Transport of patterns by Burge transpose

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Abstract

We take the first steps in developing a theory of transport of patterns from Fishburn permutations to (modified) ascent sequences. Given a set of pattern avoiding Fishburn permutations, we provide an explicit construction for the basis of the corresponding set of modified ascent sequences. Our approach is in fact more general and can transport patterns between permutations and equivalence classes of so called Cayley permutations. This transport of patterns relies on a simple operation we call the Burge transpose. It operates on certain biwords called Burge words. Moreover, using mesh patterns on Cayley permutations, we present an alternative view of the transport of patterns as a Wilf-equivalence between subsets of Cayley permutations. We also highlight a connection with primitive ascent sequences.

Keywords: Fishburn permutation, Cayley permutation, Burge word, transpose, ascent sequence, pattern avoidance.

1 Introduction

In 2010 Bousquet-Mélou, Claesson, Dukes and Kitaev [5] introduced ascent sequences, which they used as an auxiliary set of objects that most transparently embodies the recursive structure that they discovered on (2+2)-free posets, Stoimenow's matchings and a set of pattern avoiding permutations, now called Fishburn permutations. All of these objects are enumerated by the Fishburn numbers, which is sequence A022493 in the OEIS [25]. This counting sequence has a beautiful generating function [25, 27]:

$$\sum_{n\geq 0}\prod_{k=1}^{n}\big(1-(1-x)^k\big) = 1 + x + 2x^2 + 5x^3 + 15x^4 + 53x^5 + 217x^6 + \cdots$$

Since then, ascent sequences have been studied in their own right. In particular, pattern avoiding ascent sequences have been quite thoroughly investigated [3, 9, 17, 20, 21]. The study of pattern avoidance on ascent sequences has proved itself to often

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be even more intricate than its analogue on permutations and a framework capable of producing general results is missing.

Recently, Gil and Weiner [18] studied pattern avoidance on Fishburn permutations. The main purpose of this work is to initiate the development of a theory of transport of patterns from Fishburn permutations to ascent sequences, and vice versa, aiming towards a more general understanding of pattern avoidance. Instead of ascent sequences we use their modified version [5]. The main benefit is that permutations as well as modified ascent sequences are Cayley permutations, and they provide a natural setting for the transport of patterns. The necessary background on Cayley permutations and pattern avoidance is given in Section 2.

In Section 3 we introduce the Burge transpose of biwords. This operation provides a high-level description of a bijection ψ between modified ascent sequences and Fishburn permutations originally given by Bousquet-Mélou et al. [5]. In Section 4 we use the Burge transpose to define an equivalence relation on Cayley permutations and to equip its equivalence classes with a notion of pattern avoidance. The avoidance of a pattern on the quotient set is transported by Burge transposition to classical pattern avoidance on permutations, thus yielding a general result on the transport of patterns. This machinery can be specialized by suitably choosing representatives for the equivalence classes. The most striking example is a transport theorem for Fishburn permutations and modified ascent sequences, which we prove in Section 5: *Given a pattern σ we describe an explicit construction of a set of patterns $B(\sigma)$ such that Fishburn permutations avoiding σ are in one-to-one correspondence, via the Burge transpose, with modified ascent sequences avoiding all of the patterns in $B(\sigma)$.*

In Section 6, the same construction will be extensively used to derive a number of structural and enumerative results that link pattern avoiding (modified) ascent sequences to the corresponding Fishburn permutations. Table 2 contains several examples that illustrate this approach. As a corollary of the same framework, we also obtain a transport theorem for restricted growth functions and permutations avoiding the vincular pattern 23-1.

In Section 7 we "lift" the mapping ψ^{-1}, whose domain is the set of Fishburn permutations, to a new mapping η whose domains is S, the set of all permutations. The map η encodes what we call the η-active sites of a permutation. In particular, η preserves the property of transporting patterns, thus generalizing the transport theorem for Fishburn permutations. We then characterize the image set $\eta(S)$ in terms of mesh patterns on Cayley permutations. This further allows us to characterize modified ascent sequences as pattern avoiding Cayley permutations, a noteworthy consequence of which is that the transport of patterns can be regarded as a theory of Wilf-equivalence on Cayley permutations. We close Section 7 by studying the set $\eta(S) \cap S$. This set can be described as the image under η of the set of permutations in which all sites are η-active, which in turn is shown to be in bijection with primitive ascent sequences.

In Section 8 we raise some natural questions, leaving two of them as open problems.

2 Preliminaries

2.1 Cayley permutations and pattern avoidance

A word consisting of positive integers that include at least one copy of each integer between one and its maximum value is called a *Cayley permutation* [11, 22]. We will denote by Cay_n the set of Cayley permutations of length n, and by $\mathrm{Cay} = \cup_{n\geq 0} \mathrm{Cay}_n$ the set of Cayley permutations of any finite length. For example, $\mathrm{Cay}_1 = \{1\}$, $\mathrm{Cay}_2 = \{11, 12, 21\}$ and

$$\mathrm{Cay}_3 = \{111, 112, 121, 122, 123, 132, 211, 212, 213, 221, 231, 312, 321\}.$$

Equivalently, a word $x_1x_2 \dots x_n$ belongs to Cay_n precisely when there is an endofunction $x : [n] \to [n]$ such that $\mathrm{Im}(x) = [k]$ for some $k \leq n$ and $x(i) = x_i$ for each i in $[n]$. We can also view x as encoding a ballot (ordered set partition) with blocks $B_1B_2 \dots B_k$ such that $i \in B_{x(i)}$. Thus, the cardinality of Cay_n is the nth Fubini number, which is sequence A000670 in the OEIS [25].

A bijective endofunction $\pi : [n] \to [n]$ is called a *permutation* and n is said to be the length of π. We shall sometimes write permutations in so called one-line notation and thus identify π with its list of images $\pi(1)\pi(2)\cdots\pi(n)$. We will denote by id_n the identity permutation, $\mathrm{id}_n(i) = i$, in S_n. In fact, we shall often just write id (without the subscript) and let n be inferred by context. Denote by S_n the set of permutations of length n and by $S = \cup_{n\geq 0} S_n$ the set of permutations of any finite length. Note that $S \subseteq \mathrm{Cay}$.

Given two Cayley permutations x and y, we say that y is a *pattern* in x if x contains a subsequence $x(i_1)x(i_2)\cdots x(i_k)$, with $i_1 \leq i_2 \leq \cdots \leq i_k$, which is order isomorphic to y; that is, $x(i_s) < x(i_t)$ if and only if $y(s) < y(t)$ and $x(i_s) = x(i_t)$ if and only if $y(s) = y(t)$. In this case we write $y \leq x$ and $x(i_1)x(i_2)\cdots x(i_k) \simeq y$; the subsequence $x(i_1)x(i_2)\cdots x(i_k)$ is called an *occurrence* of y in x. Otherwise, x *avoids* y. Denote by $\mathrm{Cay}(y)$ the set of Cayley permutations that avoid y and by $\mathrm{Cay}_n(y)$ the set $\mathrm{Cay}(y)\cap \mathrm{Cay}_n$ of Cayley permutations of length n avoiding y. For example, $S = \mathrm{Cay}(11)$ is the set of permutations. If B is a set of patterns, $\mathrm{Cay}(B)$ denotes the set of Cayley permutations avoiding every pattern in B and $\mathrm{Cay}_n(B)$ denotes $\mathrm{Cay}_n \cap \mathrm{Cay}(B)$. We use analogous notations for subsets of Cay. For instance, $\hat{A}(212, 312)$ denotes the set of modified ascent sequences (defined in Section 2.2) avoiding the two patterns 212 and 312. The containment relation is a partial order on S and downsets in this poset are called *permutation classes*. Similarly, the containment relation is a partial order on Cay and downsets in this poset are called *Cayley permutation classes*. The *basis* of a (Cayley) permutation class is the minimal set of (Cayley) permutations it avoids. For instance, the basis for S in Cay is $\{11\}$. For a more detailed introduction to permutation patterns we refer the reader to Bevan's note "Permutation patterns: basic definitions and notations" [4].

The set S of permutations can be equipped with more general notions of patterns [2, 5, 6, 14]. A *bivincular pattern* [5] of length k is a triple (σ, X, Y), where X and Y are subsets of $\{0, 1, \dots, k\}$ and $\sigma \in S_k$. An occurrence of (σ, X, Y) in a permutation $\pi \in S_n$ is then an occurrence $\pi(i_1)\cdots\pi(i_k)$ of σ (in the classical sense) such that:

- $i_{\ell+1} = i_\ell + 1$, for each $\ell \in X$;

- $j_{\ell+1} = j_\ell + 1$, for each $\ell \in Y$,

where $\{\pi(i_1), \dots, \pi(i_k)\} = \{j_1, \dots, j_k\}$, with $j_1 < \cdots < j_k$; by convention, $i_0 = j_0 = 0$ and $i_{k+1} = j_{k+1} = n+1$. The set X identifies constraints of adjacency on the positions of the elements π, while the set Y, symmetrically, identifies constraints on their values. An example of a bivincular pattern is depicted in Figure 1. If Y is empty, then (σ, X, Y) is called a *vincular* pattern.

By allowing more general constraints on positions and values we arrive at mesh patterns. A *mesh pattern* [6] is a pair (σ, R), where $\sigma \in S_k$ is a permutation (classical pattern) and $R \subseteq [0,k] \times [0,k]$ is a set of pairs of integers. The pairs in R identify the lower left corners of unit squares in the plot of π which specify forbidden regions. An occurrence of the mesh pattern (σ, R) in the permutation π is an occurrence of the classical pattern σ such that no other points of the permutation occur in the forbidden regions specified by R.

Two subsets of Cay are *equinumerous* if they contain the same number of Cayley permutations of each length. Equivalently, if they have the same generating function. Two sets of (generalized) patterns B_1 and B_2 are *Wilf-equivalent* if $S(B_1)$ and $S(B_2)$ are equinumerous. We extend this notion to Cayley permutations by saying that B_1 and B_2 are Wilf-equivalent (over Cay) if $\mathrm{Cay}(B_1)$ and $\mathrm{Cay}(B_2)$ are equinumerous.

2.2 Ascent sequences

Let $x : [n] \to [n]$ be an endofunction. We call $i \in [n-1]$ an *ascent* of x if $x(i) < x(i+1)$. Let $\mathrm{asc}(x)$ denote the number of ascents of x. Then x is an *ascent sequence* of length n if $x(1) = 1$ and $x(i+1) \leq 2 + \mathrm{asc}\big(x \circ \mathrm{id}_{i,n}\big)$ for each $i \in [n-1]$, where $\mathrm{id}_{i,n} : [i] \to [n]$ is the inclusion map. Let A_n be the set of ascent sequences of length n. For instance, $A_3 = \{111, 112, 121, 122, 123\}$. Note that some ascent sequences are not Cayley permutations, the smallest example of which is 12124. Note also that we depart slightly from the original definition of ascent sequences [5] in that our sequences are one-based rather then zero-based. The reason for this is that we want to bring all the families of sequences considered in this paper under one umbrella, namely that of endofunctions on $[n]$.

