $v \in \mathrm{asctops}(T)$. If $v \in \mathrm{diag}(T)$, then $w \in \mathrm{diag}(T)$ as well, since $\mathrm{diag}(T) = \mathrm{lpath}(r)$. Otherwise, $v \in \overline{\mathrm{diag}}(T)$. □

The following corollary is an immediate consequence of Lemma 3.4.

Corollary 3.5. *For $T \in \mathcal{T}$ we have*

$$\mathrm{V}(T) = \bigcup_{v \in \mathrm{diag}(T)} \mathrm{rpath}(v) \;\cup\; \bigcup_{v \in \overline{\mathrm{diag}}(T)} \mathrm{rpath}\big(\mathrm{lchild}(v)\big)$$

where all the unions are disjoint and $\mathrm{lchild}(v)$ *denotes the left child of v.*

We refer to the partition of $\mathrm{V}(T)$ induced by its maximal right paths as the rpath-*decomposition* of T. Furthermore, we say that a maximal right path W is *diagonal* if its first node is diagonal; otherwise, if the first node is the left child of a non-diagonal node, W is *non-diagonal.*

By Corollary 3.5 each maximal right path W of T can be associated with a node in $\mathrm{asctops}(T)$ in the following manner:

- If W is diagonal, then it is associated with its first node, which is a diagonal node.
- If W is non-diagonal, then it is associated with the father of its first node, which is a non-diagonal node.

Conversely, each node in $\mathrm{asctops}(T)$ determines a unique maximal right path this way. The correspondence between maximal right paths of T and $\mathrm{asctops}(T)$ described above is thus bijective. Now, recall from Lemma 3.3 that $|\mathrm{asctops}(T)| = k$ and $\ell\big(\mathrm{asctops}(T)\big) = [k]$, where $k = \max(T)$. In particular, there are exactly k maximal

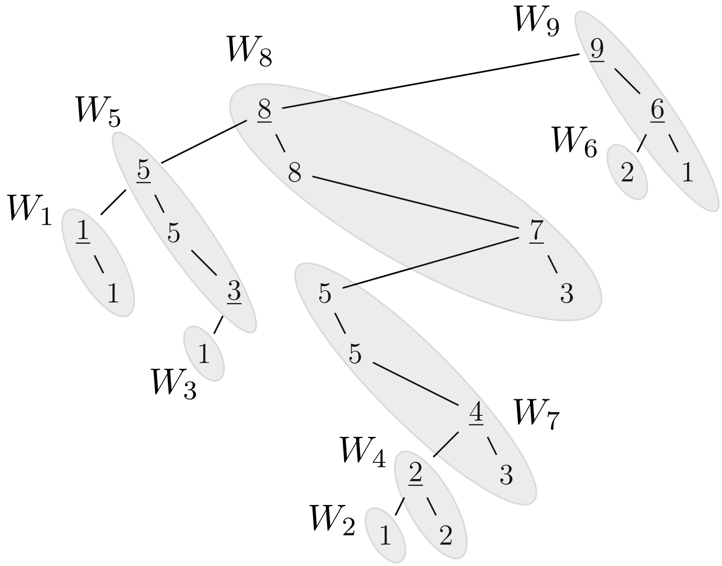


Figure 4: The rpath-decomposition of the Fishburn tree of Figure 2. Here $\operatorname{diag}(T)$ contains the underlined nodes with label 1, 5, 8 and 9. Thus the paths W_1, W_5, W_8 and W_9 are diagonal, while W_2, W_3, W_4, W_6 and W_7 are non-diagonal.

right paths in T. Moreover, if W is a maximal right path and v is its associated node in $\operatorname{asctops}(T)$, then we can assign the integer $\mathfrak{l}(v) \in [k]$ to it. Let W_i be the maximal right path assigned to $i \in [k]$ in this manner. In particular, $W_k = \operatorname{rpath}(r)$, where r is the root of T. As an example, the maximal right paths of the Fishburn tree in Figure 2 are illustrated in Figure 4. Here below, to encode them more compactly, we abuse notation and write down the corresponding labels:

$$\begin{array}{lll} W_1 = (1,1) & W_4 = (2,2) & W_7 = (5,5,4,3) \\ W_2 = (1) & W_5 = (5,5,3) & W_8 = (8,8,7,3) \\ W_3 = (1) & W_6 = (2) & W_9 = (9,6,1). \end{array}$$

Definition 3.6. For each node $u \in \mathrm{V}(T)$ we let $\mathfrak{b}(u)$ be the index of the maximal right path that contains u; i.e. $u \in W_{\mathfrak{b}(u)}$.

The label $\mathfrak{b}(u)$ can be recursively computed as follows. The label of the root r of T is $\mathfrak{b}(r) = \mathfrak{l}(r)$, and for $v \in \mathrm{V}(T)$, $v \neq r$, we have

$$\mathfrak{b}(v) = \begin{cases} \mathfrak{b}(u) & \text{if } v = \operatorname{rchild}(u), \\ \mathfrak{l}(v) & \text{if } v = \operatorname{lchild}(u) \text{ and } u \in \operatorname{diag}(T), \\ \mathfrak{l}(u) & \text{if } v = \operatorname{lchild}(u) \text{ and } u \in \overline{\operatorname{diag}}(T). \end{cases} \tag{2}$$

Here, $\operatorname{lchild}(u)$ denotes the left child of u, and $\operatorname{rchild}(u)$ denotes the right child of u. See Figure 5 for an illustration of these rules.

Let B_i be a multiset containing a copy of the integer j for each node with label j in W_i. That is, B_i is the multiset $\{\mathfrak{l}(u) : u \in W_i\}$. Given a Fishburn tree T, we denote by $\mathfrak{P}(T)$ the list of multisets

$$\mathfrak{P}(T) = B_1 \dots B_k$$

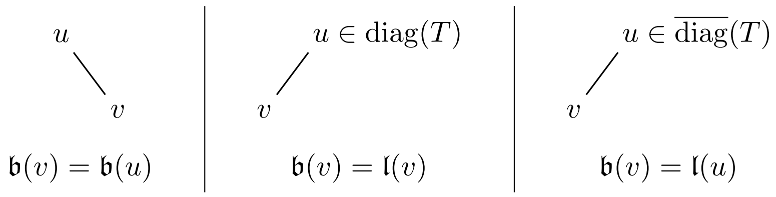


Figure 5: Rules to determine the label $\mathfrak{b}(v)$.

defined this way. Note that $\bigcup_{i\in[k]} B_i = [k]$, since $\mathrm{V}(T) = [k]$. Furthermore, $j \le i$ for each $j \in B_i$, since T is decreasing.

Definition 3.7. An ordered collection of k nonempty multisets $P = B_1 \dots B_k$ is a *Fishburn cover* if the following two conditions are satisfied:

- $\displaystyle\bigcup_{i\in[k]} B_i = [k]$;
- for all $i \in [k]$, if $j \in B_i$, then $j \le i$.

As noted above, the rpath-decomposition of a Fishburn tree T determines a Fishburn cover $\mathfrak{P}(T)$. For instance, the Fishburn cover of the tree in Figure 2 is

$$\mathfrak{P}(T) = \{1,1\}\{1\}\{1\}\{2,2\}\{5,5,3\}\{2\}\{5,5,4,3\}\{8,8,7,3\}\{9,6,1\},$$

where, for reasons that will become clear later, the elements of a block are written in weakly decreasing order. In Theorem 3.9, below, we show that the converse is true as well; that is, every Fishburn cover uniquely determines the rpath-decomposition of a Fishburn tree. First a simple lemma.

