# WEAK ASCENT SEQUENCES AND RELATED COMBINATORIAL STRUCTURES

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Abstract. In this paper we introduce *weak ascent sequences*, a class of number sequences that properly contains ascent sequences. We show how these sequences uniquely encode each of the following objects: permutations avoiding a particular length-4 bivincular pattern; upper-triangular binary matrices that satisfy a column-adjacency rule; factorial posets that contain no *weak* (3+1) subposet. We also show how weak ascent sequences are related to a class of pattern avoiding inversion sequences that has been a topic of recent research by Auli and Elizalde. Finally, we consider the problem of enumerating these new sequences and give a closed form expression for the number of weak ascent sequences having a prescribed length and number of weak ascents.

## 1. Introduction

Ascent sequences [5] are rich number sequences in that they uniquely encode four different combinatorial objects and thereby induce bijections between these objects. These objects are (2+2)-free posets; Fishburn permutations; upper-triangular matrices of non-negative integers having neither columns nor rows of only zeros; and Stoimenow matchings. Statistics on those objects have been shown to be related to natural considerations on the ascent sequences to which they correspond.

In this paper we will define a new sequence that we term a *weak ascent* sequence and study the rich connections these sequences have to other combinatorial objects that are similar in spirit to those mentioned above. Given a sequence of integers $x = (x_1, \ldots, x_n)$, we say there is a *weak ascent* at position $i$ if $x_i \leq x_{i+1}$. We denote by $\mathrm{wasc}(x)$ the number of weak ascents in the sequence $x$. Throughout this paper we will use the notation $[a, b]$ for the set $\{a, a+1, a+2, \ldots, b\}$.

**Definition 1.** We call a sequence of integers $a = (a_1, \ldots, a_n)$ a *weak ascent sequence* if $a_1 = 0$ and $a_{i+1} \in [0, 1 + \mathrm{wasc}(a_1, \ldots, a_i)]$ for all $i \in [0, n-1]$. Let $\mathrm{WAsc}_n$ be the set of weak ascent sequences of length $n$.

In Table 1 we list all weak ascent sequences of length at most four.

To contrast this with the original ascent sequences, recall that a sequence of integers $x = (x_1, \ldots, x_n)$ has an *ascent* at position $i$ if $x_i < x_{i+1}$. An *ascent sequence* is a sequence of integers $a = (a_1, \ldots, a_n)$ with $a_1 = 0$ and $a_{i+1} \in [0, 1 + \mathrm{asc}(a_1, \ldots, a_i)]$ for all $i \in [0, n-1]$, where asc denotes the number of ascents in the sequence. Clearly, ascent sequences are weak ascent sequences.

In this paper we will show how these weak ascent sequences uniquely encode each of the following objects: permutations avoiding a particular length-4 bivincular pattern (in

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*Key words and phrases.* Weak ascent sequence; (2+2)-free poset, Bivincular pattern, Pattern avoiding permutation, Enumeration.

| $n$ | $\mathrm{WAsc}_n$ |
|---|---|
| 1 | $(0)$ |
| 2 | $(0,0)$, $(0,1)$ |
| 3 | $(0,0,0)$, $(0,0,1)$, $(0,0,2)$, $(0,1,0)$, $(0,1,1)$, $(0,1,2)$ |
| 4 | $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,0,2)$, $(0,0,0,3)$, $(0,0,1,0)$, $(0,0,1,1)$, $(0,0,1,2)$, $(0,0,1,3)$, $(0,0,2,0)$, $(0,0,2,1)$, $(0,0,2,2)$, $(0,0,2,3)$, $(0,1,0,0)$, $(0,1,0,1)$, $(0,1,0,2)$, $(0,1,1,0)$, $(0,1,1,1)$, $(0,1,1,2)$, $(0,1,1,3)$, $(0,1,2,0)$, $(0,1,2,1)$, $(0,1,2,2)$, $(0,1,2,3)$ |

TABLE 1. All weak ascent sequences of length at most 4.

Section 2); upper-triangular binary matrices that satisfy a column-adjacency rule (in Section 3); factorial posets that contain no *weak* (3+1) subposet (in Section 4). We show in Section 5 how weak ascent sequences are related to a class of pattern avoiding inversion sequences that has been a topic of recent research by Auli and Elizalde [2, 3, 4]. In Section 5 we also consider the problem of enumerating these new sequences and give a closed form expression for the number of weak ascent sequences having a prescribed length and number of weak ascents.

The objects that we study in this paper are summarised in Figure 1, along with the names of the bijections that we construct and prove between these objects. In that diagram we also include the section numbers where each of the bijections may be found.

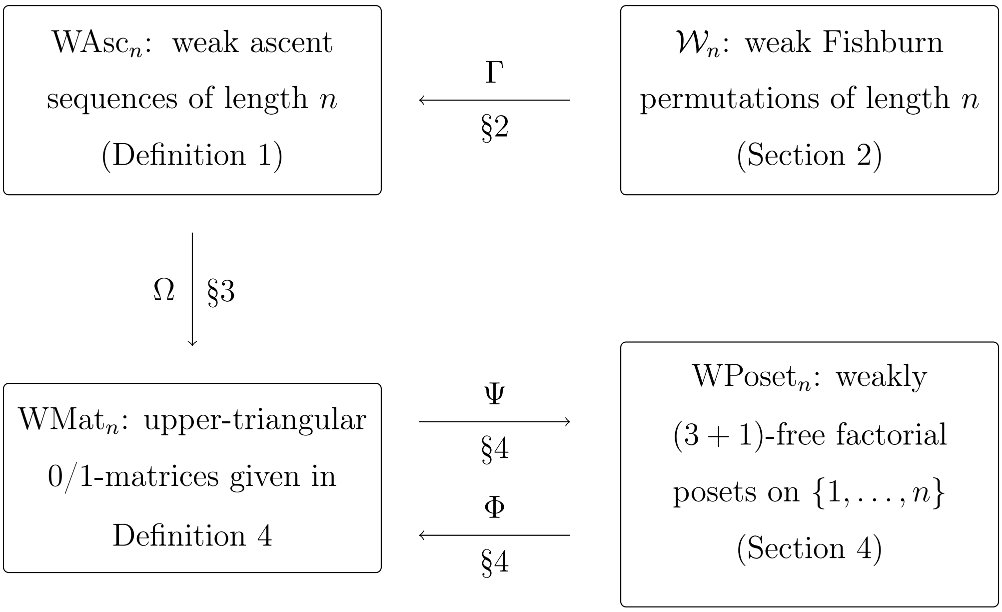
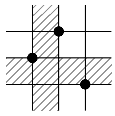



FIGURE 1. Diagrammatic summary of the sets and bijections of interest.

## 2. Weak Fishburn permutations

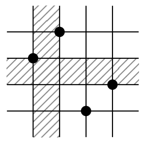
Let $S_n$ be the set of permutations of the set $\{1,\ldots,n\}$. Given a pattern $P$, in the pattern-avoidance literature the convention is to denote by $S_n(P)$ the set of permutations in $S_n$

that do not contain the pattern $P$. The set of Fishburn permutations [5, 15], $\mathcal{F}_n = S_n(F)$, are those that avoid the bivincular pattern

$$F = \big(\,231,\ [0,3]\times\{1\}\cup\{1\}\times[0,3]\,\big) =$$  ,

here defined and depicted as a mesh pattern [6]. The inclusion of shaded rows and columns indicates that in an occurrence of such a pattern in a permutation, there should be no other permutation dots in the shaded zones when this pattern is placed over a permutation. Bousquet-Mélou et al. [5] gave a bijection between ascent sequences and Fishburn permutations. More precisely, ascent sequences encode the so called active sites of the Fishburn permutations.

We define the bivincular pattern

$$W = \big(\,3412,\ [0,4]\times\{2\}\cup\{1\}\times[0,4]\,\big) =$$ 

and call $\mathcal{W}_n = S_n(W)$ the set of *weak Fishburn permutations*. For the benefit of readers not familiar with bivincular or mesh patterns we also give an elementary definition of the *weak Fishburn pattern* $W$: A permutation $\pi \in S_n$ contains $W$ if there are four indices $1 \le i < j < k < \ell \le n$ such that $j = i+1$, $\pi_i = \pi_\ell + 1$ and $\pi_k < \pi_\ell < \pi_i < \pi_j$. In this case we also say that $\pi_i\pi_j\pi_k\pi_\ell$ is an *occurrence* of $W$ in $\pi$. If there are no occurrences of $W$ in $\pi$, then we say that $\pi$ *avoids* $W$.