We shall now define the set of *modified ascent sequences* [5], denoted $\hat{A}_n$. This set, which is equinumerous with A_n, has a recursive structure that is similar to, but more complicated than, that of A_n. The definition goes as follows. There is exactly one modified ascent sequence of length zero, namely the empty word. There is also exactly one modified ascent sequence of unit length, namely the single letter word 1. Suppose $n \geq 2$. Every $x \in \hat{A}_n$ is of one of two forms depending on whether the last letter forms an ascent with the penultimate letter:

- $x = \upsilon a$ and $1 \leq a \leq b$, or
- $x = \tilde{\upsilon} a$ and $b < a \leq 2 + \mathrm{asc}(\upsilon)$,

where $\upsilon \in \hat{A}_{n-1}$, the last letter of υ is b, and $\tilde{\upsilon}$ is obtained from υ by increasing each entry $c \geq a$ by one.

Lemma 2.1. *Let $x \in \hat{A}_n$ be a modified ascent sequence. An element $x(i) = k > 1$ is the leftmost occurrence of the integer k in x if and only if $x(i-1) < x(i)$. In particular, x is a Cayley permutation.*

Proof. We proceed by induction on the length of x, using the recursive definition of $\hat{A}$. If $n = 0$ or $n = 1$, then there is nothing to prove. Suppose $n \geq 2$ and let $x \in \hat{A}$. Let $a = x(n)$. We have either

- $x = va$ and $1 \leq a \leq b$, or
- $x = \tilde{v}a$ and $b < a \leq 2 + \operatorname{asc}(v)$,

where $v \in \hat{A}_{n-1}$, the last letter of v is b and $\tilde{v}$ is obtained from v by increasing each entry $c \geq a$ by one. Note that in both cases x is a Cayley permutation since $v \in \mathrm{Cay}$ by the inductive hypothesis. Moreover, the desired property holds for v (again, by the inductive hypothesis) and it holds for $\tilde{v}$ as well since increasing each element greater than or equal to a certain value by one preserves it. Now, if $1 \leq a \leq b$, then v already contains an occurrence of a (since v is a Cayley permutation) and therefore $x(n)$ is not the leftmost occurrence of a in x. Finally, if $b < a \leq 2 + \operatorname{asc}(v)$, then $x(n)$ is the only (and thus leftmost) occurrence of a in x. $\square$

By Lemma 2.1, for any modified ascent sequence x the ascent tops of x together with the first element, $x(1) = 1$, form a permutation of length $\max(x)$. Consequently $\max(x) = 1 + \operatorname{asc}(x)$ and $\hat{A}_n \subseteq \mathrm{Cay}_n$. To see that $|A_n| = |\hat{A}_n|$ we will give a bijection $x \mapsto \hat{x}$ from A_n to $\hat{A}_n$. Given an ascent sequence x, let

$$M(x, j) = x', \text{ where } x'(i) = x(i) + \begin{cases} 1 & \text{if } i < j \text{ and } x(i) \geq x(j+1), \\ 0 & \text{otherwise,} \end{cases}$$

and extend the definition of M to multiple indices $j_1, j_2, \ldots, j_k$ by

$$M(x, j_1, j_2, \ldots, j_k) = M\big(M(x, j_1, \ldots, j_{k-1}), j_k\big).$$

We let $\hat{x} = M\big(x, \operatorname{Asc}(x)\big)$, where $\operatorname{Asc}(x) = \big(i : x(i) < x(i+1)\big)$ denotes the vector of ascents of x. For example, if $x = 121242232$, then $\operatorname{Asc}(x) = (1, 3, 4, 7)$ and we get:

$$\begin{aligned} x &= 121242232 \\ M(x,1) &= \underline{12}1242232 \\ M(x,1,3) &= 1\mathit{3}\underline{12}42232 \\ M(x,1,3,4) &= 131\underline{24}2232 \\ M(x,1,3,4,7) &= 1\mathit{4}125\mathit{5}2\underline{23}2 = \hat{x} \end{aligned}$$

The construction described above can easily be inverted and thus the mapping $x \mapsto \hat{x}$ is a bijection. The recursive definition of $\hat{A}_n$ is equivalent to saying that $\hat{A}_n$ is the image of A_n under the $x \mapsto \hat{x}$ mapping. Indeed, $\hat{A}_n$ was originally defined [5] in this manner.

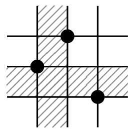
$\mathfrak{f} =$ 

Figure 1: Bivincular pattern $\mathfrak{f}$ characterizing Fishburn permutations

2.3 Fishburn permutations

Define the bivincular pattern $\mathfrak{f} = (231, \{1\}, \{1\})$, as in Figure 1. Let $F = S(\mathfrak{f})$. The elements of F are called *Fishburn permutations*. Bousquet-Mélou et al. [5] gave a length-preserving bijection between ascent sequences and Fishburn permutations. More precisely, ascent sequences encode the so called *active sites* of the Fishburn permutations. The term active site comes from the generating tree approach to enumeration. Each vertex in such a tree corresponds to a combinatorial object and the path from the root to a vertex encodes the choices made in the construction of the object. Regarding Fishburn permutations, let us construct an element of F_{n+1} by starting from an element of F_n and inserting a new maximum in some position. The avoidance of the pattern $\mathfrak{f}$ makes some of the positions forbidden, while the others are the active sites. More precisely, let $\pi \in F_n$ be a Fishburn permutation. For $i \in [n]$, if $\pi(i) = 1$ let $J(i) = 0$, otherwise let $J(i)$ be the index such that $\pi(J(i)) = \pi(i) - 1$. Counting from the position to the left of the first entry of π, position 1 is always active and position $i+1$ is active if and only if $J(i) < i$. In all the other cases, the insertion of $n+1$ immediately after $\pi(i)$ would result in an occurrence $\pi(i), n+1, \pi(J(i))$ of $\mathfrak{f}$. Now, the empty ascent sequence corresponds to the empty permutation. The ascent sequence corresponding to a nonempty Fishburn permutation $\pi \in F$ is constructed as follows. Start from the permutation 1 and the sequence 1. Record the position in which you insert the new maximum, step by step, until you get π. To illustrate this map consider the permutation $\pi = 319764825$. It is obtained by the following insertions, where the subscripts indicate the labels of the active sites, while positions between consecutive elements that have no subscript are forbidden sites.

$$
\begin{aligned}
{}_1 1_2 &\xmapsto{x_2 = 2} {}_1 1_2 2_3\\
&\xmapsto{x_3 = 1} {}_1 3\ 1_2 2_3\\
&\xmapsto{x_4 = 2} {}_1 3\ 1_2 4_3 2_4\\
&\xmapsto{x_5 = 4} {}_1 3\ 1_2 4_3 2_4 5_5\\
&\xmapsto{x_6 = 2} {}_1 3\ 1_2 6\ 4_3 2_4 5_5\\
&\xmapsto{x_7 = 2} {}_1 3\ 1_2 7\ 6\ 4_3 2_4 5_5\\
&\xmapsto{x_8 = 3} {}_1 3\ 1_2 7\ 6\ 4_3 8_4 2_5 5_6\\
&\xmapsto{x_8 = 2} {}_1 3\ 1_2 9\ 7\ 6\ 4_3 8_4 2_5 5_6.
\end{aligned}
$$

Therefore the ascent sequence corresponding to π is $x = 121242232$. This procedure can also be viewed as constructing π from a given ascent sequence by successive

$$
\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "x \mapsto \hat{x}"'] & F \\
\hat{A} \arrow[ur, "\psi"'] &
\end{tikzcd}
$$

Figure 2: How the bijections $x \mapsto \hat{x}$, φ and ψ are related

insertions of a new maximum in the active site specified by the ascent sequence. Throughout this paper we will denote this mapping from ascent sequences to Fishburn permutations by φ, so that $\varphi(x) = \pi$. For a proof that $\varphi : A \to F$ is a bijection the interested reader is again referred to Bousquet-Mélou et al. [5].

Next we recall (from [5]) the construction of a map $\psi : \hat{A} \to F$ such that $\psi(\hat{x}) = \varphi(x)$ for each ascent sequence x. It will play a central role in transporting patterns from ascent sequences to Fishburn permutations. As we will see, ψ is much easier to handle than φ. The relation between the bijections $x \mapsto \hat{x}$, φ and ψ is illustrated by the commutative diagram in Figure 2.

Let $\hat{x}$ be a modified ascent sequence. Write the integers 1 through n below it, and sort the pairs $\binom{\hat{x}(i)}{i}$ in ascending order with respect to the top entry, breaking ties by sorting in descending order with respect to the bottom entry. The resulting bottom row is the permutation $\psi(\hat{x})$. For example, with $\hat{x} = 141252232$, the modified sequence of $x = 121242232$, we have

$$
\begin{pmatrix} \hat{x} \\ \mathrm{id} \end{pmatrix} = \begin{pmatrix} 1\,4\,1\,2\,5\,2\,2\,3\,2 \\ 1\,2\,3\,4\,5\,6\,7\,8\,9 \end{pmatrix} \longmapsto \begin{pmatrix} 1\,1\,2\,2\,2\,2\,3\,4\,5 \\ 3\,1\,9\,7\,6\,4\,8\,2\,5 \end{pmatrix} = \begin{pmatrix} \upsilon(\pi) \\ \pi \end{pmatrix}
$$

To reverse this process, annotate a given Fishburn permutation π with its active sites as in $\pi = {}_1 3 1_2 9 7 6 4_3 8_4 2_5 5_6$. Write k above all entries $\pi(j)$ that lie between active sites k and $k+1$. In the example, this forms the word $\upsilon(\pi)$ above π. Then sort the pairs $\binom{k}{\pi(j)}$ in ascending order with respect to the bottom entry. This defines ψ^{-1}, the inverse of the map ψ.

It turns out that it is more natural to place the identity permutation above $\hat{x}$, rather than below it. Then

$$
\begin{pmatrix} \upsilon(\pi) \\ \pi \end{pmatrix}^T = \begin{pmatrix} \mathrm{id} \\ \hat{x} \end{pmatrix}
$$

is a special case of transposing matrices in a sense that we describe in the next section.

3 The Burge transpose

Let M_n be the set of matrices with nonnegative integer entries whose every row and column has at least one nonzero entry and are such that the sum of all entries is equal to n. For instance, M_2 consists of the following five matrices:

$$
\begin{pmatrix} 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

With each matrix $A = (a_{ij})$ in M_n we associate a biword in which every column $\binom{i}{j}$ appears a_{ij} times and the columns are sorted in ascending order with respect to the top entry, breaking ties by sorting in descending order with respect to the bottom entry. The biwords corresponding to the five matrices above are

$$\begin{pmatrix}1\,1\\1\,1\end{pmatrix}, \begin{pmatrix}1\,1\\2\,1\end{pmatrix}, \begin{pmatrix}1\,2\\1\,1\end{pmatrix}, \begin{pmatrix}1\,2\\1\,2\end{pmatrix}, \begin{pmatrix}1\,2\\2\,1\end{pmatrix}.$$

Note that if i appears in the bottom row of such a biword, then each k such that $1 \leq k < i$ also appears in the bottom row. This follows from the requirement that each column of the corresponding matrix has at least one nonzero entry. In other words, the bottom row is a Cayley permutation. Similarly, the top row is a Cayley permutation. In fact, it is a weakly increasing Cayley permutation.