Lemma 3.8. *Let T be a Fishburn tree and let $\mathfrak{P}(T) = B_1 B_2 \dots B_k$ be the Fishburn cover of T. Then, for each $i \in [k]$,*

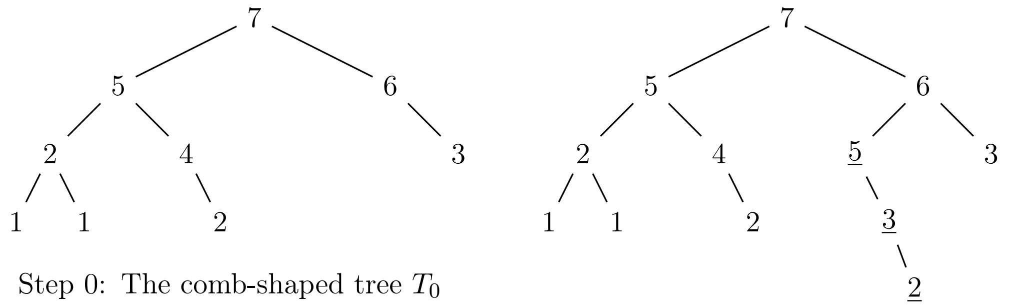
$$i \in B_i \iff W_i \text{ is diagonal.}$$

Proof. The maximal right path W_i is diagonal if and only if i is the label of the first node of W_i, that is $i \in B_i$. □

Theorem 3.9. *For each Fishburn cover P, there is a unique Fishburn tree T such that $\mathfrak{P}(T) = P$.*

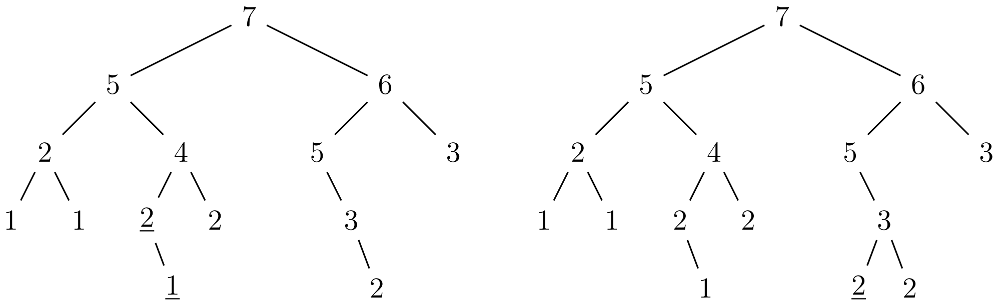
Proof. Let $P = B_1 \dots B_k$ be a Fishburn cover. We will construct, in multiple steps, a Fishburn tree T such that $\mathfrak{P}(T) = P$. For each $i, j \in [k]$, let $m_i(j)$ be the multiplicity of j in B_i. Construct a decreasing binary tree W_i consisting of a single right path with $|B_i|$ nodes in total and $m_i(j)$ nodes labeled j. It is easy to see that $\mathfrak{P}(T) = P$ if and only if the rpath-decomposition of T is given by the paths $W_1, \dots, W_k$.

Let $D = \{i \in [k] : i \in B_i\}$ and $\bar{D} = \{i \in [k] : i \notin B_i\}$. Due to Lemma 3.8, we want to construct our tree in such a way that W_i is diagonal if $i \in D$, and non-diagonal if $i \in \bar{D}$. We start by arranging the diagonal paths $\{W_i : i \in D\}$ in a comb-shaped



Step 0: The comb-shaped tree T_0 arising from the diagonal blocks $B_1 = \{1\}$, $B_2 = \{2, 1\}$, $B_5 = \{5, 4, 2\}$ and $B_7 = \{7, 6, 3\}$.

Step 1: T_1 is obtained by attaching W_6 (i.e. $B_6 = \{5, 3, 2\}$) to T_0.



Step 2: T_2 is obtained by attaching W_4 (i.e. $B_4 = \{2, 1\}$) to T_1.

Step 3: $T = T_3$ is obtained by attaching W_3 (i.e. $B_3 = \{2\}$) to T_2.

Figure 6: The step-by-step construction of Theorem 3.9 on the Fishburn cover $\mathfrak{P}(T) = \{1\}\{2, 1\}\{2\}\{2, 1\}\{5, 4, 2\}\{5, 3, 2\}\{7, 6, 3\}$. At each step, nodes in the newly appended path are underlined. Observe that, at the last step, W_3 is attached to the leftmost occurrence of 3 in T_2. This is an example where the ordering in which the paths are attached matters. Indeed, if we had started by appending W_3 to T_0, then the first node of W_3 would have been attached to a different node (the only other node labeled 3) and the result would not have been a Fishburn tree.

decreasing binary tree tree T_0—see Figure 6—so that $\mathrm{diag}(T_0) = \{w_i : i \in D\}$, where w_i is the first node of W_i.

To attach the remaining non-diagonal paths, $\{W_i : i \in \bar{D}\}$, we will specify an iterative procedure. Suppose that we are in s-th step of this procedure and that we have already constructed a tree T_{s-1}. Due to the equality $\mathrm{nub}(T) = \mathrm{asctops}(T)$ defining Fishburn trees, we shall attach W_i, with $i \in \bar{D}$, to T_{s-1} so that w_i becomes the left child of the leftmost occurrence of i. To make sure that the procedure is well-defined and that the desired property is preserved, we start with the largest index in $\bar{D}$ and proceed in decreasing order. Once again, we refer to Figure 6 for a step-by-step illustration of this construction. Assume that $\bar{D} = \{j_1, j_2, \dots, j_m\}$ with $j_1 > j_2 > \cdots > j_m$. For $s = 1, 2, \dots, m$:

- Let y_s be the leftmost occurrence of j_s in T_{s-1}.
- Let T_s be the tree obtained by attaching the path W_{j_s} to T_{s-1} so that w_{j_s}—the first node of W_{j_s}—becomes the left child of y_s.

For the succession of trees T_0, T_1, …, T_m to be well-defined we need to verify that, for $s = 1, 2, \dots, m$,

1. The tree T_{s-1} contains at least one node with label j_s.
2. The node y_s, whose label is the leftmost occurrence of j_s in T_{s-1}, has no left child.

To prove the first property, note that, since P is a Fishburn cover, we have

$$[k] = \bigcup_{i \in [k]} B_i = \bigcup_{i \in [k]} \mathfrak{l}(W_i).$$

Thus at least one path, say W_t, contains a node with label j_s. If $t \in D$, then W_t is contained in T_0. On the other hand, suppose that $t = j_q \in \bar{D}$, for some q. Note that $q < s$, or else $j_q \le j_s$ and $\mathfrak{l}(u) < j_q \le j_s$ for each $u \in W_{j_q}$, contradicting the assumption that W_t contains a node with label j_s. Hence W_{j_q} is contained in T_q, with $q < s$. In both cases T_{s-1} contains at least one node with label j_s and hence y_s is well-defined.

To prove the second property, note that

$$\mathrm{asctops}(T_{s-1}) = \{w_i : i \in D\} \cup \{y_1, \dots, y_{s-1}\}$$

and no node in $\mathrm{asctops}(T_{s-1})$ has label j_s.

Thus the succession of trees T_0, T_1, …, T_m is well-defined and we let $T = T_m$. It remains to show that T is a Fishburn tree and that it is the only Fishburn tree that satisfies $\mathfrak{P}(T) = P$. The proof is divided into four parts corresponding to following claims, in which $s \in \{0, 1, \dots, m\}$:

1. $\overline{\mathrm{diag}}(T_s) = \{y_1, \dots, y_s\}$;
2. $\mathrm{asctops}(T_s) \subseteq \mathrm{nub}(T_s)$;

3. $T = T_m$ is a Fishburn tree;

4. T is the only Fishburn tree such that $\mathfrak{P}(T) = P$.

Proof of claim 1. Note that $\overline{\text{diag}}(T_0) = \emptyset$. Assume that $s \in [m]$ and $\overline{\text{diag}}(T_{s-1}) = \{y_1, \dots, y_{s-1}\}$. Now, T_s is obtained from T_{s-1} by attaching W_{j_s} to T_{s-1} so that w_{j_s} becomes the left child of y_s. Hence,

$$\overline{\text{diag}}(T_s) = \overline{\text{diag}}(T_{s-1}) \cup \{y_s\} = \{y_1, \dots, y_{s-1}, y_s\}.$$

Proof of claim 2. Note that $\text{asctops}(T_0) = \text{diag}(T_0) \subseteq \text{nub}(T_0)$. Assume that $s \in [m]$ and $\text{asctops}(T_{s-1}) \subseteq \text{nub}(T_{s-1})$. Now,

$$\text{asctops}(T_s) = \text{asctops}(T_{s-1}) \cup \{y_s\}.$$

Observe that $y_s \in \text{nub}(T_s)$ by construction. Furthermore, each node u in the newly attached path W_{j_s} has label $\mathfrak{l}(u) < j_s$ due to Lemma 3.8 and the definition of Fishburn cover. Thus, $\mathfrak{l}(u) < j_s < j_q$ for each $q < s$ and hence $\{y_1, \dots, y_{s-1}\} \subseteq \text{nub}(T_s)$. Therefore, $\overline{\text{diag}}(T_s) = \{y_1, \dots, y_{s-1}, y_s\} \subseteq \text{nub}(T_s)$. To obtain the desired inclusion $\text{asctops}(T_s) \subseteq \text{nub}(T_s)$ it suffices to prove that $\text{diag}(T_s) \subseteq \text{nub}(T_s)$. Consider the (only) path Q from $\text{r}(T_s)$ to y_s and let i be the label of the last diagonal top contained in Q (see Figure 7). Let $v \in \text{diag}(T_s)$. Recall that $v \in \text{nub}(T_{s-1})$. If $\mathfrak{l}(v) \leq i$, then v precedes each node of W_{j_s} in the in-order traversal of T_s and thus $v \in \text{nub}(T_s)$. On the other hand, if $\mathfrak{l}(v) > i$, then $\mathfrak{l}(u) < i < \mathfrak{l}(v)$ for each $u \in W_{j_s}$, hence we have $v \in \text{nub}(T_s)$ once again.