If $\pi_i\pi_j\pi_k\pi_\ell$ is an occurrence of $W$ then $\pi_i\pi_j\pi_\ell$ is an occurrence of $F$. In other words, every Fishburn permutation is a weak Fishburn permutation and we have $\mathcal{F}_n \subseteq \mathcal{W}_n$.

We can construct permutations in $\mathcal{W}_n$ inductively: Let $\pi$ be a permutation in $\mathcal{W}_n$ with $n > 0$. Suppose that $\tau$ is obtained from $\pi$ by deleting the entry $n$. Then $\tau \in \mathcal{W}_{n-1}$. To see why this must be the case, let $\tau_i\tau_{i+1}\tau_k\tau_\ell$ be an occurrence of $W$ in $\tau$ but not in $\pi$. This can only happen if $\pi_{i+1} = n$. However, this implies that $\pi_i\pi_{i+1}\pi_{k+1}\pi_{\ell+1}$ is an occurrence of a $W$ in $\pi$.

Given $\tau \in \mathcal{W}_{n-1}$, let us call the sites where the new maximal value $n$ can be inserted in $\tau$ so as to produce an element of $\mathcal{W}_n$ *active sites*. The site before $\tau_1$ and the site after $\tau_{n-1}$ are always active. Determining whether the site between entries $\tau_i$ and $\tau_{i+1}$ is active depends on whether $\tau_i \le 2$ or if there does not exist $(\tau_i, t, \tau_i - 1)$ in $\tau$ with $t < \tau_i - 1$. This latter (non-existence) condition is somewhat hard to absorb, so let us disentangle it as follows.

The site between entries $\tau_i$ and $\tau_{i+1}$ is active if

- $\tau_i \le 2$, or
- $\tau_i - 1$ is to the left of $\tau_i$, or
- $\tau_i - 1$ is to the right of $\tau_i$ and there is no value $t < \tau_i - 1$ between $\tau_i$ and $\tau_i - 1$.

With this notion of active sites let us label the active sites, from left to right, with $\{0, 1, 2, \dots\}$.

We will now introduce a map $\Gamma$ from $\mathcal{W}_n$ to $\mathrm{WAsc}_n$, the set of weak ascent sequences of length $n$, that we then show (in Theorem 3) to be a bijection. This mapping is defined recursively. For $n = 1$, we define $\Gamma(1) = (0)$. Next let $n \ge 2$ and suppose that $\pi \in \mathcal{W}_n$ is obtained by inserting $n$ into active site labeled $i$ of $\tau \in \mathcal{W}_{n-1}$. The sequence associated with $\pi$ is then $\Gamma(\pi) = (x_1, \dots, x_{n-1}, i)$, where $(x_1, \dots, x_{n-1}) = \Gamma(\tau)$.

**Example 2.** The permutation $\pi = 62754138$ corresponds to the sequence $x = (0, 0, 2, 1, 1, 0, 1, 5)$. It is obtained through the following recursive insertion of new maximal values into active sites. The subscripts indicate the labels of the active sites.

$$\begin{aligned}
{}_0 1_1 &\xrightarrow{x_2=0} {}_0 2_1 1_2 \\
&\xrightarrow{x_3=2} {}_0 2_1 1_2 3_3 \\
&\xrightarrow{x_4=1} {}_0 2_1 4\ 1_2 3_3 \\
&\xrightarrow{x_5=1} {}_0 2_1 5_2 4\ 1_3 3_4 \\
&\xrightarrow{x_6=0} {}_0 6\ 2_1 5_2 4\ 1_3 3_4 \\
&\xrightarrow{x_7=1} {}_0 6\ 2_1 7_2 5_3 4\ 1_4 3_5 \\
&\xrightarrow{x_8=5} \ 6\ 2\ 7\ 5\ 4\ 1\ 3\ 8.
\end{aligned}$$

In our terminology, we thus have $\Gamma(6, 2, 7, 5, 4, 1, 3, 8) = (0, 0, 2, 1, 1, 0, 1, 5)$.

**Theorem 3.** *The map $\Gamma$ is a bijection from $\mathcal{W}_n$ to $\mathrm{WAsc}_n$.*

*Proof.* The integer sequence $\Gamma(\pi)$ encodes the construction of the $W$-avoiding permutation $\pi$ so the map $\Gamma$ is injective. In order to prove that $\Gamma$ is bijective we must show that the image $\Gamma(\mathcal{W}_n)$ is the set $\mathrm{WAsc}_n$. Let $\mathrm{numact}(\pi)$ be the number of active sites in the permutation $\pi$. The recursive description of the map $\Gamma$ tells us that $x = (x_1, \dots, x_n) \in \Gamma(\mathcal{W}_n)$ if and only if

$$x' = (x_1, \dots, x_{n-1}) \in \Gamma(\mathcal{W}_{n-1}) \quad \text{and} \quad 0 \le x_n \le \mathrm{numact}\big(\Gamma^{-1}(x')\big) - 1 \tag{1}$$

Recall that the leftmost active site is labeled 0 and the rightmost active site is labeled $\mathrm{numact}(\pi) - 1$. We will now prove by induction (on $n$) that for all $\pi \in \mathcal{W}_n$, with associated sequence $\Gamma(\pi) = x = (x_1, \dots, x_n)$, one has

$$\mathrm{numact}(\pi) = 2 + \mathrm{wasc}(x) \quad \text{and} \quad \mathrm{lastact}(\pi) = x_n, \tag{2}$$

where $\mathrm{lastact}(\pi)$ is the label of the site located just before the largest entry of $\pi$. This will convert the above description (1) of $\Gamma(\mathcal{W}_n)$ into the definition of weak ascent sequences, thereby concluding the proof.

Let us focus on the properties (2). It is straightforward to see that they hold for $n = 1$. Next let us assume they hold for some $n - 1$ where $n \ge 2$. Let $\pi \in \mathcal{W}_n$ be obtained by inserting $n$ into the active site labeled $i$ of $\tau \in \mathcal{W}_{n-1}$. Then $\Gamma(\pi) = x = (x_1, \dots, x_{n-1}, i)$ where $\Gamma(\tau) = x' = (x_1, \dots, x_{n-1})$.

Every entry of the permutation $\pi$ that is smaller than $n$ is followed in $\pi$ by an active site if and only if it was followed in $\tau$ by an active site. The leftmost site in $\pi$ also remains active. The label of the active site preceding $n$ in $\pi$ is $i = x_n$, and this proves the second property. In order to determine $\mathrm{numact}(\pi)$, we must see whether the site following $n$ is active in $\pi$. There are three cases to consider. Before doing this recall that, by the induction hypothesis, we have $\mathrm{numact}(\tau) = 2 + \mathrm{wasc}(x')$ and $\mathrm{lastact}(\tau) = x_{n-1}$.

*Case 1.* If $0 \le i < \mathrm{lastact}(\tau) = x_{n-1}$ then $\mathrm{wasc}(x) = \mathrm{wasc}(x')$ and the entry $n$ in $\pi$ is to the left of $n - 1$ and there is at least one element in-between these. This 'in-between' element must be $< n - 1$, so the site after $n$ in $\pi$ cannot be active since it would lead to the creation of a $W$-pattern. The number of active sites remains unchanged and $\mathrm{numact}(\pi) = \mathrm{numact}(\tau) = 2 + \mathrm{wasc}(x') = 2 + \mathrm{wasc}(x)$.

*Case 2.* If $i = \mathrm{lastact}(\tau) = x_{n-1}$ then $\mathrm{wasc}(x) = 1 + \mathrm{wasc}(x')$ and the entry $n$ in $\pi$ is immediately to the left of $n-1$ in $\pi$. Furthermore, there are no elements in-between $n$ and $n-1$. The site that follows $n$ is therefore active, and $\mathrm{numact}(\pi) = 1 + \mathrm{numact}(\tau) = 3 + \mathrm{wasc}(x') = 2 + \mathrm{wasc}(x)$.

*Case 3.* If $i > \mathrm{lastact}(\tau) = x_{n-1}$ then $\mathrm{wasc}(x) = 1 + \mathrm{wasc}(x')$ and the entry $n$ in $\pi$ is to the right of $n-1$. The site that follows $n$ is therefore active, and $\mathrm{numact}(\pi) = 1 + \mathrm{numact}(\tau) = 3 + \mathrm{wasc}(x') = 2 + \mathrm{wasc}(x)$. □

## 3. A class of upper-triangular binary matrices

Dukes and Parviainen [13] showed how the set of upper triangular integer matrices whose entries sum to $n$ and which contain no zero rows or columns are in one-to-one correspondence with ascent sequences. A property of that correspondence is that the number of ascents in an ascent sequence equals the dimension of the corresponding matrix, while the depth of the first non-zero entry in the rightmost-column corresponds one plus the final entry of the ascent sequence.