Let I_n be the subset of Cay_n consisting of the weakly increasing Cayley permutations:

$$I_n = \{u \in \mathrm{Cay}_n : u(1) \leq u(2) \leq \cdots \leq u(n)\}.$$

To ease notation we will often write biwords as pairs. As an example, the first two biwords in the list corresponding to matrices in M_2 would be written $(11, 11)$ and $(11, 21)$. In general, the set of biwords corresponding to matrices in M_n is

$$\mathrm{Bur}_n = \{(u, v) \in I_n \times \mathrm{Cay}_n : D(u) \subseteq D(v)\},$$

where $D(v) = \{i : v(i) \geq v(i+1)\}$ is the set of weak descents of v. We shall call the elements of Bur_n *Burge words*. This terminology is due to Alexandersson and Uhlin [1]. The connection to Burge is with his variant of the RSK correspondence [7]. Since u is weakly increasing we have $D(u) = \{i : u(i) = u(i+1)\}$. In particular,

$$|\mathrm{Bur}_n| = \sum_{v \in \mathrm{Cay}_n} 2^{\mathrm{des}(v)},$$

where $\mathrm{des}(v) = |D(v)|$ is the number of weak descents in v. This is sequence A120733 in the OEIS [25].

The simple operation of transposing a matrix in M_n turns out to be surprisingly useful. Assume that $A = (a_{ij}) \in M_n$ and that w is its corresponding biword in Bur_n. Let w^T denote the biword corresponding to the transpose $A^T = (a_{ji})$ of A. It is easy to compute w^T without taking the detour via the matrix A. Turn each column of w upside down and then sort the columns as previously described. In particular, if π is a permutation, then

$$\begin{pmatrix}\mathrm{id}\\ \pi\end{pmatrix}^T = \begin{pmatrix}\mathrm{id}\\ \pi^{-1}\end{pmatrix}.$$

Also, if $\pi = \psi(\hat{x})$ is the Fishburn permutation corresponding to the modified ascent sequence $\hat{x}$ and $\upsilon(\pi)$ is as described in the previous section, then

$$\begin{pmatrix}\upsilon(\pi)\\ \pi\end{pmatrix}^T = \begin{pmatrix}\mathrm{id}\\ \hat{x}\end{pmatrix},$$

Let $\mathcal{F}_n = \{(\upsilon(\pi), \pi) : \pi \in F_n\}$ and $\hat{\mathcal{A}}_n = \{(\mathrm{id}, \hat{x}) : \hat{x} \in \hat{A}_n\}$. Then the correspondence between Fishburn permutations and modified ascent sequences is the identity

$$\mathcal{F}_n^T = \hat{\mathcal{A}}_n,$$

where $\mathcal{F}_n^T = \{w^T : w \in \mathcal{F}_n\}$ is the image of $\mathcal{F}_n$ under T.

It is clear that Bur_n is closed under transpose. In fact, the definition of transposition given above applies to all biwords in $I_n \times \mathrm{Cay}_n$ and it gives an alternative characterization of the set Bur_n, as seen in the following lemma.

Lemma 3.1. *Let $w = (u, v) \in I_n \times \mathrm{Cay}_n$. Then $D(u) \subseteq D(v)$ if and only if $(w^T)^T = w$. Moreover, T is an involution on Bur_n.*

Proof. By definition of T, the biword w^T is a Burge word. Therefore $(I_n \times \mathrm{Cay}_n)^T \subseteq \mathrm{Bur}_n$. If $D(u) \subseteq D(v)$, then both w and $(w^T)^T$ are Burge words, and since they share the same set of columns, we must have $w = (w^T)^T$. Conversely, suppose that $(w^T)^T = w$. Then $w = z^T$, for $z = w^T$, and so w is a Burge word, or, equivalently, $D(u) \subseteq D(v)$, as desired. That T is an involution on Bur_n is immediate. □

It is well known that the nth Eulerian polynomial evaluated at 2 equals the nth Fubini number. That is,

$$|\mathrm{Cay}_n| = \sum_{\pi \in S_n} 2^{\mathrm{des}(\pi)}. \tag{1}$$

The following proof is taken from Stanley [26, Exercise 131(a), Chapter 1]. To each pair (π, E), with $\pi \in S_n$ and $E \subseteq D(\pi)$, we bijectively associate a ballot of $[n]$: Draw a vertical bar between $\pi(i)$ and $\pi(i+1)$ if i is an ascent or $i \in E$. Thus, if $\pi = 319764825$ and $E = \{1, 5\} \subseteq D(\pi) = \{1, 3, 4, 5, 7\}$ we get the ballot 3|1|976|4|82|5.

We shall reformulate this proof in terms of the transpose of Burge words. First a definition. The *direct sum* $u \oplus v$ of two Cayley permutations u and v is the concatenation uv', where v' is obtained from v by adding $\max(u)$ to each of its elements. For instance, $12 \oplus 1112 \oplus 11 \oplus 1 = 123334556$. We further extend the direct sum to sets U and V of Cayley permutations:

$$U \oplus V = \{u \oplus v : u \in U, v \in V\}.$$

Let us now return to the proof of Equation (1). Let π be a permutation of $[n]$. A *descending run* of π is a maximal sequence of consecutive descending letters $\pi(i) > \pi(i+1) > \cdots > \pi(i+d-1)$. Let $\pi = B_1 B_2 \cdots B_t$ be the decomposition of π into descending runs and let $\ell(i) = |B_i|$ be the length of the ith descending run. The descending runs of the example permutation $\pi = 319764825$ are 31, 9764, 82 and 5. The lengths of those runs are 2, 4, 2 and 1. The next step is to pick a weakly increasing Cayley permutation that is a direct sum of sequences of the same lengths as the descending runs. That is, we will pick u from $I_2 \oplus I_4 \oplus I_2 \oplus I_1$. Since $|I_k| = 2^{k-1}$ there are $2 \cdot 8 \cdot 2 \cdot 1 = 32$ possible choices for u. Say we pick $u = 12 \oplus 1112 \oplus 11 \oplus 1 = 123334556$. Then

$$\binom{u}{\pi}^T = \binom{1\,2\,3\,3\,3\,4\,5\,5\,6}{3\,1\,9\,7\,6\,4\,8\,2\,5}^T = \binom{1\,2\,3\,4\,5\,6\,7\,8\,9}{2\,5\,1\,4\,6\,3\,3\,5\,3} = \binom{\mathrm{id}}{v}$$

and the resulting Cayley permutation is $v = 251463353$, which encodes the same ballot, $\{3\}\{1\}\{9, 7, 6\}\{4\}\{8, 2\}\{5\}$, as in the previous example.

For $\pi \in S_n$, let

$$I(\pi) = I_{\ell(1)} \oplus \cdots \oplus I_{\ell(t)},$$

where t is the number of descending runs of π, or, equivalently,

$$I(\pi) = \{u \in I_n : D(u) \subseteq D(\pi)\}. \tag{2}$$

Define the set $B(\pi) \subseteq \mathrm{Cay}_n$ by

$$\big(I(\pi) \times \{\pi\}\big)^T = \{\mathrm{id}\} \times B(\pi).$$

We call $B(\pi)$ the *Fishburn basis* of π. The reason will become evident later. In particular,

$$|B(\pi)| = |I(\pi)| = 2^{\mathrm{des}(\pi)}. \tag{3}$$

Alternatively, let the underlying permutation of a ballot be obtained by sorting elements within blocks decreasingly and then removing the curly brackets. Thus, the underlying permutation of $\{3\}\{1\}\{9,7,6\}\{4\}\{8,2\}\{5\}$ is 319764825. This defines a natural surjection from ballots to permutations and $B(\pi)$ is exactly the collection of encodings of ballots whose underlying permutation is π. In particular,

$$\bigcup_{\pi \in S_n} B(\pi) = \mathrm{Cay}_n$$

in which the union is disjoint. Equation (1) follows in view of (3).

Example. Let $\pi = 3142$. The descending runs of π are 31 and 42. Further,

$$I(3142) = I_2 \oplus I_2 = \{1122, 1123, 1233, 1234\}.$$

The Fishburn basis $B(3142)$ is defined by

$$\big(I(3142) \times \{3142\}\big)^T = \{1234\} \times B(3142),$$

where

$$\begin{aligned}\big(I(3142) \times \{3142\}\big)^T &= \left\{ \binom{1122}{3142}^T, \binom{1123}{3142}^T, \binom{1233}{3142}^T, \binom{1234}{3142}^T \right\} \\ &= \left\{ \binom{1234}{1212}, \binom{1234}{1312}, \binom{1234}{2313}, \binom{1234}{2413} \right\}.\end{aligned}$$

At the end we get $B(3142) = \{1212, 1312, 2313, 2413\}$. The corresponding ballots, each of which has 3142 as the underlying permutation, are $\{3,1\}\{4,2\}$, $\{3,1\}\{4\}\{2\}$, $\{3\}\{1\}\{4,2\}$, and $\{3\}\{1\}\{4\}\{2\}$.

4 The transport theorem

The main goal of this section is to prove a transport theorem for Cay and S. We first define a notion of pattern containment on Bur for which the Burge transpose T behaves nicely. More specifically, T preserves pattern containment when it is used to map Cay onto S (Corollary 4.7). Transporting patterns in the other direction, that is from S to Cay, requires the introduction of a notion of equivalence on Cay. The main result of this section, Theorem 4.9, is a transport theorem for the resulting equivalence classes of Cayley permutations and classical permutations. In Section 5 we will specialize Theorem 4.9 to obtain a transport theorem for $\hat{A}$ and F (Theorem 5.1), which is the most tangible application of the proposed framework. To help the reader see where we are heading let us paraphrase this theorem here: *For any permutation σ we have*

$$F(\sigma) = \psi\big(\hat{A}(B(\sigma))\big).$$

In other words, the set $F(\sigma)$ of Fishburn permutations avoiding σ is mapped via the bijection ψ^{-1} to the set $\hat{A}(B(\sigma))$ of modified ascent sequences avoiding all patterns in the Fishburn basis $B(\sigma)$.