Proof of claim 3. Note that T is decreasing and strictly decreasing to the left by construction. Moreover, due to what proved above, we have

$$\text{asctops}(T) \subseteq \text{nub}(T) \quad \text{and} \quad |\text{asctops}(T)| = k.$$

Consequently, $|\text{nub}(T)| = k$ as well, from which the equality $\text{asctops}(T) = \text{nub}(T)$ follows and hence T is a Fishburn tree.

Proof of claim 4. Let T' be a Fishburn tree with $\mathfrak{P}(T') = P$. We will show that $T' = T$. Since $\mathfrak{P}(T') = \mathfrak{P}(T)$, the rpath-decomposition of T' is given by the same paths $W_1, \dots, W_k$. In particular, in T' the diagonal paths $\{W_i : i \in D\}$ must be arranged in a comb-shaped tree T'_0 such that $T'_0 = T_0$. We wish to prove that each of the remaining paths $\{W_i : i \in \bar{D}\}$ is attached to the same node as in T. That is, for $s = 1, \dots, m$, the path W_{j_s} is attached to the leftmost occurrence y_s of j_s in T_{s-1}, where $\bar{D} = \{j_1, \dots, j_m\}$ and $j_1 > j_2 > \cdots > j_m$. Consider the path W_{j_1}. Note that $\mathfrak{l}(u) < j_1$ for each $u \in W_{j_t}$ and $t \geq 1$, hence there are no nodes with label j_1 in the paths $W_{j_1}, \dots, W_{j_m}$. Therefore, y_1 is the leftmost occurrence of j_1 not only in $T'_0 = T_0$, but also in every tree obtained by attaching $W_{j_1}, \dots, W_{j_m}$ to T'_0. In particular, due to the usual equality $\text{asctops}(T') = \text{nub}(T')$ defining Fishburn trees, W_{j_1} must be attached to y_1 in T'. In other words, the subtree T'_1 of T' consisting of T'_0 and the path W_{j_1} is $T'_1 = T_1$. Similarly, we have $\mathfrak{l}(u) < j_2$ for each $u \in W_{j_t}$ and $t \geq 2$. Thus y_2 is the leftmost occurrence of j_2 in every tree obtained by attaching $W_{j_2}, \dots, W_{j_m}$ to T'_1, and W_{j_2} must be attached to y_2 in T'. The remaining paths can be addressed analogously. □

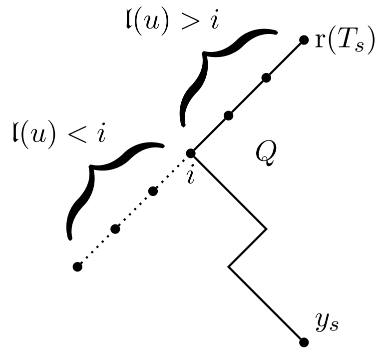


Figure 7: Referring to *claim 2* of Theorem 3.9, the only path Q from $\mathrm{r}(T_s)$ to y_s.

In the following sections, we will use Theorem 3.9 as a tool to define maps from Fishburn matrices and (**2**+**2**)-free posets to Fishburn trees. Namely, we will identify rows of matrices and down-sets of posets as the "elementary blocks" of a Fishburn cover. Then, Theorem 3.9 provides a constructive procedure to assemble the resulting blocks in order to obtain a Fishburn tree. This approach is in fact more general and could be extended to any other Fishburn structure.

4 Modified ascent sequences

Let $x : [n] \to [n]$ be an endofunction. Writing $x_i = x(i)$, as usual, we define

$$\mathrm{asctops}(x) = \{(1, x_1)\} \cup \{(i, x_i) : 1 < i \leq n, x_{i-1} < x_i\}$$

as the set of ascent tops and their indices—including the first element—and let

$$\mathrm{nub}(x) = \{(\min x^{-1}(j), j) : 1 \leq j \leq \max(x)\}$$

be the set of first occurrences and their indices. An *ascent sequence* is an endofunction $x : [n] \to [n]$ such that $x_1 = 1$ and, for $i \leq n - 1$,

$$x_{i+1} \leq |\mathrm{asctops}(x_1 \cdots x_i)| + 1.$$

Let $\mathcal{A}$ be the set of ascent sequences. Bousquet-Mélou et al. [2] defined an iterative procedure to map an ascent sequence x to its modified version $\hat{x}$, and the set $\hat{\mathcal{A}}$ of *modified ascent sequences* was originally defined as the image of $\mathcal{A}$ under the $x \mapsto \hat{x}$ bijection. We [6] have provided the following characterization of modified ascent sequences.

Lemma 4.1. *The set $\hat{\mathcal{A}}$ of modified ascent sequences is characterized by*

$$\hat{\mathcal{A}} = \{x \in \mathrm{Cay} : \mathrm{asctops}(x) = \mathrm{nub}(x)\}.$$

Alternatively, a recursive definition of $\hat{\mathcal{A}}$ can be found in [6], as well as a description of $\hat{\mathcal{A}}$ by avoidance of two Cayley-mesh patterns, defined in [5].

Recall that $\alpha(T)$ denotes the in-order sequence of the tree T. We wish to prove that $T \mapsto \alpha(T)$ is a bijective mapping from from Fishburn trees to modified ascent sequences. We shall start by showing that the statistics $\mathrm{nub}(T)$ and $\mathrm{asctops}(T)$ (on endotrees) are natural analogues of statistics $\mathrm{nub}(x)$ and $\mathrm{asctops}(x)$ (on endofunctions).

Lemma 4.2. *Let $T \in \mathrm{Tree}$ and let $x = \alpha(T)$. Then, for each $i \geq 1$,*

$$(i, x_i) \in \mathrm{nub}(x) \iff v_i \in \mathrm{nub}(T).$$

Proof. Let $i \geq 1$. Then $(i, x_i) \in \mathrm{nub}(x)$ if and only if x_i is the leftmost occurrence of the corresponding integer $\mathfrak{l}(v_i)$ in x. Equivalently, $\mathfrak{l}(v_j) \neq \mathfrak{l}(v_i)$ for each $j < i$; that is, $v_i \in \mathrm{nub}(T)$. □

Lemma 4.3. *Let $T \in \mathrm{Tree}$ and let $x = \alpha(T)$. Then, for each $i \geq 1$,*

$$(i, x_i) \in \mathrm{asctops}(x) \iff v_i \in \mathrm{asctops}(T(x)).$$

Proof. By definition we have $(1, x_1) \in \mathrm{asctops}(x)$ and $v_1 \in \mathrm{asctops}(T)$, which takes care of the case $i = 1$. Assume $i \geq 2$ and suppose, initially, that $(i, x_i) \in \mathrm{asctops}(x)$; that is, $x_{i-1} < x_i$. Since x_{i-1} and x_i are consecutive entries in the in-order sequence, the node v_i is visited immediately after v_{i-1} in the in-order traversal of T. In fact, only the following two cases are admitted:

1. v_{i-1} is the last visited node in the subtree of T with root $\mathrm{lchild}(v_i)$. In this case, $\mathrm{lchild}(v_i) \neq \emptyset$ and thus $v_i \in \mathrm{asctops}(T)$.
2. $v_i = \mathrm{rchild}(v_{i-1})$. This is however impossible since T is decreasing and $x_i > x_{i-1}$ by our assumptions.

For the converse, let $v_i \in \mathrm{asctops}(T)$. Then v_{i-1} is contained in the subtree of T with root $\mathrm{lchild}(v_i)$. In particular, $x_{i-1} < x_i$ since T is strictly decreasing to the left. □

Proposition 4.4. *Let T be an endotree and let $x = \alpha(T)$. Then*

$$T \in \mathcal{T}_n \quad \text{if and only if} \quad x \in \hat{\mathcal{A}}_n.$$

Proof. It follows immediately by Lemma 4.1, Lemma 4.2 and Lemma 4.3. Indeed, for each $i \in [n]$,

$$(i, x_i) \in \mathrm{nub}(x) \iff v_i \in \mathrm{nub}(T)$$

and

$$(i, x_i) \in \mathrm{asctops}(x) \iff v_i \in \mathrm{asctops}(T).$$

Thus the equality $\mathrm{asctops}(x) = \mathrm{nub}(x)$ is satisfied if and only if $\mathrm{nub}(T) = \mathrm{asctops}(T)$ is satisfied as well. □

Corollary 4.5. *The (restricted) map $\alpha : \mathcal{T} \to \hat{\mathcal{A}}$ and its inverse map $\bar{\alpha} : \hat{\mathcal{A}} \to \mathcal{T}$ are size-preserving bijections between Fishburn trees and modified ascent sequences. In particular, for each $n \geq 1$ we have*

$$|\mathcal{T}_n| = |\hat{\mathcal{A}}_n|.$$

A Fishburn tree and its in-order sequence are illustrated in Figure 2.