In this section we will present a similar construction for weak ascent sequences. This correspondence is different to [13] in that the matrix entries are binary and rows of zeros will be allowed. The reason for this is that we would like the dimension of a matrix to match the number of weak ascents in the corresponding weak ascent sequence. Further to this, and in keeping with the spirit of [13], we also wish to preserve the second property "the depth of the first non-zero entry in the rightmost-column corresponds to the final entry of the weak-ascent sequence".

Let us first define the class of matrices we will be interested in and then present the correspondence between these matrices and weak ascent sequences. The notation $\dim(A)$ refers to the dimension of the matrix $A$.

**Definition 4.** Let $\mathrm{WMat}_n$ be the set of upper triangular square 0/1-matrices $A$ that satisfy the following properties:

(a) There are $n$ 1s in $A$.
(b) There is at least one 1 in every column of $A$.
(c) For every pair of adjacent columns, the topmost 1 in the left column is weakly above the bottommost 1 in the right column.

All of the matrices in $\mathrm{WMat}_1, \ldots, \mathrm{WMat}_4$ are shown in Table 2.

Given a matrix $A \in \mathrm{WMat}$, let $\mathrm{topone}(A)$ be the index such that $A_{\mathrm{topone}(A),\dim(A)} = 1$ is the topmost 1 in the rightmost column of $A$. Such a value always exists since, by Definition 4(b), there is at least one 1 in every column of $A$. Let us define

$$\mathrm{reduce}(A) = \big(B,\, \mathrm{topone}(A) - 1\big)$$

where $B$ is a copy of $A$ except that $B_{\mathrm{topone}(A),\dim(A)} = 0$ (this entry was 1 in $A$), and if this results in $B$ having a final column of all zeros then we delete that column and the bottommost row so that $B$ has dimension 1 less than $A$.

**Lemma 5.** *If* $A \in \mathrm{WMat}_n$*, with* $n \geq 2$*, and* $\mathrm{reduce}(A) = \big(B, \mathrm{topone}(A) - 1\big)$*, then* $B \in \mathrm{WMat}_{n-1}$.

| $n$ | $\mathrm{WMat}_n$ |
|---|---|
| 1 | $\begin{bmatrix} 1 \end{bmatrix}$ |
| 2 | $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| 3 | $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| 4 | $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$ $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$ $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |

TABLE 2. Matrices in our class of interest.

*Proof.* Suppose $n$ and $A$ are as stated and $\mathrm{reduce}(A) = (B, \mathrm{topone}(A) - 1)$. Let us first observe that the number of 1s in $B$ is one less than the number of 1s in $A$, and is $n - 1$. This shows property (a) of Definition 4 is satisfied. Since $A \in \mathrm{WMat}_n$ there is at least one 1 in every column of $A$, let us consider what happens in the reduction from $A$ to $B$. If there was a single 1 in the rightmost column of $A$, then it is removed along with that column and bottom row so that there is at least one 1 in every column of $B$. Alternatively, if there was more than one 1 in the rightmost column of $A$, then changing the 1 at position $(\mathrm{topone}(A), \dim(A))$ to 0 will still ensure there is at least one more 1 in that column. This shows property (b) of Definition 4 is satisfied.

Showing that property (c) in Definition 4 is preserved is a little bit more delicate. Notice that in terms of our reduction we need only consider property (c) and how things change in terms of the rightmost two columns. If there was only one 1 in the rightmost column of A, then that column will not appear in $B$ so property (c) certainly holds true in this case. If there is more than one 1 in the rightmost column of $A$, then removing it does not change the depth of the bottommost one in that column, so property (c) will still hold true. In both cases, property (c) still holds. Therefore $B \in \mathrm{WMat}_{n-1}$. $\square$

Next we will define a matrix insertion operation that is complementary to the removal operation reduce.

**Definition 6.** Given a matrix $A \in \mathrm{WMat}_n$ and an integer $i \in [0, \dim(A)]$, let us define $\mathrm{expand}(A, i)$ as follows.

(a) If $i < \mathrm{topone}(A) - 1$ then let $\mathrm{expand}(A, i)$ be the matrix $A$ with $A_{i+1,\dim(A)}$ changed from 0 to 1.
(b) If $i \geq \mathrm{topone}(A) - 1$ then let $\mathrm{expand}(A, i)$ be the matrix $A$ with a new column of 0s added to the right and a new row of 0s appended to the bottom. Then change the 0 at position $(i + 1, \dim(A) + 1)$ in $\mathrm{expand}(A, i)$ to 1.

To illustrate Definition 6 let

$$A = \begin{bmatrix} 1&1&0&1&0&0 \\ 0&0&1&1&1&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \end{bmatrix}.$$

Then

$$\mathrm{expand}(A,2) = \begin{bmatrix} 1&1&0&1&0&0 \\ 0&0&1&1&1&0 \\ 0&0&1&0&0&\mathbf{1} \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \end{bmatrix} \quad \text{and} \quad \mathrm{expand}(A,4) = \begin{bmatrix} 1&1&0&1&0&0&\mathbf{0} \\ 0&0&1&1&1&0&\mathbf{0} \\ 0&0&1&0&0&0&\mathbf{0} \\ 0&0&0&0&0&1&\mathbf{0} \\ 0&0&0&0&0&0&\mathbf{1} \\ 0&0&0&0&0&1&\mathbf{0} \\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \end{bmatrix}.$$

**Lemma 7.** *Let $n \geq 2$ and $B \in \mathrm{WMat}_{n-1}$. Let $i \in [0, \dim(A)]$ and define $A = \mathrm{expand}(B, i)$. Then $A \in \mathrm{WMat}_n$ and $\mathrm{topone}(A) = i + 1$.*

*Proof.* Let $n$, $i$, and $B$ be as stated in the lemma. In Definition 6, the two operations (a) and (b) increase the number of 1s in the matrix by 1, so the number of 1s in the matrix $A = \mathrm{expand}(B, i)$ will be $n$. This means matrix $A$ satisfies Definition 4(a). Similarly, if addition rule (a) is used then the number of 1s in a column of $A$ is at least as many as in $B$, so $A$ satisfies Definition 4(b). If addition rule (b) is used then the new column that appears has precisely one 1, so again $A$ satisfies Definition 4(b). Let us now consider the cases $i < \mathrm{topone}(B) - 1$ and $i \geq \mathrm{topone}(B) - 1$ separately.

If $i < \mathrm{topone}(B)-1$ then matrix $A$ is created by inserting a 1 into position $(i+1, \dim(B))$ of $B$ which is above the topmost 1 in that column. Consequently in the new matrix $A$ one has $\mathrm{topone}(A) = i + 1$. Furthermore, the positions of the topmost 1 in the second to last column and the bottommost 1 in the final column remain unchanged, so $A$ satisfies Definition 4(c).

If $i \geq \mathrm{topone}(B) - 1$ then $A$ is created from $B$ by adding a new column and row, and inserting a 1 at position $(i+1, \dim(B)+1)$. Notice that the topmost 1 in column $\dim(B)$ is at position $(\mathrm{topone}(B), \dim(B))$. The bottommost 1 in the final column of $A$ is now at position $(i + 1, \dim(B) + 1)$. Since $\mathrm{topone}(B) - 1 \leq i$ we have $\mathrm{topone}(B) \leq i + 1$ and again $A$ satisfies Definition 4(c). □

**Lemma 8.** *Let $B \in \mathrm{WMat}_n$ and let $i \in [0, \dim(B)]$. Then*

$$\mathrm{reduce}\big(\mathrm{expand}(B, i)\big) = (B, i)$$

*and, if $n \geq 2$,*

$$\mathrm{expand}\big(\mathrm{reduce}(B)\big) = B.$$

*Proof.* Let $A = \mathrm{expand}(B, i)$. From Lemma 7 we have $\mathrm{topone}(A) = i+1$ and the removal operation when applied to $A$ will yield $(C, i)$ for some matrix $C$. We need to show $B = C$ for the two different cases of Definition 6. Suppose that $i < \mathrm{topone}(B) - 1$. Then $A$ is a copy of $B$ with a new 1 at position $(i + 1, \dim(B))$, which becomes the topmost one in that column. The reduction operation applied to $A$ removes that topmost one in the rightmost column and the resulting matrix is $C = B$. A similar argument holds for the case $i \geq \mathrm{topone}(B) - 1$. This establishes the first part of our lemma.