Consider the map $\Gamma : \mathrm{Bur}_n \to \mathrm{Cay}_n$ defined by

$$\binom{u}{v}^T = \binom{y}{\Gamma(u,v)},$$

for any $(u, v) \in \mathrm{Bur}_n$. Let us write $\mathrm{sort}(v)$ for the word obtained by sorting v in weakly increasing order. Then $y = \mathrm{sort}(v)$ and, since T is an involution, $u = \mathrm{sort}(\Gamma(u, v))$. For example, in the previous section we computed

$$\begin{pmatrix} 1\,2\,3\,3\,3\,4\,5\,5\,6 \\ 3\,1\,9\,7\,6\,4\,8\,2\,5 \end{pmatrix}^T = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 2\,5\,1\,4\,6\,3\,3\,5\,3 \end{pmatrix},$$

that is $\Gamma(123334556, 319764825) = 251463353$.

Definition 4.1. Let $E \subseteq \mathrm{Cay}$ be a set of Cayley permutations. A *Burge labeling* on E is a map $\lambda : E \to I$ such that $(\lambda(x), x)$ is a Burge word for each $x \in E$. Equivalently, $D(\lambda(x)) \subseteq D(x)$ for each $x \in E$.

Let λ be a Burge labeling on E. Then λ induces a map $\Gamma_\lambda : E \to \mathrm{Cay}$ by

$$\Gamma_\lambda(x) = \Gamma(\lambda(x), x).$$

If λ is injective, then Γ_λ is also injective. Indeed suppose that $\Gamma_\lambda(x) = \Gamma_\lambda(y)$. Then $\lambda(x) = \mathrm{sort}(\Gamma_\lambda(x)) = \mathrm{sort}(\Gamma_\lambda(y)) = \lambda(y)$ and thus $x = y$, if λ is injective.

This construction becomes particularly meaningful for specific labelings. Let ι : $\mathrm{Cay} \to I$ be defined by $\iota(x) = \mathrm{id}_n$, for each $x \in \mathrm{Cay}_n$. Since $D(\iota(x)) = \emptyset$, the mapping ι is clearly a Burge labeling on Cay. From now on, let $\gamma = \Gamma_\iota$.

Lemma 4.2. *We have* $\mathrm{sort}(x) = x \circ \gamma(x)$ *for each Cayley permutation x.*

Proof. To ease notation, let $\pi = \gamma(x)$. By definition of γ we have $\big(\mathrm{id}_n, x\big)^T = \big(\mathrm{sort}(x), \pi\big)$. The ith column of this biword is $\big(\mathrm{sort}(x)(i), \pi(i)\big)$. By definition of Burge transpose, $\big(\pi(i), \mathrm{sort}(x)(i)\big)$ must be a column of $\big(\mathrm{id}_n, x\big)$. Indeed, it is the $\pi(i)$th column of $\big(\mathrm{id}_n, x\big)$, but that column is plainly also equal to $\big(\pi(i), x(\pi(i))\big)$, and hence $\mathrm{sort}(x)(i) = x(\pi(i))$ as claimed. □

Remark 4.3. If $\pi \in S_n$, then $\mathrm{id}_n = \mathrm{sort}(\pi) = \pi \circ \gamma(\pi)$ by Lemma 4.2. That is, $\gamma(\pi) = \pi^{-1}$. In this sense, $\gamma : \mathrm{Cay} \to S$ generalizes the permutation inverse to Cay.

Remark 4.4. Recall that, for any modified ascent sequence x, we have

$$\big(\mathrm{id}, x\big)^T = \big(\mathrm{sort}(x), \psi(x)\big),$$

where $\psi : \hat{A} \to F$ is the bijection described in Section 2.3. Thus, restricting ι to $\hat{A}$ gives the map $\Gamma_{\iota_{|\hat{A}}} = \psi$. That is, $\gamma_{|_{\hat{A}}} = \psi$ and in this sense γ generalizes ψ to Cay. On the other hand, consider the map $\upsilon : F \to I$ introduced in Section 2.3. It is easy to see that υ is a Burge labeling on F and $\Gamma_\upsilon : F \to \hat{A}$ is equal to ψ^{-1}, the inverse map of ψ.

Next, we extend the pattern containment relation from Cayley permutations to Burge words.

Definition 4.5. Let (u', v') in Bur_k and (u, v) in Bur_n. Then $(u', v') \leq (u, v)$ if there is an increasing injection $\alpha : [k] \to [n]$ such that $u \circ \alpha$ and $v \circ \alpha$ are order isomorphic to u' and v', respectively. In other words, there is a subset of columns determined by the indices $\alpha([k]) = \{i_1, \dots, i_k\}$, with $i_1 < \cdots < i_k$, such that both $u(i_1) \cdots u(i_k) \simeq u'$ and $v(i_1) \cdots v(i_k) \simeq v'$. We also say that $\big(u(i_1) \cdots u(i_k), v(i_1) \cdots v(i_k)\big)$ is an *occurrence* of (u', v') in (u, v).

As an important special case, $(\mathrm{id}_k, v') \leq (\mathrm{id}_n, v)$ if and only if $v' \leq v$. The next lemma shows that the Burge transpose behaves well with respect to pattern containment on biwords.

Lemma 4.6. *Let* $(u, x) \in \mathrm{Bur}_n$ *and* $(v, y) \in \mathrm{Bur}_k$. *Then:*

$$(v, y) \leq (u, x) \iff (v, y)^T \leq (u, x)^T.$$

Proof. Suppose that $(v, y) \leq (u, x)$ and let $(u', x') = (u(i_1) \cdots u(i_k), x(i_1) \cdots x(i_k))$ be an occurrence of (v, y) in (u, x). The relative order of any pair of columns is not affected by the remaining columns when T is applied. In other words, the effect of T on (u', x') is the same as the one on the (order isomorphic) biword (v, y). Therefore (u', v') is mapped by T to an occurrence of $(v, y)^T$ in $(u, x)^T$, as desired. The converse follows from T being an involution. $\square$

Corollary 4.7. *Let* $x \in \mathrm{Cay}_n$.

1. *If* $y \in \mathrm{Cay}_k$ *and* $y \leq x$, *then* $\gamma(y) \leq \gamma(x)$.
2. *If* $\sigma \in S_k$ *and* $\sigma \leq \gamma(x)$, *then there exists* $y \in \mathrm{Cay}_k$ *such that* $y \leq x$ *and* $\gamma(y) = \sigma$.

Proof. The first statement follows by letting $u = \mathrm{id}_n$ and $v = \mathrm{id}_k$ in Lemma 4.6. For the second statement, suppose that $\sigma \leq \gamma(x)$ and let $\gamma(x)(i_1) \cdots \gamma(x)(i_k)$ be an occurrence of σ in $\gamma(x)$. We have:

$$\binom{\mathrm{id}_n}{x}^T = \binom{\mathrm{sort}(x)}{\gamma(x)} = \begin{pmatrix} \dots & \mathrm{sort}(x)(i_1) & \dots & \mathrm{sort}(x)(i_k) & \dots \\ \dots & \gamma(x)(i_1) & \dots & \gamma(x)(i_k) & \dots \end{pmatrix}.$$

Let $v \in I_k$ be the only weakly increasing Cayley permutation that is order isomorphic to $\mathrm{sort}(x)(i_1) \dots \mathrm{sort}(x)(i_k)$. Note that $(\mathrm{sort}(x), \gamma(x)) \geq (v, \sigma)$. Thus, again by Lemma 4.6, we have:

$$\binom{\mathrm{id}_n}{x} = \binom{\mathrm{sort}(x)}{\gamma(x)}^T \geq \binom{v}{\sigma}^T = \binom{\mathrm{id}_k}{\Gamma(v, \sigma)}.$$

Therefore, for $y = \Gamma(v, \sigma)$ we have $x \geq y$ and $\gamma(y) = \sigma$, as desired. $\square$

Example. Let $x = 251463353$. We have

$$\begin{pmatrix}\mathrm{id}\\ x\end{pmatrix}^T = \begin{pmatrix}\underline{1}\,\underline{2}\,3\,4\,5\,6\,\underline{7}\,\underline{8}\,9\\ \underline{2}\,\underline{5}\,1\,4\,6\,3\,\underline{3}\,\underline{5}\,3\end{pmatrix}^T = \begin{pmatrix}1\,\underline{2}\,3\,\underline{3}\,3\,4\,\underline{5}\,\underline{5}\,6\\ 3\,\underline{1}\,9\,\underline{7}\,6\,4\,\underline{8}\,\underline{2}\,5\end{pmatrix} = \begin{pmatrix}\mathrm{sort}(x)\\ \gamma(x)\end{pmatrix}.$$

The underlined subset of columns highlights that T maps the occurrence $(1278, 2535)$ of $(1234, 1323)$ to an occurrence $(2355, 1782)$ of $(1233, 1342) = (1234, 1323)^T$ as per Lemma 4.6, with $(v, y) = (1234, 1323)$, and Corollary 4.7, with $y = 1323$. On the other hand, every occurrence of 1342 in $\gamma(x)$ does not necessarily come from an occurrence of 1323 in x. For example, $(2356, 1785)$ is the image under T of $(1578, 2635)$. Here $2635 \simeq 1423$ whereas in the previous case the occurrence 1782 of 1342 resulted from $2535 \simeq 1323$. Note that $\gamma(1423) = \gamma(1323) = 1342$ as per Corollary 4.7 with $\sigma = 1342$.

The first item of Corollary 4.7 shows that when the Burge transpose is used as a map from Cay to S—by means of γ—it preserves pattern containment. The second item of the same corollary shows that when the Burge transpose is used as a map from S to Cay—by means of the computation $(\mathrm{sort}(x), \gamma(x))^T = (\mathrm{id}_n, x)$—an occurrence of a pattern σ in $\gamma(x)$ is mapped to an occurrence of some y in x, with $\gamma(y) = \sigma$. This leads us to define an equivalence relation $\sim$ on Cay by $y \sim y'$ if and only if $\gamma(y) = \gamma(y')$. Denote by $[y]$ the equivalence class

$$[y] = \{y' \in \mathrm{Cay} : y \sim y'\}$$

of y, and denote by [Cay] the quotient set

$$[\mathrm{Cay}] = \{[y] : y \in \mathrm{Cay}\}.$$

Since $\sim$ is the equivalence relation induced by γ, there is a unique injective map $\tilde{\gamma}$ such that the diagram

$$\begin{tikzcd}
\mathrm{Cay} \arrow[r, "\gamma"] \arrow[d, "x \mapsto [x]"'] & S \\
{[\mathrm{Cay}]} \arrow[ur, "\tilde{\gamma}"'] &
\end{tikzcd}$$

commutes. Furthermore, since γ is surjective, $\tilde{\gamma}$ is surjective too. Indeed, for any permutation σ, we have $\tilde{\gamma}([\sigma^{-1}]) = \gamma(\sigma^{-1}) = \sigma$. Thus $\tilde{\gamma}$ is a bijection and the quotient set [Cay] is equinumerous with S, the set of permutations. By slight abuse of notation we will write γ for $\tilde{\gamma}$ as well. That is, we have two functions $\gamma : \mathrm{Cay} \to S$ and $\gamma : [\mathrm{Cay}] \to S$, and it should be clear from the context which one is referred to. Equivalence classes of Cayley permutations up to length four are listed in Table 1.

Next we extend the notion of pattern containment to [Cay].