5 Fishburn matrices

A *Fishburn matrix* is a lower triangular matrix with non-negative integer entries such that every row and column contains at least one nonzero entry. Let $\mathcal{M}$ denote the set of Fishburn matrices. The size of $A \in \mathcal{M}$ is the sum of its entries, and $\mathcal{M}_n$ denotes the set of Fishburn matrices of size n. When displaying a Fishburn matrix we will for the sake of readability leave the region above the diagonal empty and zeros on or below the diagonal will be denoted by a dot. For example, we have $\mathcal{M}_1 = \{[1]\}$,

$$\mathcal{M}_2 = \left\{[2], \begin{bmatrix}1 & \\ \cdot & 1\end{bmatrix}\right\} \text{ and } \mathcal{M}_3 = \left\{[3], \begin{bmatrix}2 & \\ \cdot & 1\end{bmatrix}, \begin{bmatrix}1 & \\ \cdot & 2\end{bmatrix}, \begin{bmatrix}1 & \\ 1 & 1\end{bmatrix}, \begin{bmatrix}1 & & \\ \cdot & 1 & \\ \cdot & \cdot & 1\end{bmatrix}\right\}.$$

Our definition of a Fishburn matrix is a slight departure from the original definition [12] in that our matrices are lower triangular rather than upper triangular.

We wish to define a mapping $\beta : \mathcal{T} \to \mathcal{M}$ by regarding the label $\mathfrak{l}(u)$ of each node u as a column index and the index $\mathfrak{b}(u)$ of the maximal right path containing u as a row index.

Let T be a Fishburn tree with n nodes. Let $k = \max(T)$ and let $\mathfrak{P}(T) = B_1 \dots B_k$ be the Fishburn cover of T. For each $i, j \in [k]$, let $m_i(j)$ be the multiplicity of j in B_i. Equivalently, $m_i(j)$ is equal to the number of nodes with label j contained in W_i, where W_i is the i-th maximal right path in the rpath-decomposition of T. Then we let $\beta(T)$ be the $k \times k$ matrix whose (i, j)-th entry is equal to $m_i(j)$.

A simple high-level description of β is the following:

- Compute the rpath-decomposition of T.
- Use the i-th path W_i to "fill" the i-th row of $\beta(T)$.

Proposition 5.1. *Let T be a Fishburn tree of size n and with* $\max(T) = k$. *Then $\beta(T)$ is a $k \times k$ Fishburn matrix of size n.*

Proof. Since B_i is nonempty, the i-th row of $\beta(T)$ contains at least one nonzero entry. Also, $\bigcup_{i\in[k]} B_i = [k]$ and hence each column contains at least one nonzero entry. Furthermore, $\beta(T)$ is lower triangular, since $j \leq i$ for each $j \in B_i$ and $i \in [k]$. Finally, the number of nodes in T is equal to the sum of entries of $\beta(T)$. □

For instance, recall the Fishburn cover associated with the Fishburn tree in Figure 2:

$$\mathfrak{P}(T) = \{1,1\}\{1\}\{1\}\{2,2\}\{5,5,3\}\{2\}\{5,5,4,3\}\{8,8,7,3\}\{9,6,1\}.$$

Its corresponding Fishburn matrix $A = \beta(T)$ is

$$A = \begin{bmatrix}
2 & & & & & & & & \\
1 & \cdot & & & & & & & \\
1 & \cdot & \cdot & & & & & & \\
\cdot & 2 & \cdot & \cdot & & & & & \\
\cdot & \cdot & 1 & \cdot & 2 & & & & \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & 1 & 1 & 2 & \cdot & \cdot & & \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 2 & \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1
\end{bmatrix}$$

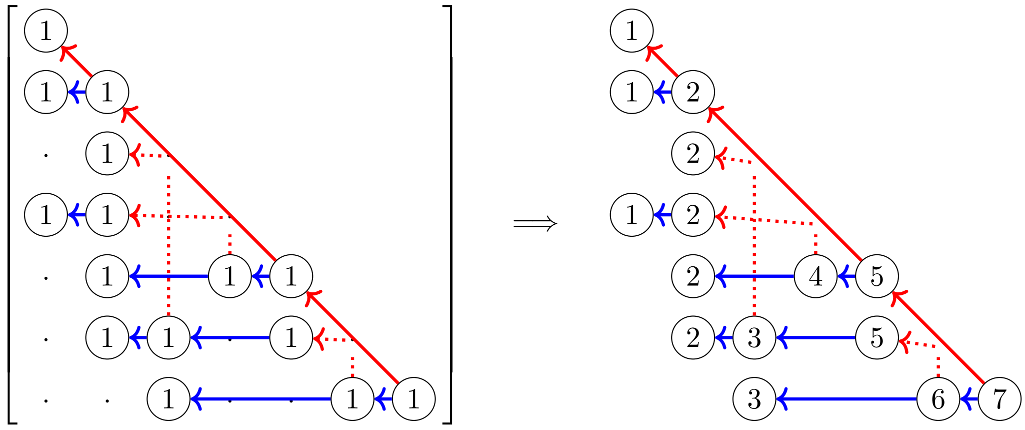


Figure 8: The Fishburn tree illustrated in Figure 6 drawn on the corresponding binary Fishburn matrix. The rightmost entry in the bottom row is the root of the tree. Red arrows point to left children and blue arrows point to right children. Dotted arrows indicate the "bouncing" construction that determines the father of non-diagonal rightmost entries.

where, for instance, the penultimate row of A corresponds to the penultimate block of $\mathfrak{P}(T)$:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 0 \end{bmatrix} \quad \longleftrightarrow \quad \{8, 8, 7, 3\}.$$

Defining the inverse map $\bar{\beta}$ is now straightforward. Let $A = (a_{i,j})$ be a $k \times k$ Fishburn matrix. Let $\mathfrak{P}(A) = B_1 \dots B_k$, where B_i is a multiset containing $a_{i,j}$ copies of the integer j, for $i, j \in [k]$. Then $\mathfrak{P}(A)$ is a Fishburn cover. Indeed $\bigcup_{i \in [k]} B_i = [k]$, since A does not contain null columns; each multiset B_i is nonempty, since A does not contain null rows; and $j \leq i$ for each $j \in B_i$, since A is lower-triangular. Now, due to Theorem 3.9, there is a unique Fishburn tree T such that $\mathfrak{P}(T) = \mathfrak{P}(A)$, and we let $\bar{\beta}(A) = T$. Finally, it is clear that $\beta(T) = A$.

We have thus proved the following result.

Corollary 5.2. *The map $\beta : \mathcal{T} \to \mathcal{M}$ and its inverse map $\bar{\beta} : \mathcal{M} \to \mathcal{T}$ are size-preserving bijections between Fishburn trees and Fishburn matrices. In particular, for each $n \geq 1$ we have*

$$|\mathcal{T}_n| = |\mathcal{M}_n|.$$

Note that $\bar{\beta}(A)$ can be constructed using Theorem 3.9: each path W_i is obtained by reading the entries in the i-th row of A, and the paths W_i are then assembled as illustrated in Theorem 3.9.

Remark 5.3. The Fishburn tree $\bar{\beta}(A)$ can be drawn directly on the Fishburn matrix A. This construction is most significant on binary matrices, where each nonzero entry of A is identified with exactly one node of $\beta(A)$. Instead of giving the full details, we refer the reader to the example in Figure 8. One interesting aspect is that if the rightmost entry of a row is not on the diagonal of A, then its father can be determined by "bouncing" off of the diagonal; indeed if $a_{i,i} = 0$, then W_i is non-diagonal and the topmost node of W_i (i.e. the rightmost entry of the i-th row of A) is the left child of a node with label i (i.e. in column i). In general, more than one entry could be hit by bouncing off of the diagonal. To determine the correct one is rather tricky, and involves defining a notion of in-order traversal of matrices which we have decided to omit.