Suppose $B \in \mathrm{WMat}_n$ is a matrix that has only one 1 entry in the last column. Then $\mathrm{reduce}(B) = (C, i)$, where $C$ is the matrix that we obtain by deleting the rightmost column and bottommost row of $B$ and $i = \mathrm{topone}(B) - 1$. By the property (c) of Definition 4 $\mathrm{topone}(C) \leq \mathrm{topone}(B)$. So, $i \geq \mathrm{topone}(C) - 1$ and by Definition 6(b) we have that the matrix $A = \mathrm{expand}(C, i)$ is the matrix that we obtain by appending a column with a single 1 in row $i + 1$ to the right and an all-zeros row to the bottom of $C$. Hence $A = B$. Suppose now that $B \in \mathrm{WMat}_n$ is a matrix with more than one 1 in the rightmost column. Then $\operatorname{reduce}(B) = (C, i)$, where $C$ is the matrix that we obtain by exchanging the topmost 1 in the rightmost column to 0 and $i = \operatorname{topone}(B) - 1$. Note that due to this we have $\operatorname{topone}(C) > \operatorname{topone}(B)$ and $i < \operatorname{topone}(C) - 1$. By Definition 6(a) $A = \operatorname{expand}(C, i)$ is the matrix that we obtain by changing the 0 in the $i + 1$th row in the rightmost column of $C$ to 1, hence we have $A = B$. □

Let us now define a mapping $\Omega$ from $\mathrm{WMat}_n$ to integer sequences of length $n$.

**Definition 9.** For $n = 1$ let $\Omega([\,1\,]) = (0)$. Now let $n \geq 2$ and suppose that the removal operation, when applied to $A \in \mathrm{WMat}_n$ gives $\operatorname{reduce}(A) = (B, i)$. Then the sequence associated with $A$ is $\Omega(A) = (x_1, \ldots, x_{n-1}, i)$ where $(x_1, \ldots, x_{n-1}) = \Omega(B)$.

**Theorem 10.** *The mapping $\Omega : \mathrm{WMat}_n \to \mathrm{WAsc}_n$ is a bijection.*

*Proof.* Since the sequence $\Omega(A)$ encodes the construction of the matrix $A$, the mapping $\Omega$ is injective. We have to prove that the image of $\mathrm{WMat}_n$ is the set $\mathrm{WAsc}_n$. By definition, $x = (x_1, \ldots, x_n) \in \Omega(\mathrm{WMat}_n)$ if and only if

$$x' = (x_1, \ldots, x_{n-1}) \in \Omega(\mathrm{WMat}_{n-1}) \quad \text{and} \quad x_n \in [0, \dim(\Omega^{-1}(x'))]. \tag{3}$$

We will prove by induction on $n$ that for all $A \in \mathrm{WMat}_n$, with associated sequence $\Omega(A) = x = (x_1, \ldots, x_n)$, one has

$$\dim(A) = \operatorname{wasc}(x) \quad \text{and} \quad \operatorname{topone}(A) = x_n + 1. \tag{4}$$

This will convert the description (3) above into the definition of weak ascent sequences, thus concluding the proof.

Let us examine the two statements of (4) more closely. They hold for $n = 1$. Assume they hold for $n - 1$ with $n \geq 2$ and let $A = \operatorname{expand}(B, i)$ for $B \in \mathrm{WMat}_{n-1}$. If $\Omega(B) = (x_1, \ldots, x_{n-1})$ then $\Omega(A) = (x_1, \ldots, x_{n-1}, i)$. Lemma 7 gives us that $\operatorname{topone}(A) = i + 1$ and it follows that

$$\dim(A) = \begin{cases} \dim(B) = \operatorname{wasc}(x') = \operatorname{wasc}(x) & \text{if } i < x_{n-1} \\ \dim(B) + 1 = \operatorname{wasc}(x') + 1 = \operatorname{wasc}(x) & \text{if } i \geq x_{n-1}. \end{cases}$$

The result follows from this. □

**Example 11.** Let us construct the matrix $A$ that corresponds to the weak ascent sequence $x = (0, 0, 2, 1, 1, 0, 1, 5) \in \mathrm{WAsc}_8$; that is, $\Omega(A) = x$. To begin, we have $\Omega([\,1\,]) = (0)$. From this,

$$\begin{bmatrix} 1 \end{bmatrix} \xrightarrow{x_2=0} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{x_3=2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{x_4=1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{x_5=1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{x_6=0} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{x_7=1} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{x_8=5} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = A$$

It is straightforward to see how several simple statistics get translated through the bijection $\Omega$.

**Proposition 12.** *Let $w = (w_1, \ldots, w_n) \in \mathrm{WAsc}_n$ and suppose that $M \in \mathrm{WMat}_n$ is such that $\Omega(M) = w$. Then*

- *the number of zeros in $w$ equals the sum of the entries in the top row of $M$,*
- *the number of weak ascents in $w$ equals the dimension of $M$,*
- *the number of right to left maxima of the sequence $w$ equals the sum of the entries in the rightmost column of $M$, and*
- *$w_n$, the last entry of $w$, is equal to* $\operatorname{topone}(M) - 1$.

3.1. **The image of the set of ascent sequences.** In this subsection we will characterise those matrices that correspond, via our bijection $\Omega$, to ascent sequences. First, we need some definitions. Let a 1 entry in a 0/1-matrix be called *weak* entry if (a) there are only 0s below the 1 entry in its column, and (b) there is a 1 to the left of it in the same row such that there are only 0s above this 1. More precisely,

**Definition 13.** Given a matrix $A \in \mathrm{WMat}_n$ the entry $A_{k,\ell} = 1$ is weak, if

(a) $A_{j,\ell} = 0$ for all $j > k$, and
(b) $A_{k,\ell-1} = 1$ and $A_{i,\ell-1} = 0$ for all $i < k$.

**Example 14.** In matrix $C$ the entry at position $(2,4)$ is the only weak entry, while in the matrix $D$ the entries at positions $(1,2)$ and $(2,5)$ are the weak entries.

$$C = \begin{bmatrix} 1&1&\mathbf{0}&1&0&0\\ 0&1&\mathbf{1}&\mathbf{1}&1&0\\ 0&0&1&\mathbf{0}&0&0\\ 0&0&0&\mathbf{0}&0&0\\ 0&0&0&\mathbf{0}&0&0\\ 0&0&0&\mathbf{0}&0&1 \end{bmatrix} \qquad D = \begin{bmatrix} \mathbf{1}&\mathbf{1}&0&\mathbf{0}&1&1\\ 0&\mathbf{0}&1&\mathbf{1}&\mathbf{1}&0\\ 0&\mathbf{0}&0&1&\mathbf{0}&1\\ 0&\mathbf{0}&0&0&\mathbf{0}&0\\ 0&\mathbf{0}&0&0&\mathbf{0}&0\\ 0&\mathbf{0}&0&0&\mathbf{0}&0 \end{bmatrix}$$

Next we introduce the merge operation that acts on an $n \times n$ matrix and results in an $(n-1) \times (n-1)$.

**Definition 15.** We define $A' = \mathrm{merge}(A,(k,\ell))$ as follows.

(a) Let $A'_{i,\ell-1} = A_{i,\ell-1} + A_{i,\ell}$ for $i < k$ (we add the entries in the two columns)
(b) Let $A'_{k,\ell-1} = 1$
(c) Delete the final row of $A'$. Delete the entries $A'_{i,\ell}$ for $i \leq \ell$ and $A'_{j,j}$ for $j > \ell$.

**Example 16.**

$$C = \begin{bmatrix} 1&1&0&1&0&0\\ 0&1&1&\mathbf{1}&1&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1 \end{bmatrix} \qquad \Longrightarrow \qquad \mathrm{merge}(C,(2,4)) = \begin{bmatrix} 1&1&1&0&0\\ 0&1&1&1&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix}$$

We say that a matrix $A$ is *mergeable at an entry $A_{k,\ell}$* if the entries that we delete during the merging operation $A' = \mathrm{merge}(A,(k,\ell))$ are all zeros.