Definition 4.8. Let $[x]$ and $[y]$ in [Cay]. Then $[x] \geq [y]$ if $x' \geq y'$ for some $x' \in [x]$ and $y' \in [y]$.

Corollary 4.7 can then be reformulated as a transport theorem between permutations and equivalence classes of Cayley permutations.

Equivalence class $[y]$	$\gamma(y)$
$\underline{1}$	$\underline{1}$
$\underline{12}$	$\underline{12}$
$\underline{11}, 21$	$\underline{21}$
$\underline{123}$	$\underline{123}$
$\underline{122}, 132$	$\underline{132}$
$\underline{112}, 213$	$\underline{213}$
$212, 312$	231
$\underline{121}, 231$	$\underline{312}$
$\underline{111}, 211, 221, 321$	$\underline{321}$
$\underline{1234}$	$\underline{1234}$
$\underline{1233}, 1243$	$\underline{1243}$
$\underline{1223}, 1324$	$\underline{1324}$
$1323, 1423$	1342
$\underline{1232}, 1342$	$\underline{1423}$
$\underline{1222}, 1322, 1332, 1432$	$\underline{1432}$

Equivalence class $[y]$	$\gamma(y)$
$\underline{1123}, 2134$	$\underline{2134}$
$\underline{1122}, 1132, 2133, 2143$	$\underline{2143}$
$2123, 3124$	2314
$3123, 4123$	2341
$2132, 3142$	2413
$2122, 3122, 3132, 4132$	2431
$\underline{1213}, 2314$	$\underline{3124}$
$\underline{1212}, 1312, 2313, 2413$	$\underline{3142}$
$\underline{1112}, 2113, 2213, 3214$	$\underline{3214}$
$2112, 3112, 3213, 4213$	3241
$2312, 3412$	3412
$2212, 3212, 3312, 4312$	3421
$\underline{1231}, 2341$	$\underline{4123}$
$\underline{1221}, 1321, 2331, 2431$	$\underline{4132}$
$\underline{1121}, 2131, 2231, 3241$	$\underline{4213}$
$2121, 3121, 3231, 4231$	4231
$\underline{1211}, 2311, 2321, 3421$	$\underline{4312}$
$\underline{1111}, 2111, 2211, 2221, 3211, 3221, 4321$	$\underline{4321}$

Table 1: Equivalence classes $[y] \in [\mathrm{Cay}]$ and their image $\gamma(y)$ for $1 \le |y| \le 4$. Modified ascent sequences and Fishburn permutations are underlined.

Theorem 4.9 (The transport theorem). *Let $x, y \in \mathrm{Cay}$. Then*

$$[x] \ge [y] \iff \gamma(x) \ge \gamma(y)$$

or, equivalently,

$$\gamma\big([\mathrm{Cay}][y]\big) = S\big(\gamma(y)\big).$$

Proof. Suppose that $[x] \ge [y]$, that is, $x' \ge y'$ for some $x' \in [x]$ and $y' \in [y]$. Then $\gamma(x') \ge \gamma(y')$ by Corollary 4.7. By definition of the equivalence relation we have $\gamma(x') = \gamma(x)$ and $\gamma(y') = \gamma(y)$, and hence $\gamma(x) \ge \gamma(y)$. Conversely, if $\gamma(x) \ge \gamma(y)$, then by Corollary 4.7 there exists y' such that $\gamma(y') = \gamma(y)$, that is, $y' \in [y]$ and $x \ge y'$, from which $[x] \ge [y]$ follows. $\square$

The equivalence class of y is none other than the Fishburn basis of $\sigma = \gamma(y)$:

Lemma 4.10. *For $y \in \mathrm{Cay}$ we have $[y] = B(\gamma(y))$. Moreover, for each permutation $\sigma \in S$, we have $B(\sigma) = [\sigma^{-1}]$.*

Proof. Let $\sigma = \gamma(y) \in S_n$. We will start by showing the inclusion $B(\sigma) \subseteq [y]$. Let $x \in B(\sigma)$. By definition of Fishburn basis we have

$$(\mathrm{id}, x)^T = (\mathrm{sort}(x), \sigma)$$

and $\mathrm{sort}(x) \in I(\sigma)$. On the other hand, by definition of γ we have

$$(\mathrm{id}, x)^T = (\mathrm{sort}(x), \gamma(x)).$$

Thus $\gamma(x) = \sigma = \gamma(y)$ and $x \in [y]$. Conversely, let $x \in [y]$; that is, $\gamma(x) = \sigma$. Then

$$(\mathrm{id}, x)^T = (\mathrm{sort}(x), \gamma(x)) = (\mathrm{sort}(x), \sigma).$$

We need to show that $\mathrm{sort}(x) \in I(\sigma)$. Since $(\mathrm{sort}(x), \sigma) \in \mathrm{Bur}_n$ we have $D(\mathrm{sort}(x)) \subseteq D(\sigma)$, which in turn is equivalent to $\mathrm{sort}(x) \in I(\sigma)$ by Equation (2). Thus $B(\sigma) = [y]$ as claimed. Finally, $\gamma(\sigma^{-1}) = \Gamma(\mathrm{id}, \sigma^{-1}) = \sigma$, and therefore $\sigma^{-1} \in B(\sigma)$. □

If σ is a permutation, then—by Lemma 4.10—we can choose σ^{-1} as representative for the Fishburn basis of σ and thus we have the following corollary.

Corollary 4.11. *If σ is a permutation, then $\gamma(S(\sigma)) = [\mathrm{Cay}][\sigma^{-1}]$.*

A remarkable consequence is that the sets $S(\sigma)$ and $[\mathrm{Cay}][\sigma^{-1}]$ are equinumerous. In fact, we can say a bit more. By Lemma 4.10 and Equation (3) we have $|[y]| = 2^{\mathrm{des}(\gamma(y))}$, which leads to the following result relating the Eulerian polynomial on $S_n(\sigma)$ to a polynomial recording the distribution of (the logarithm of) sizes of equivalence classes in $[\mathrm{Cay}_n][\sigma^{-1}]$. The special case $t = 2$ can be seen as a generalization of Equation (1).

Corollary 4.12. *For any natural number n and permutation σ,*

$$\sum_{\pi \in S_n(\sigma)} t^{\mathrm{des}(\pi)} = \sum_{[y] \in [\mathrm{Cay}_n][\sigma^{-1}]} t^{\log |[y]|},$$

in which the logarithm is with respect to the base 2.

We end this section by providing some results on the equivalence relation $\sim$ introduced in this section. First we show that $\sim$ does not depend on our choice of ι as Burge labeling in the definition of $\gamma = \Gamma_\iota$.

Lemma 4.13. *Let (u, x) and (u, y) be Burge words that have the same top row. If $x \sim y$, then $\Gamma(u, x) = \Gamma(u, y)$.*

Proof. Let $\pi = \gamma(x) = \gamma(y)$. Note that $\pi \in S_n$. Let $u \in I_n$. By definition of Γ we have $(u, x)^T = (\mathrm{sort}(x), \Gamma(u, x))$. Moreover, by Lemma 4.2, $\mathrm{sort}(x) = x \circ \pi$ and hence

$$(u, x)^T = \big(x \circ \pi, \Gamma(u, x)\big).$$

It follows that $\Gamma(u, x) = u \circ \pi$. Similarly, $\Gamma(u, y) = u \circ \pi$, concluding the proof. □

Our next goal is to prove that the pattern containment relation is a partial order on the set $[\mathrm{Cay}]$.

Lemma 4.14. *Let $x, y \in \mathrm{Cay}$. The following two statements are equivalent:*

1. *$[x] \geq [y]$.*
2. *For each $x' \in [x]$, there exists $y' \in [y]$ such that $x' \geq y'$.*

Proof. Suppose that $[x] \geq [y]$. That is, there are two Cayley permutations $x' \in [x]$ and $y' \in [y]$ such that $x' \geq y'$. By Lemma 4.7, we have $\gamma(x') \geq \gamma(y')$. Let $\bar{x} \in [x]$. Then $\gamma(\bar{x}) = \gamma(x') \geq \gamma(y')$. Thus, again by Lemma 4.7, there exists $\bar{y} \in [y'] = [y]$ such that $\bar{x} \geq \bar{y}$, as desired. The other implication is trivial. □

Proposition 4.15. *The containment relation is a partial order on* [Cay].

Proof. Reflexivity is trivial. To show transitivity, suppose that $[x] \geq [y]$ and $[y] \geq [z]$. Then there are $x' \in [x]$ and $y' \in [y]$ such that $x' \geq y'$. Further, since $[y] \geq [z]$ and $y' \in [y]$ there is $z' \in [z]$ such that $y' \geq z'$ by Lemma 4.14. Thus $x' \geq z'$ and $[x] \geq [z]$. It remains to show antisymmetry. If $[x] \geq [y]$ and $[y] \geq [x]$, then, as in the proof of transitivity, there are three elements $x' \in [x]$, $y' \in [y]$ and $x'' \in [x]$ such that $x' \geq y' \geq x''$. But then $x' = x''$, since Cayley permutations in the same equivalence class have the same length. Thus $x' = y' = x''$ and $[x] = [y]$. $\square$

5 Transport of patterns from F to $\hat{A}$

Theorem 4.9 can be specialized by choosing a representative in each equivalence class of [Cay]. Among the resulting examples, the most significant one is that of transport of patterns between Fishburn permutations and modified ascent sequences. Consider the bijection $\psi = \gamma_{|\hat{A}}$ of Remark 4.4. If σ is a Fishburn permutation, then $\psi^{-1}(\sigma) \in [\sigma^{-1}]$. In other words, the map ψ^{-1} picks exactly one representative in the equivalence class $[\sigma^{-1}]$, see Table 1. Let σ and π be Fishburn permutations. By Theorem 4.9

$$\pi \geq \sigma \iff [\psi^{-1}(\pi)] \geq [\psi^{-1}(\sigma)] = [\sigma^{-1}].$$

By Lemma 4.14 we therefore have $\pi \geq \sigma$ if and only if $\psi^{-1}(\pi) \geq \sigma'$ for some $\sigma' \in [\sigma^{-1}]$. Since $\psi : \hat{A} \to F$ is bijective and $[\sigma^{-1}] = B(\sigma)$ is the Fishburn basis of σ we obtain the promised transport theorem.

Theorem 5.1 (Transport of patterns from F to $\hat{A}$)**.** *For any permutation σ and Cayley permutation y we have*

$$F(\sigma) = \gamma\big(\hat{A}[\sigma^{-1}]\big) \quad \textit{and} \quad \gamma\big(\hat{A}[y]\big) = F\big(\gamma(y)\big)$$

In other words, the set $F(\sigma)$ of Fishburn permutations avoiding σ is mapped via the bijection ψ^{-1} to the set $\hat{A}(B(\sigma))$ of modified ascent sequences avoiding all patterns in the Fishburn basis $B(\sigma)$.