6 (2+2)-free posets

In this paper we consider two posets to be equal up to isomorphism, that is, if there is an order preserving bijections between them. The isomorphism class is called an *unlabeled poset*. An unlabeled poset is (**2**+**2**)-*free* if it does not contain an induced subposet order isomorphic to **2**+**2**, the union of two disjoint 2-element chains. The *size* of a poset is the number of its elements and we let $\mathcal{P}_n$ denote the set of unlabeled (**2**+**2**)-free posets of size n. Given $Q \in \mathcal{P} = \cup_{n\geq 0}\mathcal{P}_n$ and $u \in Q$, let

$$D(u) = \{v : v < u\}$$

be the strict down-set of u. Fishburn [14] showed that a poset is (**2**+**2**)-free if and only if it is order isomorphic to an interval order. Alternatively (see [2] for a proof), a poset is (**2**+**2**)-free if and only if its strict down-sets can be linearly ordered by inclusion. That is, the strict down-sets of Q form a chain

$$\emptyset = D_1 \subset D_2 \subset \cdots \subset D_k.$$

For convenience, we let $D_{k+1} = Q$. If $D(u) = D_i$, we say that the element u is at *level* i and we write $\mathrm{lev}(u) = i$. Finally, we let

$$L_i = \{u : \mathrm{lev}(u) = i\}$$

denote the i-th level of Q. It is clear that a (**2**+**2**)-free poset is completely determined by its levels and strict down-sets. Indeed, any poset Q is determined by the list of its strict down-sets $\{D(u) : u \in Q\}$, and $D(u) = D_i$ if $\mathrm{lev}(u) = i$. An element u of Q is *maximal* if no other element of Q is greater than u. It is *minimal* if no other element is smaller than u. We let $\max(Q)$ and $\min(Q)$ denote the set of maximal and minimal elements, respectively. It is easy to see that if Q is (**2**+**2**)-free, then $\min(Q) = L_1$ and $\max(Q) = D_{k+1} \setminus D_k$.

We wish to define a bijection $\gamma : \mathcal{T} \to \mathcal{P}$. Let T be a Fishburn tree and let $k = \max(T)$. Recall from Definition 3.6 that, given $u \in \mathrm{V}(T)$, the index of the maximal right path that contains u in the rpath-decomposition of T is denoted by $\mathfrak{b}(u)$. Recall also that $\mathfrak{l}(u) \leq \mathfrak{b}(u)$ for each $u \in \mathrm{V}(T)$, a fact that will be used repeatedly in this section. We wish to define a (**2**+**2**)-free poset $Q = \gamma(T)$ by associating each node of T with an element of Q. That is, we let $V(T)$ be the set of elements of Q. Then we define a partial order on Q by letting, for any u and v in Q,

$$u < v \iff \mathfrak{b}(u) < \mathfrak{l}(v).$$

Let us prove that this relation is a strict partial order. Irreflexivity is an immediate consequence of the inequality $\mathfrak{l}(u) \leq \mathfrak{b}(u)$. To prove antisymmetry, suppose that $u < v$; i.e. $\mathfrak{b}(u) < \mathfrak{l}(v)$. For a contradiction, suppose also that $v < u$; i.e. $\mathfrak{b}(v) < \mathfrak{l}(u)$. Then

$$\mathfrak{b}(u) < \mathfrak{l}(v) \leq \mathfrak{b}(v) < \mathfrak{l}(u),$$

from which we get $\mathfrak{b}(u) < \mathfrak{l}(u)$, which is impossible. Finally, to prove transitivity, suppose that $u < v$ and $v < w$. Then

$$\mathfrak{b}(u) < \mathfrak{l}(v) \leq \mathfrak{b}(v) < \mathfrak{l}(w),$$

from which $\mathfrak{b}(u) < \mathfrak{l}(w)$, and thus $u < w$, follows. To prove that Q is $(\mathbf{2}+\mathbf{2})$-free, we show that its strict down-sets are linearly ordered by inclusion. Let us first determine its strict down-sets. Let $u \in Q$ and suppose that $\mathfrak{l}(u) = i$. The strict down-set of u is

$$D(u) = \{v \in Q : \mathfrak{b}(v) < i\}.$$

In other words, all the elements with vertex label i have the same strict down-set, namely $\{v \in Q : \mathfrak{b}(v) < i\}$. For $i \in [k]$, let $D_i = \{v \in Q : \mathfrak{b}(v) < i\}$. Note that there is at least one element in Q whose down-set is D_i since $\mathfrak{l}(Q) = [k]$ by definition of Fishburn tree. Now, it is clear by definition that $D_i \subseteq D_{i+1}$. Furthermore, the inclusion is strict since $\mathfrak{b}(Q) = [k]$; thus, there is at least one element in $D_{i+1} \setminus D_i = \{u \in Q : \mathfrak{b}(u) = i\}$. Therefore, the down-sets of Q are precisely the sets D_i, $i \in [k]$, which are strictly ordered by inclusion. We have now proved that Q is $(\mathbf{2}+\mathbf{2})$-free. Note that the levels of Q are

$$L_i = \{u \in Q : D(u) = D_i\} = \{u \in Q : \mathfrak{l}(u) = i\}.$$

In fact, an alternative way to define Q is to let its levels and strict down-sets be

$$L_i = \{u \in \mathrm{V}(T) : \mathfrak{l}(u) = i\} \quad \text{and} \quad D_i = \{u \in \mathrm{V}(T) : \mathfrak{b}(u) < i\}.$$

The inverse map $\bar{\gamma}$ of γ is defined as follows. Given a poset $Q \in \mathcal{P}$, we define a canonical labeling of Q by setting, for each $u \in Q$,

$$\mathfrak{l}(u) = \mathrm{lev}(u) \quad \text{and} \quad \mathfrak{b}(u) = \min\{i : u \in D_i\} - 1.$$

Note that $\{\mathfrak{l}(u) : u \in Q\} = \{\mathfrak{b}(u) : u \in Q\} = [k]$, where k is the number of levels of Q. Moreover, we have $\mathfrak{l}(u) \leq \mathfrak{b}(u)$ for each $u \in Q$. In fact, to the poset Q we have associated the Fishburn cover $\mathfrak{P}(Q) = B_1 \ldots B_k$, where B_i contains a copy of the integer j for each $u \in Q$ with labels $\mathfrak{b}(u) = i$ and $\mathfrak{l}(u) = j$. We can thus use Theorem 3.9 to construct a Fishburn tree $T = \bar{\gamma}(Q)$ in which each node u has labels $\mathfrak{l}(u)$ and $\mathfrak{b}(u)$. Finally, it is easy to see that $\gamma(T) = Q$ and it follows that $\bar{\gamma}$ is the inverse map of γ. Indeed, $\mathfrak{l}(u) = \mathrm{lev}(u)$ and $\mathfrak{b}(u) = i+1-1 = i$, where $u \in D_{i+1} \setminus D_i$. In the end we obtain the following result.

Corollary 6.1. *The map $\gamma : \mathcal{T} \to \mathcal{P}$ and its inverse map $\bar{\gamma} : \mathcal{P} \to \mathcal{T}$ are size-preserving bijections between Fishburn trees and $(\mathbf{2}+\mathbf{2})$-free posets. In particular, for each $n \geq 1$ we have*

$$|\mathcal{T}_n| = |\mathcal{P}_n|.$$

A Fishburn tree and the canonical labeling of the corresponding poset are illustrated in Figure 9.

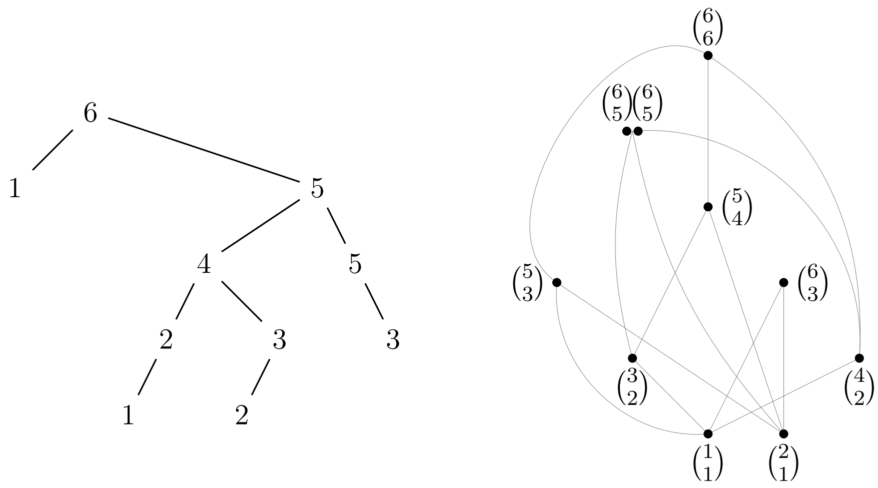


Figure 9: A Fishburn tree T and the poset $\gamma(T)$. The poset is equipped with its canonical labeling; that is, each node u is equipped with the pair of labels $\binom{\mathfrak{b}(u)}{\mathfrak{l}(u)}$.