**Example 17.** Matrix $C$ is not mergeable at $C_{2,4}$, because of the 1 in the bottom right corner, while $D$ is mergeable at both weak entries.

$$C = \begin{bmatrix} 1&1&0&1&0&0\\ 0&1&1&\mathbf{1}&1&0\\ 0&0&1&\mathbf{0}&0&0\\ 0&0&0&\mathbf{0}&0&0\\ 0&0&0&0&\mathbf{0}&0\\ 0&0&0&0&0&\mathbf{1} \end{bmatrix} \qquad D = \begin{bmatrix} 1&\mathbf{1}&0&0&1&1\\ 0&\mathbf{0}&1&1&1&0\\ 0&0&\mathbf{0}&1&0&1\\ 0&0&0&\mathbf{0}&0&0\\ 0&0&0&0&\mathbf{0}&0\\ 0&0&0&0&0&\mathbf{0} \end{bmatrix} \qquad D = \begin{bmatrix} 1&1&0&0&1&1\\ 0&0&1&1&\mathbf{1}&0\\ 0&0&0&1&\mathbf{0}&1\\ 0&0&0&0&\mathbf{0}&0\\ 0&0&0&0&\mathbf{0}&0\\ 0&0&0&0&0&\mathbf{0} \end{bmatrix}$$

**Lemma 18.** *Let $A \in \mathrm{WMat}_n$ and let $A_{k,\ell}$ be a weak entry. If $A$ is mergeable at $A_{k,\ell}$, then $A' = merge(A,(k,\ell)) \in \mathrm{WMat}_{n-1}$.*

*Proof.* We obtain $A'$ from the matrix $A$ by deleting only zeros and keeping all the 1 entries (by column addition of columns $\ell - 1$ and $\ell$) except at the entry $A'_{\ell-1,k}$. The entry $A'_{\ell-1,k}$ is defined to be 1. So we *loose* only one 1 entry, and we have that the number of 1s in $A'$

is $n-1$, which verifies Definition 4(a). As there is at least one 1 in every column of $A$, there is also at least one 1 in every column of $A'$ (because we deleted only zero elements in the merging operation), and this implies Definition 4(b).

Property (c) of Definition 4 only needs to be checked for the $\ell-1$th and $\ell$th column of $A'$ pair since the merge operation has, essentially, only changed these these columns. We applied the merge operation at a weak entry, and have added the $\ell$th column to the $\ell-1$th column, so the topmost 1 in the $\ell-1$th column of $A'$ is the same as the topmost 1 in the $\ell$th column of $A$. The entries in the $\ell$th column of $A$ above the diagonal are added to the left, and then deleted. Hence the $\ell$th column of $A'$ is the same as the $\ell+1$th column of $A$. Since $A \in \mathrm{WMat}_n$, the bottommost 1 in the $\ell+1$th column is weakly below the topmost 1 in the $\ell$th column. These imply that in the matrix $A'$ the topmost 1 in $\ell-1$th column is weakly above the bottommost 1 in $\ell$th column, and this implies Definition 4(c).

Since the three properties of Definition 4 hold for $A'$ we have $A' \in \mathrm{WMat}_{n-1}$. $\square$

Given a matrix $A \in \mathrm{WMat}_n$ and let $e_1, \dots, e_r$ be the sequence of weak entries in the order of their occurrences in the columns from left to right. Further, let $A^{(1)}$, $A^{(2)}$, ..., $A^{(r)}$ denote the matrices that we obtain by the operations $A^{(i+1)} = \mathrm{merge}(A^{(i)}, e_i)$. If all $A^{(i)}$ are mergeable at $e_i$, we say that the matrix $A$ is *mergeable*.

**Example 19.** The sequence of operations are shown in the next example, where $D$ is a mergeable matrix.

$$D = \begin{bmatrix} 1 & \mathbf{1} & 0 & 0 & 1 & 1 \\ 0 & \mathbf{0} & 1 & 1 & 1 & 0 \\ 0 & 0 & \mathbf{0} & 1 & 0 & 1 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \to D^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & \mathbf{1} & 0 \\ 0 & 0 & 1 & \mathbf{0} & 1 \\ 0 & 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \to D^{(2)} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**Theorem 20.** $\Omega^{-1}(\mathrm{Asc}_n)$ *is the set of mergeable matrices of* $\mathrm{WMat}_n$.

*Proof.* Given a word $w = w_1 \cdots w_n$ let an entry $w_i$ with $w_{i-1} = w_i$ be called a *plateau*. By the map $\Omega^{-1}$ a plateau in a weak ascent sequence $w$ corresponds to a weak entry in the matrix $\Omega^{-1}(w)$. Note that in an ascent sequence the appearance of a plateau $w_i$ restricts the possible values of $w_j$ for $j \geq i+1$ , though it does not influence the values in a weak ascent sequence. Suppose $w_{n-1}$ is a plateau, then the corresponding restriction in matrices (under the bijection $\Omega$) is exactly the property that the matrix is mergeable at the $n-1$th entry (the last 1 is not in the bottom right corner). Since for a matrix being mergeable is a successive application of this property the theorem follows. $\square$

## 4. A class of factorial posets

In this section we will define a mapping from the set of matrices WMat to a set of labeled posets and prove that this mapping is a bijection. First let us recall the definition of a factorial poset from [8]. To begin, a poset $P$ on the elements $\{1, \dots, n\}$ is *naturally labeled* if $i <_P j$ implies $i < j$. The poset whose Hasse diagram is depicted in Figure 2 is a naturally labeled poset.

**Definition 21** ([8])**.** A naturally labeled poset $P$ on $[1, n]$ such that, for all $i, j, k \in [1, n]$, we have

$$i < j \text{ and } j <_P k \implies i <_P k$$

is called a *factorial poset*.

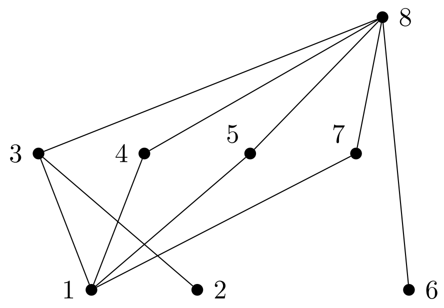



FIGURE 2. A weakly $(3+1)$-free factorial poset.

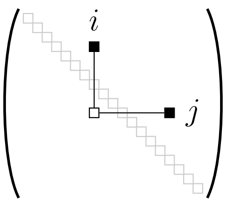
Factorial posets are (2+2)-free. This fact, and further properties of factorial posets can be found in [8].

**Definition 22** (The mapping $\Psi$)**.** Let $A \in \mathrm{WMat}_n$. Form a matrix $B$ as follows. Make a copy of $A$. Beginning with the leftmost column, and within each column one goes from bottom to top, replace every 1 that appears with the elements $1, 2, \ldots, n$. Further, define a partial order $(P, <)$ on $[1, n]$ as follows: $i <_P j$ if the index of the column that contains $i$ is strictly less than the index of the row that contains $j$. Let

$$P = \Psi(A)$$

be the resulting poset.

Diagrammatically the relation in Definition 22 is equivalent to $i$ being north-west of $j$ in the matrix and the "lower hook" of $i$ and $j$ being strictly beneath the diagonal:




Note that the set of entries contained in the first $s$ columns for an $s$ is the complete set $\{1, 2, \ldots, s_k\}$ for some $s_k$.

**Example 23.** Consider the matrix $A$ from Example 11. Form matrix $B$ by relabeling the 1s in the matrix according to the rule.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \mapsto \quad B = \begin{bmatrix} 1 & 2 & 0 & 6 & 0 & 0 \\ 0 & 0 & 4 & 5 & 7 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

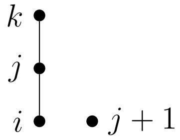
This gives the poset $(P, <) = \Psi(A)$ with the following relations:

- $1 <_P 3, 4, 5, 7, 8$
- $2 <_P 3, 8$
- $3, 4, 5, 6, 7 <_P 8$

The Hasse diagram of this poset is illustrated in Figure 2.

The mapping $\Psi$ is a mapping from $\mathrm{WMat}_n$ to a set of labeled (2+2)-free posets on the set $[1, n]$, which we will now define.

Let $P$ be a factorial poset on $[1, n]$. We say that $P$ contains a *special 3+1* if there exist four distinct elements $i < j < j+1 < k$ such that the poset $P$ restricted to $\{i, j, j+1, k\}$ induces the 3+1 poset with $i <_P j <_P k$:




If $P$ does not contain a special 3+1 we say that $P$ is *weakly* $(3+1)$*-free*. Let $\mathrm{WPoset}_n$ be the set of weakly $(3+1)$-free factorial posets on $[1, n]$.

**Theorem 24.** *Let $\Psi$ be as in Definition 22. If $A \in \mathrm{WMat}_n$, then the poset $P = \Psi(A)$ is factorial and weakly $(3+1)$-free. That is $P \in \mathrm{WPoset}_n$ so that*

$$\Psi : \mathrm{WMat}_n \to \mathrm{WPoset}_n.$$

*Proof.* Let $A \in \mathrm{WMat}_n$ and $P = \Psi(A)$. Given $i \in [1, n]$, we define the strict downset $D(i) = \{j \in [1, n] : j <_P i\}$ of $i$. A defining characteristic of a (2+2)-free poset is that the collection $\{D(i) : i \in [1, n]\}$ of strict downsets can be linearly ordered by inclusion. Similarly, a defining characteristic of a factorial poset is that each strict downset is of the form $[1, k]$ for some $k < n$. From Definition 22, it is clear why this must be the case for $P$: In constructing $P$ from the matrix $A$, the intermediate matrix $B$ contains all entries in $[1, n]$ exactly once. If $j$ appears in row $t$ of $B$, then the strict downset $D(j)$ will consist of all $i$'s that appear in columns 1 through to $t-1$ (inclusive). All elements that appear in row $t$ of $B$ have the same strict downset. Furthermore, since the entries 1, 2, …, $n$ in $B$ are such that their indices appear from left to right in increasing order, the strict downset of every element must be $[1, k]$ for some $k < n$. Thus $P$ is factorial.