Corollary 5.2. *For any permutation σ we have $|F_n(\sigma)| = |\hat{A}_n(B(\sigma))|$.*

Recall that a constructive procedure for determining the Fishburn basis $B(\sigma) = [\sigma^{-1}]$ was described at the end of Section 3. In Table 1 we listed all the equivalence classes $[y]$ and the corresponding image $\gamma(y)$ up to length four. In each case, as stated in Theorem 5.1, we have $\gamma\big(\hat{A}[y]\big) = F\big(\gamma(y)\big)$. Equivalently, if $\sigma = \gamma(y)$ then $[y] = [\sigma^{-1}]$ and $\gamma\big(\hat{A}[\sigma^{-1}]\big) = F(\sigma)$.

Theorem 5.1 can be easily generalized to Fishburn permutations avoiding a set of patterns:

Theorem 5.3. *Let Ω be a set of permutation patterns. Then*

$$F(\Omega) = \gamma\big(\hat{A}[\Omega^{-1}]\big), \ \textit{where} \ [\Omega^{-1}] = \bigcup_{\sigma \in \Omega} [\sigma^{-1}].$$

6 Examples

Theorems 5.1 provides a bijection between the set $F(\sigma)$ of Fishburn permutations avoiding σ and the set $\hat{A}(B(\sigma))$ of modified ascent sequences avoiding all the patterns in the Fishburn basis $B(\sigma)$. A construction for $B(\sigma)$ has been explicitly described at the end of Section 3. In this section we collect many examples where this approach can be pushed further by interpreting this correspondence in terms of (plain) ascent sequences. As a corollary of the same framework we obtain, in Section 6.3, a transport theorem for set partitions encoded as restricted growth functions, and the set of permutations avoiding the vincular pattern 23-1.

Let Ω be a set of patterns and define

$$A_\Omega = \big\{x \in A : \hat{x} \in \hat{A}[\Omega^{-1}]\big\}.$$

The following diagram illustrates the bijections described in Figure 2 and Theorem 5.3.

$$\begin{tikzcd}
A_\Omega \arrow[rr, "\varphi"] \arrow[d, "x \mapsto \hat{x}"'] & & F(\Omega) \\
\hat{A}[\Omega^{-1}] \arrow[urr, "\gamma = \psi"'] & &
\end{tikzcd}$$

Below we provide several cases where the set A_Ω can be described in terms of pattern avoidance, that is, where $A_\Omega = A(C(\Omega))$ for a set of Cayley permutations $C(\Omega)$. In these cases, we obtain a transport theorem that links ascent sequences directly to Fishburn permutations:

$$\varphi\big(A(C(\Omega))\big) = F(\Omega).$$

An overview of some of the results obtained this way can be found in Table 2. In what follows we just sketch some of the ideas and proofs, leaving the details to the reader.

6.1 Transport of a single pattern

When $\Omega = \{\sigma\}$ is a singleton it is sometimes possible to show that there is a pattern $y \in \mathrm{Cay}$ such that $A_{\{\sigma\}} = A(y)$. Equivalently, $x \in A(y)$ if and only if $\hat{x} \in \hat{A}[\sigma^{-1}]$. As a result we get the transport of a single pattern:

$$\varphi\big(A(y)\big) = F(\sigma).$$

Below we show the case $\sigma = 2134$, the other cases being similar.

Example. Let $\sigma = 2134$. Observe that $[2134^{-1}] = \{1123, 2134\}$. We shall prove that $A_{\{2134\}} = A(1123)$, that is, $x \in A(1123)$ if and only if $\hat{x} \in \hat{A}[2134^{-1}] = \hat{A}(1123, 2134)$. We first prove that $\hat{A}(1123, 2134) = \hat{A}(1123)$.

Lemma 6.1. *We have $\hat{A}(1123, 2134) = \hat{A}(1123)$.*

Ω	$C(\Omega)$	Counting Sequence
21	11	$1, 1, \ldots$
12	12	$1, 1, \ldots$
213	112	2^{n-1}
312	121	2^{n-1}
132	122	2^{n-1}
123	123	2^{n-1}
231	212	Catalan
3142	1212	Catalan
2134	1123	Catalan
1423	132	Catalan
3412	312	$A202062$
$231, 4132$	$212, 221$	Odd indexed Fibonacci
$231, 4123$	$212, 231$	$A116703$
$231, 4312$	$212, 211, 321$	Odd indexed Fibonacci

Table 2: Sets of patterns Ω and $C(\Omega)$ such that $F(\Omega) = \varphi\big(A(C(\Omega))\big)$.

Proof. The inclusion $\hat{A}(1123, 2134) \subseteq \hat{A}(1123)$ is trivial. To prove the opposite inclusion, we proceed by contraposition. Suppose that $\hat{x} \geq 2134$ and let $\hat{x}(i_1)\hat{x}(i_2)\hat{x}(i_3)\hat{x}(i_4)$ be an occurrence of 2134 in $\hat{x}$. Let j be the index of the leftmost occurrence of the integer $\hat{x}(i_2)$ in $\hat{x}$. If $j < i_2$, then $\hat{x}(j)\hat{x}(i_2)\hat{x}(i_3)\hat{x}(i_4)$ is an occurrence of 1123. Otherwise, if $j = i_2$, then $\hat{x}(i_2)$ is an ascent top by Lemma 2.1, that is, $\hat{x}(i_2 - 1) < \hat{x}(i_2)$. Therefore we can repeat the same argument on the occurrence $\hat{x}(i_1)\hat{x}(i_2-1)\hat{x}(i_3)\hat{x}(i_4)$ of 2134 until we eventually find an occurrence of 1123. □

Lemma 6.2. *Let $x \in A$. Then $x \geq 1123$ if and only if $\hat{x} \geq 1123$.*

Proof. We proceed by induction on the length n of x. If $n \leq 3$ there is nothing to prove. Otherwise, consider the following two cases:

- If x contains exactly one occurrence of the integer 1, then $x = 1y^{+1}$, where y^{+1} is obtained by increasing by one each entry of some ascent sequence y. Similarly, $\hat{x} = 1\hat{y}^{+1}$. By the inductive hypothesis, $y \geq 1123$ if and only if $\hat{y} \geq 1123$. Thus the same holds for y^{+1} and $\hat{y}^{+1}$, and also for x and $\hat{x}$, as desired.

- Suppose that x contains at least two occurrences of the integer 1. Let i be the index of the rightmost occurrence of 1. Note that $x(i-1) \leq x(i)$, and hence the sequence $y = x(1)\cdots x(i-1)x(i+1)\cdots x(n)$ obtained from x by removing $x(i)$ is an ascent sequence. In particular, $\hat{y}$ is obtained by removing the entry $\hat{x}(i)$ from $\hat{x}$. We conclude by applying the inductive hypothesis to y and $\hat{y}$. □

As a corollary of the previous two results, we have $A_{\{2134\}} = A(1123)$ and hence

$$\varphi\big(A(1123)\big) = F(2134).$$

In 2011 Duncan and Steingrímsson [17] conjectured that $|A_n(1123)| = C_n$, the nth Catalan number. Three years later this conjecture was settled in the affirmative by Mansour and Shattuck [20]. Gil and Weiner [18] independently proved that $|F_n(2134)| = C_n$ in 2019.

$$
\begin{tikzcd}
A_{\{231,4123\}} \arrow[rr, "\varphi"] \arrow[d, "x \mapsto \hat{x}"'] & & F(231,4123) \\
\hat{A}([231^{-1}],[4123^{-1}]) \arrow[urr, "\gamma"'] & &
\end{tikzcd}
$$

Figure 3: The diagram illustrating the transport of patterns between $F(231,4123) = S(231,4123)$ and $A_{231,4123} = A(212,231)$.

Example. The sets $F(1423)$ and $F(3412)$ can be dealt with similarly. For instance, we have $A_{\{1423\}} = A(132)$ and hence

$$\varphi\big(A(132)\big) = F(1423).$$

The set $A(132)$ has been enumerated in [17], whereas the enumeration of $F(1423)$ can be found in [18]. Both sets are enumerated by the Catalan numbers. Analogously, $A_{\{3412\}} = A(312)$, which leads to

$$\varphi\big(A(312)\big) = F(3412).$$

No formula is currently known for the enumeration of $A(312)$ or $F(3412)$. It has been conjectured [13] that 312-sortable permutations share the same enumeration.

6.2 Transport of sets of patterns

Baxter and Pudwell [3] showed that $A(212,221)$ is enumerated by the odd indexed Fibonacci numbers (sequence A001519 in the OEIS [25]) by providing a generating tree for this set. They also showed that $A(212,231)$ and $S(231,4123)$ admit an isomorphic generating tree (via the transfer matrix method and using Maple). An alternative proof of both results can be obtained as a simple corollary of our framework. Indeed,

$$\varphi\big(A(212,221)\big) = S(231,4132),$$

which is well known to be enumerated by the odd indexed Fibonacci numbers, and

$$\varphi\big(A(212,231)\big) = S(231,4123).$$

Note that $F(231,4132) = S(231,4132)$ and $F(231,4123) = S(231,4123)$, since 231 is the underlying permutation of $\mathfrak{f}$. Therefore, as outlined in the previous examples (see also Figure 3 for the pair $(231,4123)$), to prove both results it is sufficient to show that $A_{\{231,4132\}} = A(212,221)$ and $A_{\{231,4123\}} = A(212,231)$, respectively. This can be achieved with a simple case by case analysis, whose details are omitted.

In a similar fashion, an example of transport of a triple of patterns is the following:

$$\varphi\big(A(212,211,321)\big) = S(231,4312),$$

where it is easy to derive that $|S_n(231,4312)|$ is an odd indexed Fibonacci number.

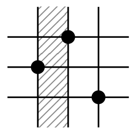
23-1 = 

Figure 4: Vincular pattern such that $\gamma(\mathrm{RGF}) = S(23\text{-}1)$.

6.3 RGF as set of representatives for [Cay]

In Section 5 we specialized Theorem 4.9 by choosing $\hat{A}$ as set of representatives for the equivalence classes $[x] \in [\mathrm{Cay}]$ such that $\gamma(x) \in F$. The same approach can be replicated in order to obtain a transport theorem for restricted growth functions and the set of permutations avoiding the vincular pattern $23\text{-}1 = (231, \{1\}, \{\})$, which is depicted in Figure 4. Note that every permutation avoiding 23-1 is a Fishburn permutation; in symbols, $S(23\text{-}1) \subseteq F$. Below we outline the main ideas and give a couple of examples, leaving a deeper investigation of this notion of transport for future work.

A Cayley permutation $x \in \mathrm{Cay}_n$ is a *restricted growth function* if $x(1) = 1$ and $x(i+1) \leq \max\{x(1), \ldots, x(i)\} + 1$, for each $i = 1, 2, \ldots, n-1$. Let RGF_n be the set of restricted growth functions of length n and let $\mathrm{RGF} = \cup_{n\geq 0} \mathrm{RGF}_n$. It is well known that restricted growth functions encode set partitions in the same manner as Cayley permutations encode ballots. Therefore the cardinality of RGF_n is equal to the nth Bell number (sequence A000110 in [25]). Pattern avoidance on RGF has been extensively studied [10, 19, 24].