7 Flip and sum operations on $\hat{A}$

Duality acts as an involution on $(\mathbf{2}+\mathbf{2})$-free posets. Dukes and Parviainen [12] showed that this operation is equivalent to computing the reflection of a Fishburn matrix in its antidiagonal. On the other hand, it is difficult to infer how duality acts on the corresponding ascent sequences. Similarly, the sum of two Fishburn matrices is a Fishburn matrix, but to describe the corresponding sum operation on ascent sequences is a challenging problem. In this section we solve both problems in terms of modified ascent sequences.

Let $A = (a_{i,j})$ be a $k \times k$ matrix. We denote by $\mathrm{flip}(A)$ the reflection of A in its antidiagonal; that is, the (i,j)-th entry of $\mathrm{flip}(A)$ is equal to

$$\mathrm{flip}(A)(i,j) = a_{k+1-j,k+1-i}.$$

Let $A = (a_{i,j})$ and $A' = (a'_{i,j})$ be two matrices of dimension $p \times p$ and $q \times q$, respectively, with $p \leq q$. We denote by $A + A'$ the $q \times q$ matrix obtained by summing A and A' entry by entry; that is, the (i,j)-th entry of $A + A'$ is equal to

$$(A+A')(i,j) = \begin{cases} a_{i,j} + a'_{i,j}, & \text{if } i \leq p \text{ and } j \leq p; \\ a'_{i,j}, & \text{if } i > p \text{ or } j > p. \end{cases}$$

It is easy to see that if A is a Fishburn matrix, then $\mathrm{flip}(A)$ is a Fishburn matrix as well. Similarly, the sum $A + A'$ of two Fishburn matrices A and A' is a Fishburn matrix. The flip and sum problems are formulated in terms of modified ascent sequences as follows:

- Let x be a modified ascent sequence and let $A = (\beta \circ \bar{\alpha})(x)$ be the corresponding Fishburn matrix. What is the modified ascent sequence $\mathrm{flip}(x)$ that corresponds to $\mathrm{flip}(A)$?

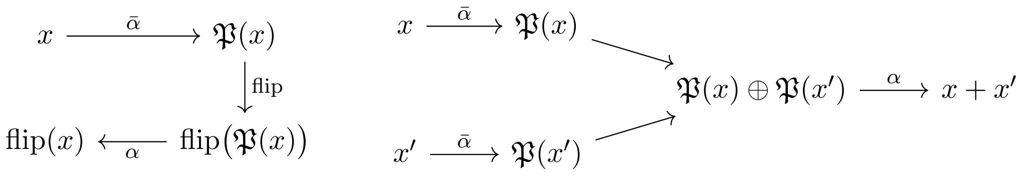


Figure 10: Diagrams to compute the flip and sum operations on modified ascent sequences.

- Let x and x' be modified ascent sequences and let $A = (\beta \circ \bar{\alpha})(x)$ and $A' = (\beta \circ \bar{\alpha})(x')$ be the corresponding Fishburn matrices. What is the modified ascent sequence $x + x'$ that corresponds to $A + A'$?

An answer to the previous two questions could be obtained by composing the bijection α, defined in Section 3, with the bijection β, defined in Section 5. For instance, we could first determine the Fishburn matrix $A = \beta\big(\bar{\alpha}(x)\big)$ associated with the modified ascent sequence x, then compute $\mathrm{flip}(A)$, and finally obtain $\mathrm{flip}(x)$ as $\alpha\big(\bar{\beta}(\mathrm{flip}(A))\big)$. However, we have defined α in terms of Fishburn trees, while β was defined in terms of Fishburn covers. To make the whole construction more straightforward, we wish to reinterpret α and its inverse $\bar{\alpha}$ in terms of Fishburn covers (see also Figure 10).

Let $x \in \hat{\mathcal{A}}$ and $A \in \mathcal{M}$. With slight abuse of notation, we denote by $\mathfrak{P}(x)$ the Fishburn cover of x; that is, we let $\mathfrak{P}(x) = \mathfrak{P}\big(\bar{\alpha}(x)\big)$. Similarly, we let $\mathfrak{P}(A) = \mathfrak{P}\big(\bar{\beta}(A)\big)$ be the Fishburn cover of A. Our first goal is to describe the composition

$$x \xmapsto{\;\bar{\alpha}\;} \mathfrak{P}(x) \xmapsto{\;\beta\;} A$$

and its inverse

$$A \xmapsto{\;\bar{\beta}\;} \mathfrak{P}(A) \xmapsto{\;\alpha\;} x,$$

thus bypassing the construction of the intermediate Fishburn tree. We spell out the main ideas below, leaving some details to the reader.

We start by redefining $\alpha : \mathcal{T} \to \hat{\mathcal{A}}$ in terms of Fishburn covers. Let $P = B_1 \dots B_k$ be a Fishburn cover. For each i, let $\overrightarrow{B_i}$ be the sequence obtained by arranging B_i in weakly decreasing order. Following Theorem 3.9, let

$$D = \{i \in [k] : i \in B_i\} \quad \text{and} \quad \bar{D} = \{i \in [k] : i \notin B_i\}.$$

Write

$$D = \{i_1, i_2, \dots, i_p\} \quad \text{and} \quad \bar{D} = \{j_1, j_2, \dots, j_q\},$$

with $p + q = k$, $i_1 < i_2 < \cdots < i_p$ and $j_1 > j_2 > \cdots > j_m$. The modified ascent sequence x associated with P is defined as follows:

1. Define $x^{(0)} = \overrightarrow{B_{i_1}}\overrightarrow{B_{i_2}} \dots \overrightarrow{B_{i_p}}$ as the sequence obtained by juxtaposing the "diagonal blocks".
2. For $s = 1, 2, \dots, q$, let $x^{(s)}$ be obtained from $x^{(s-1)}$ by inserting $\overrightarrow{B_s}$ immediately before the leftmost occurrence of the integer j_s.

Finally, the desired modified sequence is $x = x^{(q)}$. Referring once again to Theorem 3.9, the initial sequence $x^{(0)}$ is the in-order sequence of the "comb-shaped" tree T_0. The second item produces a succession of sequences $x^{(0)} \subset x^{(1)} \subset \cdots \subset x^{(q)}$, where $x^{(s)}$ is the in-order sequence of T_s, for $s = 1, 2, \ldots, q$. In particular, the insertion of $\overrightarrow{B_s}$ is analogous to the operation of attaching T_s: each insertion creates a new ascent top; ascent tops have distinct labels; and ascent tops are preserved when new blocks are appended. To illustrate this construction, let

$$P = \{1\}\{2,1\}\{2\}\{2,1\}\{5,4,2\}\{5,3,2\}\{7,6,3\}.$$

be the Fishburn cover of Figure 6. The diagonal blocks are

$$\overrightarrow{B_1} = 1, \quad \overrightarrow{B_2} = 21, \quad \overrightarrow{B_5} = 542, \quad \overrightarrow{B_7} = 763.$$

The non-diagonal blocks are

$$\overrightarrow{B_3} = 2, \quad \overrightarrow{B_4} = 21, \quad \overrightarrow{B_6} = 532.$$

By juxtaposing the diagonal blocks, we obtain

$$x^{(0)} = 1\ 21\ 542\ 763.$$

Then we insert the non-diagonal blocks, each one immediately before the leftmost occurrence of its index, starting from the one with biggest index:

$$\begin{array}{rrl} & x^{(0)} & = 121542763 \\ \overrightarrow{B_6} \longrightarrow & x^{(1)} & = 1215427\ \underline{532}\ 63 \\ \overrightarrow{B_4} \longrightarrow & x^{(2)} & = 1215\ \underline{21}\ 427\ 532\ 63 \\ \overrightarrow{B_3} \longrightarrow & x^{(3)} & = 1215\ 21\ 427\ 5\ \underline{2}\ 32\ 63 \end{array}$$

In the end, we get the modified ascent sequence

$$x = x^{(3)} = 121521427523263.$$

As expected, x is the in-order sequence of the Fishburn tree of Figure 6.