Let us now suppose that $P$ contains an induced subposet on the four elements $i < j < j+1 < k$ that forms a special 3+1. In particular, $i <_P j <_P k$. Consider the matrix entries in $B$ that correspond to $i$, $j$, $j+1$ and $k$. Suppose that $i$ is at position $(i_1, i_2)$ in $B$, and that $j$ and $k$ are at positions $(j_1, j_2)$ and $(k_1, k_2)$, respectively. The hooks formed from $(i, j)$ and $(j, k)$ must be beneath the diagonal, so we must have $i_1 \le i_2 < j_1 \le j_2 < k_1 \le k_2$. Consider now $\ell = j+1$ that is at position $(\ell_1, \ell_2)$ in $B$. This element must appear either (a) in the same column as $j$ and strictly above it, or (b) in the next column and in a row weakly below $j$. For case (a) this means $\ell_1 \le \ell_2 = j_2$, from which we find that $\ell <_P k$, but this cannot happen since $\ell$ and $k$ are incomparable. For case (b) this means $j_2 \le \ell_1 \le \ell_2$, from which we find that $i <_P \ell$, but this cannot happen since $\ell$ and $i$ are incomparable. Therefore $P$ cannot contain a special 3+1. In other words, $P$ is weakly (3+1)-free and $\Psi : \mathrm{WMat}_n \to \mathrm{WPoset}_n$. □

We next define a function $\Phi$ that maps posets in $\mathrm{WPoset}_n$ to matrices. We shall show that $\Phi$ is the inverse of $\Psi$.

**Definition 25.** Given $P \in \mathrm{WPoset}_n$, suppose there are $k$ different strict downsets of the elements of $P$, and that these are $D_0 = \emptyset$, $D_1$, …, $D_{k-1}$. By convention we also let $D_k = [1, n]$. Suppose that $L_i$ is the set of elements $p \in P$ such that $D(p) = D_i$; these are called *level sets*. Let $C$ be the matrix with $C_{i,j} = L_{i-1} \cap (D_j \backslash D_{j-1})$ for all $i, j \in [1, k]$, see [11] for details. Start with $B$ as a copy of $C$ and then repeat the following steps until every entry in $B$ is at most a singleton:

A1. Choose the first column, $i$ say, of $B$ that contains a set of size $> 1$.
A2. With respect to the usual order on $\mathbb{N}$, let $\ell$ be the smallest entry in column $i$.
A3. Introduce a new empty column between columns $i-1$ and $i$ so that the old column $i$ is now column $i+1$.
A4. Move $\ell$ one column to the left (to the current column $i$) and set $j = 1$.
A5. If $\ell+j$ is strictly above $\ell+j-1$, then move it one column to the left, increase $j$ by 1, and repeat A5. Otherwise, go to A6. (The outcome of step A5 will be that the non-empty singleton sets in column $i$, from bottom to top, are $\ell, \ell+1, \ldots, \ell+t$ for some $t \geq 0$.)
A6. Introduce a new row of empty sets between rows $i-1$ and $i$ of $B$. Matrix $B$ will have increased in dimension by 1.

Finally, let $\Phi(P)$ be the result of replacing the singletons in $B$ with ones and the empty sets in $B$ with zeros.

**Example 26.** Let $P$ be the poset in Figure 2. The strict downsets of $P$ are

$$D_0 = \emptyset, D_1 = \{1\}, D_2 = \{1,2\} \text{ and } D_3 = \{1,2,3,4,5,6,7\}.$$

The level sets of $P$ are

$$L_0 = \{1,2,6\}, L_1 = \{4,5,7\}, L_2 = \{3\} \text{ and } L_3 = \{8\}.$$

This gives the matrix

$$C = \begin{bmatrix} \{1\} & \{2\} & \{6\} & \emptyset \\ & \emptyset & \{4,5,7\} & \emptyset \\ & & \{3\} & \emptyset \\ & & & \{8\} \end{bmatrix}.$$

Before continuing we would like to point out that the matrix obtained from $C$ by replacing each set $C_{i,j}$ by its cardinality $|C_{i,j}|$ corresponds, via [11, §3.1], to the unlabeled version of the poset which is (2+2)-free. Continuing, by applying A1 we find that $i = 3$ is the first column containing a set of size $> 1$. The smallest number in this column is 3. Furthermore, 4 is above 3 so we move 4 to the 3rd column, but 5 is in the same row as 4. So the largest contiguous sequence starting from 3 as we go up from it in that column (and skip rows if we wish) is $\{3,4\}$. We next go to step A6 and insert a new row with empty sets below the row of 3. Hence, the outcome of A6 will be

$$B = \begin{bmatrix} \{1\} & \{2\} & \emptyset & \{6\} & \emptyset \\ & \emptyset & \{4\} & \{5,7\} & \emptyset \\ & & \{3\} & \emptyset & \emptyset \\ & & & \emptyset & \emptyset \\ & & & & \{8\} \end{bmatrix}.$$

Since there is still a non-singleton set in this matrix we start over with A1 and find that $i = 4$ is the first column containing a set of size $> 1$. Going through the process, the outcome of A6 is

$$B = \begin{bmatrix} \{1\} & \{2\} & \emptyset & \{6\} & \emptyset & \emptyset \\ & \emptyset & \{4\} & \{5\} & \{7\} & \emptyset \\ & & \{3\} & \emptyset & \emptyset & \emptyset \\ & & & \emptyset & \emptyset & \emptyset \\ & & & & \emptyset & \emptyset \\ & & & & & \{8\} \end{bmatrix}.$$

Since there are no sets of size at least two in this matrix, we replace singletons with 1s and emptysets with 0s to find

$$\Phi(P) = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

**Theorem 27.** *Let $\Phi$ be the mapping from Definition 25. If $P$ is a poset in* $\mathrm{WPoset}_n$*, then the matrix $\Phi(P)$ is in* $\mathrm{WMat}_n$ *so that*

$$\Phi : \mathrm{WPoset}_n \to \mathrm{WMat}_n.$$

*Proof.* Let $P \in \mathrm{WPoset}_n$. Since $P$ is a factorial poset, we know that the strict downsets of elements all have the same contiguous form $[1, \ell]$ for some $\ell \in [0, n-1]$. Let us suppose that there are $k$ different strict downsets of the elements of $P$, and that these are $D_0 = \emptyset$, $D_1$, …, $D_{k-1}$. Let us further suppose that $L_i$ is the set of elements $p \in P$ such that $D(p) = D_i$, the elements at *level* $i$ of the poset. Next let $C$ be the matrix with

$$C_{i,j} = L_{i-1} \cap (D_j \backslash D_{j-1}) \text{ for all } i, j \in [1, k].$$

The matrix $C$ is upper triangular and is in fact a partition matrix. A partition matrix is an upper triangular matrix whose entries form a set partition of an underlying set with the additional property: for all $1 \leq a < b \leq n$, the column containing $b$ cannot be right of that containing $a$. See [7] for further details on partition matrices.

Moreover, since $P$ is weakly (3+1)-free, the structure of the matrix $C$ is further restricted in the following sense:

**Property 1:** The matrix $C$ does not contain four entries $i$, $j$, $j+1$ and $k$ such that the hooks for the pairs $(i, j)$ and $(j, k)$ are both below the main diagonal, whereas the hooks between $j+1$ and each of $i, j, k$ are on or above the main diagonal.

As $C$ is a partition matrix, the top left entry must be non-empty and contains the entry 1. Similarly, the bottom right entry is non-empty and contains some entry, $v$ say. Note that it is not necessarily the largest entry (with respect to the order on $\mathbb{N}$). Consider now the entry $j$ in Property 1 above, and how the condition translates with respect to these "extremal" matrix entries 1 and $v$ (in other words we are considering $i = 1$ and $k = v$). Property 1 is equivalent to

**Property 1':** The matrix $C$ does not contain an entry $j$ such that (a) $j$ is not in the top row and not in the rightmost column, and (b) $j+1$ is in the top right position of the matrix.

Again, since $C$ is a partition matrix, for all $w \in [1, n-1]$ the element $w+1$ must be in the same column as $w$, or in the one to its right. This observation, combined with Property 1' allows us to write the following equivalent statement:

**Property 1":** If $C$ contains an entry $j+1$ as a top right entry, then $j$ must either be in the top row (in that same position or immediately to its left) or in the rightmost column of $C$ beneath $j$.