Recall from Section 3 that $B(\pi)$ is the set of Cayley permutations that encode ballots whose underlying permutation is π. More precisely, let $x \in B(\pi)$ and let P_x be the ballot encoded by x, where as usual $x(i) = j$ if $i \in B_j$. Then $\gamma(x) = \pi$ is the permutation obtained from P_x by sorting blocks decreasingly and removing the curly brackets. For instance, $x = 112132341$ is the Cayley permutation that encodes the ballot $P_x = \{9, 4, 2, 1\}\{6, 3\}\{7, 5\}\{8\}$ and we have

$$\begin{pmatrix}\mathrm{id} \\ x\end{pmatrix}^T = \begin{pmatrix}1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 1\,1\,2\,1\,3\,2\,3\,4\,1\end{pmatrix}^T = \begin{pmatrix}1\,1\,1\,1\,2\,2\,3\,3\,4 \\ 9\,4\,2\,1\,6\,3\,7\,5\,8\end{pmatrix} = \begin{pmatrix}\mathrm{sort}(x) \\ \gamma(x)\end{pmatrix}.$$

Note that $\gamma(x) = 942163758$ is the permutation underlying P_x and $\mathrm{sort}(x)$ is a Burge labeling of $\gamma(x)$.

Now, let π be a permutation. Let $\pi = B_1 B_2 \cdots B_k$ be the decomposition of π into descending runs and let $\ell(i) = |B_i|$. Denote by $1^{\ell(i)} = 1 \ldots 1$ the concatenation of $\ell(i)$ copies of 1. We call $\pi \mapsto 1^{\ell(1)} \oplus \cdots \oplus 1^{\ell(k)}$ the *minimal Burge labeling* of π because it is using the least number of distinct integers. For an alternative way of viewing this, observe that $\max(\lambda(\pi)) \geq \mathrm{asc}(\pi) + 1$ for any Burge labeling $\lambda : S \to I$ and that the minimal Burge labeling is the unique labeling for which equality is obtained. An overview of the Burge labelings used in this paper can be found in Table 3. Continuing the previous example (with $x = 112132341$) we have $x \in \mathrm{RGF}$ and $\gamma(x) \in S(23\text{-}1)$. The decomposition of $\gamma(x)$ into decreasing runs is $\gamma(x) = 9421|63|75|8$, where minima of blocks are in increasing order and $\mathrm{sort}(x) = 1111 \oplus 11 \oplus 11 \oplus 1 = 111122334$ is the minimal Burge labeling of $\gamma(x)$. The following proposition, whose easy proof is omitted, characterizes the cases where a ballot is encoded by a restricted growth function.

Set E	Labeling λ	$\Gamma_\lambda(E)$
S	Maximal labeling (ι)	S^{-1}
F	υ	$\hat{A}$
S	$\tilde{\upsilon}$	X
$S(23\text{-}1)$	Minimal labeling	RGF

Table 3: Burge labelings $\lambda : E \to I$ on $E \subseteq S$ and the resulting sets $\Gamma_\lambda(E) \subseteq \mathrm{Cay}$.

Proposition 6.3. *Let $x \in \mathrm{Cay}$ and let $(u, \pi) = (\mathrm{id}, x)^T$. Then $x \in \mathrm{RGF}$ if and only if $\pi \in S(23\text{-}1)$ and u is the minimal Burge labeling of π.*

An immediate corollary of Proposition 6.3 is that the map γ is injective on RGF and

$$\gamma(\mathrm{RGF}) = S(23\text{-}1).$$

This gives an alternative perspective on the original proof [14] that $S(23\text{-}1)$ is enumerated by the Bell numbers. The operation that associates the permutation $\pi = \gamma(x)$ to the set partition $x \in \mathrm{RGF}$ is similar to the *flattening* of set partitions considered by Callan [8], with the difference that blocks are sorted increasingly in their case. According to our notion, a permutation is the flattening of a set partition (through γ) if and only if it avoids 23-1.

From now on, let $\mathrm{Flat} = S(23\text{-}1)$. Since γ is injective on RGF, we can specialize Theorem 4.9 by choosing RGF as a set of representatives for [Cay] to get a transport theorem for RGF and Flat:

Theorem 6.4 (Transport of patterns from Flat to RGF)**.** *For any permutation σ and Cayley permutation y we have*

$$\mathrm{Flat}(\sigma) = \gamma\big(\mathrm{RGF}[\sigma^{-1}]\big) \quad \textit{and} \quad \gamma\big(\mathrm{RGF}[y]\big) = \mathrm{Flat}\big(\gamma(y)\big)$$

In other words, the set $\mathrm{RGF}[\sigma^{-1}]$ of restricted growth functions avoiding all the patterns in $[\sigma^{-1}]$ is mapped via the bijection γ to the set $\mathrm{Flat}(\sigma)$ of flattened partitions avoiding σ. In particular, we have $|\mathrm{Flat}_n(\sigma)| = |\mathrm{RGF}_n[\sigma^{-1}]|$.

We end this section by providing two instances of transport between RGF and Flat.

Example. By Theorem 6.4 we have

$$\mathrm{Flat}(2341) = \gamma\big(\mathrm{RGF}[2341^{-1}]\big) = \gamma\big(\mathrm{RGF}(4123, 3123)\big).$$

It is easy to show that $\mathrm{RGF}(4123, 3123) = \mathrm{RGF}(3123)$ and $\mathrm{Flat}(2341) = F(2341)$. Thus,

$$\gamma\big(\mathrm{RGF}(3123)\big) = F(2341).$$

Note that $\mathrm{RGF}(3123) = \mathrm{RGF}(123123)$. Restricted growth functions avoiding 123123 are sometimes called 3-noncrossing set partitions [19]. The enumeration of both $\mathrm{RGF}(123123)$ and $F(2341)$ is currently unknown.

Example. By Theorem 6.4 we have

$$\mathrm{Flat}(1342) = \gamma\big(\mathrm{RGF}[1342^{-1}]\big) = \gamma\big(\mathrm{RGF}(1423, 1323)\big).$$

Moreover, $\mathrm{RGF}(1423, 1323) = \mathrm{RGF}(1323)$ and $\mathrm{Flat}(1342) = F(1342)$. Thus,

$$\gamma\big(\mathrm{RGF}(1323)\big) = F(1342).$$

The set $F(1342)$ is enumerated by the binomial transform of the Catalan numbers [18].

7 Picking a representative for each equivalence class

In Theorem 5.1 we exploited the map $\psi : \hat{A} \to F$ and its inverse ψ^{-1} to transport patterns from F to $\hat{A}$. It seems natural to push this approach further by "lifting" ψ^{-1} to a map whose domain is S, the set of all permutations, thus extending the reach of Theorem 5.1. In effect, we will define a map, called η, that picks a representative for each equivalence class in [Cay]. The set of representatives will be called X and the lifted map will be $\eta : S \to X$. We now detail this construction.

Remark 4.4 shows that $\psi^{-1} = \Gamma_\upsilon$ is the map induced by the Burge labeling υ of Fishburn permutations described in Section 2.3. Let π be a Fishburn permutation. Recall that $\upsilon(\pi)$ is obtained by

1. annotating π with its active sites with respect to the Fishburn pattern $\mathfrak{f}$;
2. writing k above all entries $\pi(j)$ that lie between active sites k and $k+1$.

Due to the avoidance of $\mathfrak{f}$, the site between $\pi(i)$ and $\pi(i+1)$ is active if and only if $J(i) < i$, where $\pi(J(i)) = \pi(i) - 1$. In addition, the sites before $\pi(1)$ and after $\pi(j) = 1$ are always considered active. From now on, we call these sites $\mathfrak{f}$-*active*.

We wish to lift the map $\psi^{-1} : F \to \hat{A}$ to a map η, with S as its domain, by extending the labeling υ to a labeling $\tilde{\upsilon}$ on S. The lifted map η will then be $\eta = \Gamma_{\tilde{\upsilon}}$.

Let π be a permutation. We stipulate that the site between $\pi(i)$ and $\pi(i+1)$ is η-*active* if $J(i) < i$ or $\pi(i) < \pi(i+1)$. In addition, the sites before $\pi(1)$ and after $\pi(j) = 1$ are always considered η-active. The labeling $\tilde{\upsilon} : S_n \to I_n$ is defined by

1. annotating π with its η-active sites;
2. writing k above all entries $\pi(j)$ that lie between η-active sites k and $k+1$.

Now, $\tilde{\upsilon}$ is a Burge labeling on S. Indeed the site between $\pi(i)$ and $\pi(i+1)$ is η-active if $\pi(i) < \pi(i+1)$, therefore $D(\tilde{\upsilon}(\pi)) \subseteq D(\pi)$. Next we prove that $\tilde{\upsilon} = \upsilon$ on Fishburn permutations.

Lemma 7.1. *If π is a Fishburn permutation, then $\upsilon(\pi) = \tilde{\upsilon}(\pi)$.*

Proof. We will show that each site of a Fishburn permutation π is $\mathfrak{f}$-active if and only if it is η-active. The sites before $\pi(1)$ and after $\pi(n)$ are both $\mathfrak{f}$-active and η-active by definition. Consider the site between $\pi(i)$ and $\pi(i+1)$, for $1 \leq i < n$. If the site is $\mathfrak{f}$-active, then $J(i) < i$ and thus it is also η-active. Conversely, suppose that the site is η-active. If $J(i) < i$, then it is also $\mathfrak{f}$-active. Otherwise, if $J(i) > i$, we must have $\pi(i) < \pi(i+1)$. But then $\pi(i)\pi(i+1)\pi(J(i))$ is an occurrence of $\mathfrak{f}$ in π, which is impossible. $\square$

Since $\tilde{\upsilon}$ is a Burge labeling of S and the restriction of $\tilde{\upsilon}$ to F coincides with υ, the map $\eta = \Gamma_{\tilde{\upsilon}} : S \to \mathrm{Cay}$ lifts the map $\Gamma_\upsilon : F \to \hat{A}$. Moreover, since $\tilde{\upsilon}$ is injective, η is also injective, as shown below Definition 4.1. In other words, η picks one representative in the equivalence class $[\pi^{-1}]$, for each permutation π. If π is a Fishburn permutation, η chooses the same element as ψ^{-1}. Let

$$X = \eta(S).$$

$\pi \in S$	$\eta(\pi) \in X$	$\eta(\pi) \in \hat{A}$?
1	1	✓

$\pi \in S$	$\eta(\pi) \in X$	$\eta(\pi) \in \hat{A}$?
12	12	✓
21	11	✓

$\pi \in S$	$\eta(\pi) \in X$	$\eta(\pi) \in \hat{A}$?
123	123	✓
132	122	✓
213	112	✓
231	312	
312	121	✓
321	111	✓

$\pi \in S$	$\eta(\pi) \in X$	$\eta(\pi) \in \hat{A}$?
1234	1234	✓
1243	1233	✓
1324	1223	✓
1342	1423	
1423	1232	✓
1432	1222	✓
2134	1123	✓
2143	1122	✓
2314	3124	
2341	4123	
2413	2132	
2431	3122	
3124	1213	✓
3142	1312	✓
3214	1112	✓
3241	3112	
3412	3412	
3421	3312	
4123	1231	✓
4132	1221	✓
4213	1121	✓
4231	3121	
4312	1211	✓
4321	1111	✓

Table 4: Permutations and corresponding members of X

Note that $\hat{A} \subseteq X$ by Lemma 7.1. In Table 4 we list permutations and members of X of length one through four. We also indicate which ones are Fishburn permutations and modified ascent sequences, respectively.