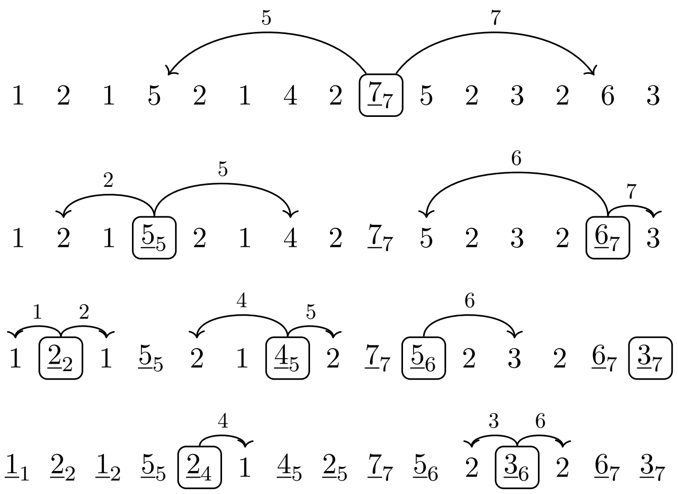
Conversely, we wish to define the Fishburn cover $\mathfrak{P}(x)$ directly on x. Equivalently, for each entry x_i we determine the index $\mathfrak{b}(x_i)$ of the maximal right path that contains the corresponding node v_i in $\bar{\alpha}(x)$. To do so, we recursively apply the max-decomposition to x, as in the definition of $\bar{\alpha}$. The label of the leftmost occurrence x_m of $\max(x)$ is $\mathfrak{b}(x_m) = x_m$. Let $y = \mathrm{pref}(y) y_m \mathrm{suff}(y)$ be the current sequence in the max-decomposition of x. Then the leftmost occurrence y_j of $\max(\mathrm{pref}(y))$ in $\mathrm{pref}(y)$ gets label

$$\mathfrak{b}(y_j) = \begin{cases} y_j & \text{if } y_m \text{ is a left-to-right maximum of } x, \\ y_m & \text{otherwise.} \end{cases}$$

While the leftmost occurrence y_j of $\max(\mathrm{suff}(y))$ in $\mathrm{suff}(y)$ gets label

$$\mathfrak{b}(y_j) = \mathfrak{b}(y_m).$$

It is not hard to see that these rules are analogous to the rules given in Equation 2 and illustrated in Figure 5. Below we apply this procedure to the modified ascent sequence $x = 121521427523263$ obtained before. At each step, the current leftmost maxima are highlighted; arrows starting from a current leftmost maximum carry the $\mathfrak{b}$-label of the target node; and $\mathfrak{b}$-labels are recorded as subscripts.



In the end, we get

$$x = 1_1\ 2_2\ 1_2\ 5_5\ 2_4\ 1_4\ 4_5\ 2_5\ 7_7\ 5_6\ 2_3\ 3_6\ 2_6\ 6_7\ 3_7,$$

and, as expected, the corresponding Fishburn cover is

$$\mathfrak{P}(x) = \{1\}\{2,1\}\{2\}\{2,1\}\{5,4,2\}\{5,3,2\}\{7,6,3\}.$$

We are now able to compute the flip and sum operations on modified ascent sequences. For convenience, we represent a Fishburn cover $P = B_1 \dots B_k$ as a biword containing a column $\binom{i}{j}$ for each $j \in B_i$, with entries in the top row sorted in increasing order, and breaking ties by sorting the bottom row in decreasing order. For instance, the Fishburn cover obtained above is written as

$$\mathfrak{P}(x) = \begin{pmatrix} 1 & 2\,2 & 3 & 4\,4 & 5\,5\,5 & 6\,6\,6 & 7\,7\,7 \\ 1 & 2\,1 & 2 & 2\,1 & 5\,4\,2 & 5\,3\,2 & 7\,6\,3 \end{pmatrix}.$$

A biword whose entries are sorted this way is called a *Burge word* [1, 6]. It is well known that Burge words are in bijection with nonnegative integer matrices whose every row and column has at least one nonzero entry: each biword is associated to a matrix whose (i,j)-th entry is equal to the number of columns $\binom{i}{j}$ contained in the biword. The map β is simply the restriction of this correspondence on Fishburn covers and Fishburn matrices.

The flip operation.

Let $A = (a_{i,j})$ be a $k \times k$ Fishburn matrix. By applying the flip operation, the (i,j)-th entry of A is mapped to the $(k+1-j, k+1-i)$-th entry of $\mathrm{flip}(A)$. In terms of Fishburn covers, the flip operation acts on the columns of $\mathfrak{P}(A)$ by

$$\begin{pmatrix} i \\ j \end{pmatrix} \longmapsto \begin{pmatrix} k+1-j \\ k+1-i \end{pmatrix}.$$

Given a Fishburn cover P, let $\mathrm{flip}(P)$ be the biword obtained by applying the above operation to each column of P (and then sorting the resulting columns accordingly). It is clear from the preceding discussion that $\mathrm{flip}(x)$ is the modified sequence of $\mathrm{flip}\big(\mathfrak{P}(x)\big)$; that is,

$$\mathrm{flip}(x) = \alpha\Big(\mathrm{flip}\big(\mathfrak{P}(x)\big)\Big)$$

Example 1. Let $x = 1612423553$ be a modified ascent sequence (note that this is the in-order sequence of the Fishburn tree depicted in Figure 9). We shall compute $\mathrm{flip}(x)$. The Fishburn cover of x is

$$\mathfrak{P}(x) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5\,5 & 6\,6\,6\,6 \\ 1 & 1 & 2 & 2 & 4\,3 & 6\,5\,5\,3 \end{pmatrix}$$

and

$$\mathrm{flip}\big(\mathfrak{P}(x)\big) = \begin{pmatrix} 1 & 2\,2 & 3 & 4\,4 & 5\,5 & 6\,6 \\ 1 & 1\,1 & 2 & 2\,1 & 4\,3 & 6\,5 \end{pmatrix}.$$

Finally, we apply α to obtain

$$\mathrm{flip}(x) = 1611214235.$$

To check that $\mathrm{flip}(x)$ is the correct sequence, it is easy to compute the corresponding matrices

$$(\beta\circ\bar{\alpha})(x) = \begin{bmatrix} 1 & & & & & \\ 1 & \cdot & & & & \\ \cdot & 1 & \cdot & & & \\ \cdot & 1 & \cdot & \cdot & & \\ \cdot & \cdot & 1 & 1 & \cdot & \\ \cdot & \cdot & 1 & \cdot & 2 & 1 \end{bmatrix} \quad\text{and}\quad (\beta\circ\bar{\alpha})\big(\mathrm{flip}(x)\big) = \begin{bmatrix} 1 & & & & & \\ 2 & \cdot & & & & \\ \cdot & 1 & \cdot & & & \\ 1 & 1 & \cdot & \cdot & & \\ \cdot & \cdot & 1 & 1 & \cdot & \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \end{bmatrix}$$

to see that each one is the flip of the other.

The sum operation.

Let A and A' be two Fishburn matrices of dimension $j \times j$ and $k \times k$, respectively, with $j < k$. Let

$$\mathfrak{P}(A) = \begin{pmatrix} 1\cdots 1 & 2\cdots 2 & \dots & j\cdots j \\ \overrightarrow{B_1} & \overrightarrow{B_2} & \dots & \overrightarrow{B_k} \end{pmatrix} \text{ and } \mathfrak{P}(A') = \begin{pmatrix} 1\cdots 1 & 2\cdots 2 & \dots & k\cdots k \\ \overrightarrow{B_1'} & \overrightarrow{B_2'} & \dots & \overrightarrow{B_k'} \end{pmatrix}.$$

It is easy to see that the Fishburn cover associated to $A + A'$ contains the union of columns of A and A'. We define a sum operation on Fishburn covers accordingly. That is, if $P = B_1 \cdots B_j$ and $P' = B_1' \cdots B_k'$ are Fishburn covers, we let

$$P + P' = \begin{pmatrix} 1\cdots 1 & 2\cdots 2 & \dots & j\cdots j & j+1\cdots j+1 & \dots & k\cdots k \\ \overrightarrow{B_1 \cup B_1'} & \overrightarrow{B_2 \cup B_2'} & \dots & \overrightarrow{B_j \cup B_j'} & \overrightarrow{B_{j+1}'} & \dots & \overrightarrow{B_k'} \end{pmatrix}.$$