Next let us consider $B$ that is constructed from $C$ in Definition 25. When a column of $B$ is split into two (as per A3), a new empty row is added in step A6 which preserves the upper-triangular property. Also, there can be no empty columns in $B$ since the dissection step A4 ensures a set of size at least 2 is split into a singleton set (that will appear in the new left column) and the set difference (that will appear one place to its right). Since $C$ is upper triangular, the construction of $B$ ensures it is upper triangular.

Furthermore, by construction, it can never be the case that on completion of all A1–A6, the entry $a+1$ appears above $a$ in the column to its right. If it were then then rule A5 would not have been executed properly. So the matrix $B$ is such that the highest indexed entry in every column is the highest entry in that column, $a$ say, is weakly above the smallest entry (with respect to the order on $\mathbb{N}$) in the subsequent column $a+1$ (which is also the lowest in that column). This condition absorbs Property 1" when one considers the final pair of columns and the entries $j$ and $j+1$.

The replacement of all singleton sets with ones and empty sets with zeros results in a matrix $\Phi(P)$ with the following propoerties:

- $\Phi(P)$ contains $n$ ones and is upper triangular.
- There are no columns consisting of all-zeros, but there can be rows of all-zeros.
- The topmost one in every column is weakly above the bottommost 1 in the column to its right.

Therefore, we have $\Phi(P) \in \mathrm{WMat}_n$. □

**Theorem 28.** *The mapping* $\Psi : \mathrm{WMat}_n \to \mathrm{WPoset}_n$ *is a bijection.*

*Proof.* We start by showing that $\Psi$ is injective. Suppose that $A$ and $A'$ are two different matrices in $\mathrm{WMat}_n$. As there are $n$ 1s in each of the matrices $A$ and $A'$, and they are different, there must be at least two positions in which they differ. In $A$ there is a 1 in position $(x_1, y_1)$ but no 1 in position $(x_2, y_2)$, however in $A'$ there is no 1 in position $(x_1, y_1)$ but a 1 in position $(x_2, y_2)$. Consider next the intermediate matrices $B$ and $B'$ in the construction and the entries $B_{x_1,y_1} = a$ and $B'_{x_2,y_2} = b$. Depending on the relative values of $x_1, y_1, x_2, y_2$, the strict downsets and strict upsets of the elements $a$ and $b$ ensure that different posets are constructed via $\Psi$.

To prove that $\Psi$ is surjective, we will show that $\Psi(\Phi(P)) = P$, thereby establishing $\Phi$ as the inverse of $\Psi$. Let $P \in \mathrm{WPoset}_n$ that has $k$ different levels $L_0, \ldots, L_{k-1}$ and down sets $D_0, \ldots, D_{k-1}$. Let $M = \Phi(P)$ be the matrix that satisfies Definition 4. Suppose that $Q = \Psi(\Phi(P)) = \Psi(M)$.

As $M \in \mathrm{WMat}_n$ we know, by Theorem 27, that $Q \in \mathrm{WPoset}_n$. The poset $Q = \Psi(M)$ that we construct using Definition 22 is such that the level $j$ of the poset $Q$ corresponds to the set of elements in the $j+1$th non-zero row of $M$ once the labelleing of the 1s in $M$ that is described in Definition 22 has taken place. Given that the $j+1$th non-zero row of $M$ corresponded to the $j$th level of $P$, we have that $L_{j+1}(Q) = L_{j+1}(P)$ for all $j$. Let $i_1, \ldots, i_k$ denote the indices of the $k$ non-empty rows of $M$. The elements of $D_j(Q)$ are all of those matrix entries (with the labelling of Definition 22) weakly to the right of column $i_j$. As $M$ was constructed from $P$ using a process of separating out columns while creating empty rows, the $j$th downset of $D_j(P)$ will coincide with that of $D_j(Q)$. □

Just as we did in the previous section, we can see how statistics between these two sets are translated:

**Proposition 29.** *Suppose that $M \in \mathrm{WMat}_n$ and $P = \Psi(M)$. Then*

- *the sum of the top row of $M$ is the number of minimal elements in $P$,*
- *the number of non-zero rows in $M$ equals the number of levels in the poset $P$,*
- *the sum of the entries in the rightmost column of $M$ equals the number of maximal elements of $P$.*

## 5. Pattern-avoiding inversion sequences and enumeration

The study of patterns in inversion sequences was recently considered by Corteel et al. [10] and continued throughout several papers [1, 2, 3, 4, 18, 19]. In a recent paper Auli and Elizalde [4] focused on vincular patterns in inversion sequences. We recall some important definitions.

It is well known that a permutation of $[1, n]$ can be encoded by an integer sequence $e_1e_2 \dots e_n = (e_1, \dots e_n)$, where $e_i$ is the number of larger elements to the left of the entry $\pi_i$. It is easy to see that every sequence $e_1e_2 \dots e_n$ with the property that $e_i \in [0, i-1]$ for all $i$ corresponds uniquely to a permutation. Such a sequence is called an *inversion sequence.*

A *vincular pattern* is a sequence $p = p_1p_2 \dots p_r$, where some disjoint subsequences of two or more adjacent entries may be underlined, satisfying $p_i \in \{0, 1, \dots, r-1\}$ for each $i$, where any value $j > 0$ can only appear in $p$ if $j-1$ appears as well. The *reduction* of a word $w = w_1w_2 \dots w_k$ is the word obtained by replacing all instances of the $i$th smallest entry of $w$ with $i > 1$, for all $i$.

An inversion sequence $e$ *avoids* the vincular pattern $p$ if there is no subsequence $e_{i_1}e_{i_2} \dots e_{i_r}$ of $e$ whose reduction is $p$, and such that $i_{s+1} = i_s + 1$ whenever $p_{i_s}$ and $p_{i_{s+1}}$ are part of the same underlined subsequence. $\mathcal{I}_n(p)$ denotes the set of inversion sequences of size $n$ that avoid the vincular pattern $p$ and $I_n(p)$ denotes the size of this set.

Auli and Elizalde [4] showed that $I_n(\underline{10}0) = I_n(\underline{10}1)$ and this number sequence ([20, A336070]) begins

$$1, 2, 6, 23, 106, 567, 3440, 23286, \dots$$

The main result of this section is the following theorem.

**Theorem 30.** *The number of length-n weak ascent sequences is the same as the number of length-n inversion sequences that avoid the vincular pattern $\underline{10}0$, and also the same as the number of length-n inversion sequences that avoid the vincular pattern $\underline{10}1$.*

We prove Theorem 30 by establishing a bijection between the sets $\mathrm{WAsc}_n$ and $\mathcal{I}(\underline{10}0)$. First, we recall a crucial property of the elements of $\mathcal{I}(\underline{10}0)$ from Auli and Elizalde [4]. Let $\mathrm{desbot}(e)$ denote the set of descent bottoms of $e$, i.e. the set of $e_i$'s with $e_{i-1} > e_i$. For each $e \in \mathcal{I}_n(\underline{10}0)$, $e_n \in \{0, 1, \dots, n-1\} \setminus \mathrm{desbot}(\bar{e})$, where $\bar{e}$ denotes the inversion sequence that we obtain by deleting the last entry of $e$.

Another important observation is that the descent bottoms are distinct elements in $e \in \mathcal{I}_n(\underline{10}0)$. Assume namely, that $e_i$ and $e_j$, with $i < j$ are descent bottoms, and $e_i = e_j$. Since $e_i$ is a descent bottom, $e_{i-1} > e_i$. Then the elements $e_{i-1}$, $e_i$, and $e_j$ would form the pattern $\underline{10}0$. On the other hand, by definition, given a weak ascent sequence $w = w_1 \cdots w_n$, we have $w_n \in \{0, 1, \dots, \mathrm{wasc}(\bar{w})\}$, and $\mathrm{wasc}(\bar{w})$ is equal to $n - 1 - |\mathrm{desbot}(\bar{w})|$.