Therefore, $x + x'$ is the modified sequence of $\mathfrak{P}(x) + \mathfrak{P}(x')$; that is,

$$x + x' = \alpha\big(\mathfrak{P}(x) + \mathfrak{P}(x')\big)$$

Example 2. Let $x = 1612423553$ and $x' = 113312443$ be modified ascent sequences. We wish to compute their sum $x + x'$. We have

$$\mathfrak{P}(x) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5\,5 & 6\,6\,6\,6 \\ 1 & 1 & 2 & 2 & 4\,3 & 6\,5\,5\,3 \end{pmatrix} \quad \text{and} \quad \mathfrak{P}(x') = \begin{pmatrix} 1\,1 & 2 & 3\,3\,3 & 4\,4\,4 \\ 1\,1 & 1 & 3\,3\,2 & 4\,4\,3 \end{pmatrix}.$$

Thus

$$\mathfrak{P}(x) + \mathfrak{P}(x') = \begin{pmatrix} 1\,1\,1 & 2\,2 & 3\,3\,3\,3 & 4\,4\,4\,4 & 5\,5 & 6\,6\,6\,6 \\ 1\,1\,1 & 1\,1 & 3\,3\,2\,2 & 4\,4\,3\,2 & 4\,3 & 6\,5\,5\,3 \end{pmatrix}$$

and

$$x + x' = \alpha\big(\mathfrak{P}(x) + \mathfrak{P}(x')\big) = 1113311224432643553.$$

Again, it is easy to check that the equality $\big(\beta \circ \bar{\alpha}\big)(x + x') = \big(\beta \circ \bar{\alpha}\big)(x) + \big(\beta \circ \bar{\alpha}\big)(x')$ between the corresponding matrices holds. Indeed,

$$\big(\beta \circ \bar{\alpha}\big)(x) = \begin{bmatrix} 1 & & & & & \\ 1 & \cdot & & & & \\ \cdot & 1 & \cdot & & & \\ \cdot & 1 & \cdot & \cdot & & \\ \cdot & \cdot & 1 & 1 & \cdot & \\ \cdot & \cdot & 1 & \cdot & 2 & 1 \end{bmatrix}, \qquad \big(\beta \circ \bar{\alpha}\big)(x') = \begin{bmatrix} 2 & & & \\ 1 & \cdot & & \\ \cdot & 1 & 2 & \\ \cdot & \cdot & 1 & 2 \end{bmatrix}$$

and

$$\big(\beta \circ \bar{\alpha}\big)(x + x') = \begin{bmatrix} 3 & & & & & \\ 2 & \cdot & & & & \\ \cdot & 2 & 2 & & & \\ \cdot & 1 & 1 & 4 & & \\ \cdot & \cdot & 1 & 1 & \cdot & \\ \cdot & \cdot & 1 & \cdot & 2 & 1 \end{bmatrix}.$$

8 Final remarks

We have introduced Fishburn trees and Fishburn covers, two classes of objects that transparently embody the combinatorial structure of modified ascent sequences, Fishburn matrices and (**2**+**2**)-free posets. The bijections relating these families are illustrated in Figure 11. Fishburn trees act as a central hub from which every other Fishburn structure can be easily derived. Modified ascent sequences arise from the in-order traversal of Fishburn trees; if a tree is drawn accordingly, then the sequence is simply obtained by letting the labels of the tree fall under the action of gravity (as in Figure 2). In this sense, modified ascent sequences can be regarded as vertical projections of Fishburn trees. On the other hand, Fishburn matrices and (**2**+**2**)-free posets stem from the rpath-decomposition of Fishburn trees. Referring once again to the usual representation of trees adopted in this paper, matrices and posets can be seen as the projection of Fishburn trees along their maximal right paths, that is, along the NW-SE branches.

One may view Fishburn covers as encodings of the other Fishburn structures in the following manner. Given a Fishburn cover $P = B_1 \dots B_k$, the other Fishburn objects are obtained by suitably arranging the "elementary blocks" $B_1, \dots, B_k$:

- $\mathcal{T}$: To obtain a Fishburn tree T, each elementary block B_i is encoded as a maximal right path W_i. Diagonal paths form the tree T_0. Then the other paths are attached, one by one, to the leftmost occurrence of the corresponding label. This construction has been described in Theorem 3.9.

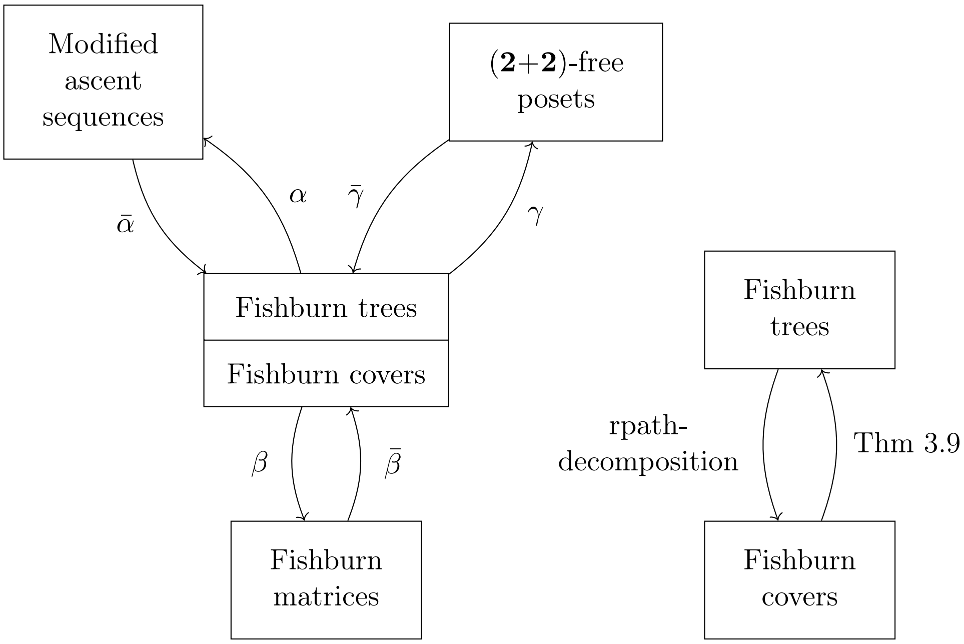


Figure 11: Bijections relating Fishburn trees and Fishburn covers to modified ascent sequences, Fishburn matrices and (**2**+**2**)-free posets.

- $\hat{\mathcal{A}}$: To obtain a modified ascent sequence x, each elementary block B_i is encoded as a decreasing sequence $\overrightarrow{B_i}$. Diagonal sequences are juxtaposed to obtain $x^{(0)}$. Then the remaining sequences are inserted one by one, each one immediately before the leftmost occurrence of the corresponding integer. This construction has been described in Section 7.
- $\mathcal{M}$: To obtain a Fishburn matrix A, each elementary block B_i is simply encoded as the i-th row of A under the action of β.
- $\mathcal{P}$: To obtain a (**2**+**2**)-free poset Q, each elementary block B_i is encoded as the difference between two consecutive strict down-sets of Q, which are strictly ordered by inclusion, under the action of γ.

In light of this, we could say that the Fishburn structures considered here fall into two categories: Fishburn trees and modified sequences are obtained by arranging their elementary blocks as dictated by the leftmost occurrences of labels or integers. On the other hand, the most trivial way of arranging elementary blocks—listing one block above the other, as rows of a matrix or as strict down-sets of a poset—leads to Fishburn matrices and (**2**+**2**)-free posets, respectively.

As a first application of our framework, we have provided a more direct solution to the flip and sum problems. With a similar approach, we can use Fishburn trees to determine how several statistics and subfamilies of Fishburn structures are related to each other. We sketch some preliminary results below, leaving a deeper investigation for future work.

A *flat step* in a modified ascent sequence x is a pair of consecutive entries $x_{i+1} = x_i$. Two elements of a poset Q are *indistinguishable* if they have the same down-set and up-set. A modified ascent sequence is *primitive* if it does not contain flat steps and a poset is *primitive* if it has no pairs of indistinguishable elements. Furthermore, a modified ascent sequence is *self-modified* if it is equal to the corresponding (plain) ascent sequence.

Proposition 8.1. *Let T be a Fishburn tree. Let $x = \alpha(T)$, $A = \beta(T)$ and $Q = \gamma(T)$ be the corresponding modified ascent sequence, Fishburn matrix, and* $(\mathbf{2}+\mathbf{2})$*-free poset, respectively. Then the following four conditions are equivalent:*

1. *T is strictly-decreasing;*
2. *x is primitive;*
3. *A is binary;*
4. *Q is primitive.*

Similarly, the following four conditions are equivalent:

1. *T is comb-shaped;*
2. *x is self-modified;*
3. *The main diagonal of A is strictly positive;*
4. *Q contains a chain of maximum length.*

We end with an open problem: Dukes and Parviainen [12] described the set of ascent sequences corresponding to bidiagonal Fishburn matrices. What is the corresponding set of Fishburn trees?

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