*Proof of Theorem 30.* We define a map $\phi$ from the set $\mathcal{I}_n(\underline{10}0)$ to the set $\mathrm{WAsc}_n$. Let $e = e_1 \dots e_n$ be an inversion sequence from the set $\mathcal{I}_n(\underline{10}0)$. We map $e$ entry by entry such that $x = \phi(e)$ is becoming a weak ascent sequence. We map in each step the possible set for $e_n$ to the possible set for $w_n$. For the initial values $i = 1, 2, 3$ $e_i$ are the same as $w_i$. Let $n > 3$. We know that $e_n$ is from the set $\{0, 1, 2, \dots, n-1\} \setminus \mathrm{desbot}(\bar{e})$. But, we know also that $w_n$ is from the set $\{0, 1, \dots, n-1-|\mathrm{desbot}(\bar{w})|\}$. Since the two sets are equinumerous

$$|\{0, 1, 2, \dots, n-1\} \setminus \mathrm{desbot}(\bar{e})| = |\{0, 1, \dots, n-1-|\mathrm{desbot}(\bar{w})\}|,$$

it is easy to define a one-to-one correspondence between them that preserves the number of descent bottoms for all cases. So, if the possible values for the last entry in $e$ are $\ell_1 < \ell_2 < \dots < \ell_k$ then the map $\phi$ restricted to the last entries maps $\ell_1 \to 1$, $\ell_2 \to 2$, etc. Hence, the last entry of $e$ will be mapped to $w_n = \phi(e_n)$. In particular, if $e_n = \ell_i$, then $w_n = i$. □

**Example 31.** Let $e = 010213$.

- $e_1e_2e_3 = w_1w_2w_3 = 010$
- The only descent bottom in the word $e_1e_2e_3 = 010$ is the second 0, hence $e_4$ is from the set $\{1, 2, 3\}$, (0 is not allowed). Similarly, the possible set for $w_4$ is $\{0, 1, 2\}$ here 3 is forbidden. We map the two possible sets, $1 \to 0$, $2 \to 1$, $3 \to 2$. Now we have $e_4 = 2$, so we have $w_4 = 1$. So, we have $w_1w_2w_3w_4 = 0101$. Note that neither $e_4$ nor $w_4$ is a descent.
- The only descent bottom in the word $e_1e_2e_3e_4 = 0102$ is still the 0, so the possible set for $e_5$ is $\{1, 2, 3, 4\}$. Similarly, the possible set for $w_5$ is $\{0, 1, 2, 3\}$, 4 is not in the set. We map these two possible sets as before, and since $e_5 = 1$ we have $w_5 = 0$. Note that $w_5$ is a descent bottom, just as $e_5$.
- There are two descent bottoms in the word $e_1e_2e_3e_4e_5 = 01021$, just as in the word $w_1w_2w_3w_4w_5 = 01010$. Hence, $e_6$ is from the set $\{2, 3, 4, 5\}$, 0 and 1 are deleted from the ground set. Similarly, the possible set for $w_6$ is $\{0, 1, 2, 3\}$, where 4 and 5 are deleted from the ground set. We map the two possible sets, and obtain that $e_6 = 3$ is mapped to $w_6 = 1$. Hence, we have $w = 010101$.

We give another set of inversion sequences that are also in one-to-one correspondence with weak ascent sequences. Let $\mathcal{I}_n(\mathrm{posdt})$ denote the sequences of integers $w = w_1 \cdots w_n$ with

$$w_i \in \{0, 1, \dots, i-1\} \setminus \{j : w_j > w_{j+1}\},$$

i.e., the set of length-$n$ inversion sequences where the positions of the descent tops are forbidden as entries. Another way to describe this is as the set of inversion sequences that contain only entries at which positions a weak ascent occur. The sequence 0102 is not in $\mathcal{I}_4(\mathrm{posdt})$, because $w_2$ is a descent top, hence the value 2 is forbidden. All other length-4 inversion sequences are in $\mathcal{I}_4(\mathrm{posdt})$.

While we do not give a formal proof of the following result, let us note that it follows by an argument similar to that given for Theorem 30 in conjunction with the bijection $\Lambda$ from partition matrices to inversion sequences that was given in [7].

**Proposition 32.** *The set $\mathcal{I}_n(posdt)$ is equinumerous with the set* $\mathrm{WAsc}_n$.

Note that the inductive construction is very similar in each case, weak-ascent sequences, inversion sequences avoiding the vincular pattern, and the weak Fishburn permutations. In each case there is a set of possible values for the $j$'th entry that is determined in the prefix of length $j-1$. Auli and Elizalde [4] use the method of generating trees to derive an expression for the generating function

$$A(z) = \sum_{n\geq 0} I_n(\underline{10}0)z^n = \sum_{n\geq 0} I_n(\underline{10}1)z^n.$$

**Proposition 33** ([4] Proposition 3.12)**.** *We have that $A(z) = G(1,z)$, where $G(u,z)$ is defined recursively by*

$$G(u,z) = u(1-u) + uG(u(1+z-uz), z).$$

This expression and the bijection from the proof of Theorem 30 imply that if we denote by $A_n$ the enumeration sequence that counts the number of weak ascent sequences of length $n$, we have $A_n = \sum_{k=0}^{n} a_{n,k}$, where $a_{n,k}$ is given by the following formula. The initial values $a_{0,0}=1$, $a_{n,0}=a_{0,k}=0$ and

$$a_{n,k} = \sum_{i=0}^{n}\sum_{j=0}^{k-1}(-1)^j \binom{k-j}{i}\binom{i}{j} a_{n-i,k-j-1}. \tag{5}$$

**Proposition 34.** *The number of weak ascent sequences of length $n$ having $k$ weak ascents is $a_{n,k}$.*

*Proof.* Let $b_{n,k}$ denote the number of weak ascent sequences of length $n$ and having $k$ weak ascents. A weak ascent is an ascent, or a plateau, where we mean under a plateau an entry that is followed by the same entry. ($\pi_i$ with $\pi_i = \pi_{i-1}$.) Consider weak ascent sequences with $k$ weak ascents of length $n$ such that in the last $i$ entries there are only plateaux and descents. Let $j$ be the number of plateaux of the sequence among the last $i$ entries. One can construct such a weak ascent sequence by the following procedure. Take a weak ascent sequence of length $n-i$ with $k-j-1$ weak ascents. This means that for the entry at the $n-i+1$'th position we have a restriction that the weak ascents that are among the last $i$ entries are all plateaux (no ascents), it is at most $k-j$. However, since we have at the remaining positions only descents and plateaux, all of the entries at positions, $n-i+1, n-i+2, \ldots, n$ are at most $k-j$. Choose first $i$ distinct entries from the available $k-j$ values, which we list in a decreasing order. To create the plateaux, choose now $j$ positions (from the $i$) to make it a plateau, to equal it to the previous value. Clearly, what we get from this procedure is a desired weak ascent sequence. However, there are overlappings, we obtain each such sequence several times, so we apply the inclusion exclusion principle.

We obtain for $b_{n,k}$ the same formula as for $a_{n,k}$ in (5). Thus, $b_{n,k} = a_{n,k}$. □

Note that $a_{n,n}$ are the Catalan numbers, which is clear, since weak ascent sequences that have only ascents are in a trivial bijection for instance with Dyck paths.

**Remark 35.** Since the sequence $A_n$ has a rapid growth, greater than $n^{n/2}$, the series $\sum_{n=0}^{\infty} A_n z^n$ converges only for $z=0$. On the other hand, since $A_n \leq n!$, the exponential generating function $\sum_{n=0}^{\infty} A_n \frac{t^n}{n!}$ determines an analytic function on a certain domain. However, it could be difficult to represent it as a function by using classical functions. We did not manage to derive a nice closed formula for it.

## 6. Concluding remarks

Experimentation with restricted classes of weak ascent sequences has shown that there are relationships to other known number sequences. As an example, we offer the following simple Catalan result:

**Proposition 36.** *The number of weak ascent sequences $w = (w_1, \ldots, w_n)$ that are weakly-increasing, i.e. $w_i \leq w_{i+1}$ for all $i$, is given by the Catalan numbers.*

*Proof.* If a weak-ascent sequence is weakly increasing then there is no restriction on the entries, so the set of weakly-increasing weak ascent sequences is the same as the set of nondecreasing sequences of integers $a_i$ with $0 \leq a_i \leq i$ which are known to be enumerated by the Catalan numbers. □

A slightly different restriction gives rise to the following conjecture:

**Conjecture 37.** *The number of weak ascent sequences $w = (w_1, \ldots, w_n)$ that satisfy $w_{i+1} \geq w_i - 1$ for all $i$ equals OEIS* [20, A279567] *"Number of length $n$ inversion sequences avoiding the patterns 100, 110, 120, and 210."*

The paper [12] probed restrictions on ascent sequences and how such restrictions played out in the bijective correspondences. The above proposition and conjecture represent a first step in that direction for weak ascent sequences.

Research into pattern avoidance in ascent sequences (see Duncan and Steingrímsson [14]) proved to be a fruitful avenue of research that produced a wealth of enumerative identities and conjectures, some of which are still open. The asymptotics of generating functions for these has recently been investigated by Conway et al. [9]. We posit that a similarly rich collection of results are to be discovered by exploring pattern avoidance for weak ascent sequences.

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