We defined η as the function $\Gamma_{\tilde{v}} : S \to \mathrm{Cay}$. We then defined its range to be X. From now on we will consider η as a function $\eta : S \to X$. It is clearly a bijection[1] and we have the following transport theorem.

Theorem 7.2 (Transport of patterns from S to X)**.** *For any permutation σ and Cayley permutation y we have*

$$S(\sigma) = \gamma\big(X[\sigma^{-1}]\big) \quad \textit{and} \quad \gamma\big(X[y]\big) = S\big(\gamma(y)\big).$$

In other words, $S(\sigma)$ is mapped via the bijection η to $X(B(\sigma))$.

As a direct consequence, $S(\sigma)$ and $X(B(\sigma))$ are equinumerous subsets of Cay.

Example. For each natural number n, we have

$$|S_n(1324)| = |X_n(1223, 1324)|;$$
$$|S_n(4231)| = |X_n(2121, 3121, 3231, 4231)|.$$

The rest of this section is devoted to describing the set X. Mesh patterns on Cayley permutations were recently introduced by Cerbai [12]. They are defined like mesh patterns on permutations, but with additional regions to account for the possibility

[1] Is there some easier way (than using this bijection) to see that $|X_n| = n!$?

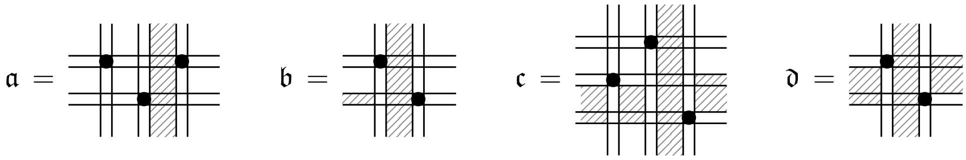


Figure 5: Mesh patterns such that $\hat{A} = \mathrm{Cay}(\mathfrak{a}, \mathfrak{b})$ and $X = \mathrm{Cay}(\mathfrak{a}, \mathfrak{c}, \mathfrak{d})$

of having repeated elements. Instead of giving a formal definition, we refer the reader to [12] and Figure 5. From now on, let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ be the mesh patterns depicted in Figure 5.

Recall from Lemma 2.1 that the ascent tops of a modified ascent sequence x together with the first element, $x(1) = 1$, form a permutation of length $\max(x)$. The converse is also true. To be precise, let

$$\mathrm{top}(x) = \{(1, 1)\} \cup \{(i, x(i)) : 1 < i \leq n,\, x(i-1) < x(i)\}$$

be the set of ascent tops and their indices—including the first element—and let

$$\mathrm{nub}(x) = \{(\min x^{-1}(j), j) : 1 \leq j \leq \max(x)\}$$

be the set of first occurrences and their indices. Then

$$\hat{A} = \{x \in \mathrm{Cay} : \mathrm{top}(x) = \mathrm{nub}(x)\},$$

giving an arguably simpler characterization of modified ascent sequences than the one given in Section 2.2. This can be equivalently expressed in terms of avoidance of the two mesh patterns $\mathfrak{a}$ and $\mathfrak{b}$.

Theorem 7.3. *We have $\hat{A} = \mathrm{Cay}(\mathfrak{a}, \mathfrak{b})$, and hence the two sets $\{\mathfrak{a}, \mathfrak{b}\}$ and $\{11, \mathfrak{f}\}$ are Wilf-equivalent.*

Proof. Let $x \in \mathrm{Cay}_n$ be a Cayley permutation. We start by showing that if x contains $\mathfrak{a}$ or $\mathfrak{b}$, then x is not a modified ascent sequence. Suppose that $x(i)x(j)x(j+1)$ is an occurrence of $\mathfrak{a}$ in x. Then $x(j+1)$ is an ascent top and $x(j+1) = x(i)$ with $i < j+1$. Thus $x \notin \hat{A}_n$ by Lemma 2.1. Suppose that $x(i)x(i+1)$ is an occurrence of $\mathfrak{b}$ and let $k = x(i+1)$. Then $x(i+1)$ is the leftmost occurrence of k in x, but $x(i+1)$ is not an ascent top. Again, $x \notin \hat{A}_n$ by Lemma 2.1.

Conversely, suppose that x avoids both $\mathfrak{a}$ and $\mathfrak{b}$. We shall use the recursive definition of $\hat{A}$ to prove that x is a modified ascent sequence. Let $v = x(1) \cdots x(n-1)$ and let $a = x(n)$. Note that v avoids $\mathfrak{a}$ and $\mathfrak{b}$, but v is not necessarily a Cayley permutation. We distinguish the following three cases.

- If $x(n-1) > x(n)$, then $x(n)$ is not the leftmost occurrence of a in x (since x avoids $\mathfrak{b}$). Thus v is a Cayley permutation: it contains all the integers from 1 to $\max(v) = \max(x)$. By the inductive hypothesis, v is a modified ascent sequence. Since $x = va$, with $1 \leq a \leq x(n-1)$, we have that x is also a modified ascent sequence.

- If $x(n-1) = x(n)$, then v is again a Cayley permutation and we can proceed as in the previous case.
- If $x(n-1) < x(n)$, then $x(n)$ must be the only occurrence of a in x (since x avoids $\mathfrak{a}$). Because x is a Cayley permutation, the string w obtained from v by decreasing each entry $c > a$ by one must also be a Cayley permutation (that still avoids $\mathfrak{a}$ and $\mathfrak{b}$). By the inductive hypothesis, w is a modified ascent sequence and $x(n) \leq \max(w) + 1 = \mathrm{asc}(w) + 2$. Therefore x is a modified sequence (since $x = \tilde{w}x(n)$ with $x(n-1) < x(n) \leq \mathrm{asc}(w) + 2$).

Finally, $\hat{A}$ and the set of Fishburn permutations, $F = \mathrm{Cay}(11, \mathfrak{f})$, are equinumerous. Therefore the two sets $\{\mathfrak{a}, \mathfrak{b}\}$ and $\{11, \mathfrak{f}\}$ are Wilf-equivalent. $\square$

Lemma 7.4. *We have* $X = \big\{x \in \mathrm{Cay} : (\tilde{v} \circ \gamma)(x) = \mathrm{sort}(x)\big\}$.

Proof. For any Cayley permutation x we have $(\mathrm{id}, x)^T = (\mathrm{sort}(x), \gamma(x))$. For any permutation π we have $(\tilde{v}(\pi), \pi)^T = (\mathrm{sort}(\pi), \eta(\pi)) = (\mathrm{id}, \eta(\pi))$. Thus

$$\begin{aligned} x \in X &\iff x = \eta(\pi) && \text{for some } \pi \in S \\ &\iff (\tilde{v}(\pi), \pi)^T = (\mathrm{id}, x) && \text{for some } \pi \in S \\ &\iff (\tilde{v}(\pi), \pi) = (\mathrm{sort}(x), \gamma(x)) && \text{for some } \pi \in S \\ &\iff \tilde{v}(\gamma(x)) = \mathrm{sort}(x). \end{aligned}$$

$\square$

Theorem 7.5. *We have* $X = \mathrm{Cay}(\mathfrak{a}, \mathfrak{c}, \mathfrak{d})$*, and hence the set* $\{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\}$ *is Wilf-equivalent to the pattern* 11.

Proof. Let $x \in \mathrm{Cay}_n$ be a Cayley permutation. We start by showing that if x contains $\mathfrak{a}$, $\mathfrak{c}$ or $\mathfrak{d}$, then $x \notin X$. Let $\pi = \gamma(x)$. By Lemma 7.4, it suffices to show that $\tilde{v}(\pi) \neq \mathrm{sort}(x)$. To ease notation, let $v = \tilde{v}(\pi)$.

Suppose that $x(i)x(j)x(j+1)$ is an occurrence of the pattern $\mathfrak{a}$ in x. Then $x(i) = x(j+1) > x(j)$ and

$$\begin{aligned} \begin{pmatrix} \mathrm{id} \\ x \end{pmatrix}^T &= \begin{pmatrix} \cdots & i & \cdots & j & j+1 & \cdots \\ \cdots & x(i) & \cdots & x(j) & x(j+1) & \cdots \end{pmatrix}^T \\ &= \begin{pmatrix} \cdots & x(j) & \cdots & x(j+1) & \cdots & x(i) & \cdots \\ \cdots & j & \cdots & j+1 & \cdots & i & \cdots \end{pmatrix} = \begin{pmatrix} \mathrm{sort}(x) \\ \pi \end{pmatrix}. \end{aligned}$$

For $\ell \in [n]$, let $K(\ell)$ be the index of the column $(x(\ell), \ell)$ in $(\mathrm{id}, x)^T$. In particular, $\mathrm{sort}(x)(K(\ell)) = x(\ell)$. Note that $K(j) < K(j+1) < K(i)$. In particular, the site in π immediately after $K(j+1)$ is η-active. Therefore $v(K(i)) > v(K(j+1))$, whereas $\mathrm{sort}(x)(K(i)) = x(i) = x(j+1) = \mathrm{sort}(x)(K(j+1))$, and hence $\mathrm{sort}(x) \neq v$.

Next, suppose that $x(i)x(j)x(j+1)$ is an occurrence of the pattern $\mathfrak{c}$ in x. Then $x(i) = x(j+1) + 1 < x(j)$ and

$$\begin{aligned} \begin{pmatrix} \mathrm{id} \\ x \end{pmatrix}^T &= \begin{pmatrix} \cdots & i & \cdots & j & j+1 & \cdots \\ \cdots & x(i) & \cdots & x(j) & x(j+1) & \cdots \end{pmatrix}^T \\ &= \begin{pmatrix} \cdots & x(j+1) & x(K(t)) & \cdots & x(i) & \cdots & x(j) & \cdots \\ \cdots & j+1 & t & \cdots & i & \cdots & j & \cdots \end{pmatrix} = \begin{pmatrix} \mathrm{sort}(x) \\ \pi \end{pmatrix}. \end{aligned}$$

Note that $K(j+1) < K(i) < K(j)$. Now, if $K(i) = K(j+1)+1$, then $j+1 > i$ is a descent in π and, since $K(j) > K(j+1)$, the site in π immediately after $K(j+1)$ is not η-active. Therefore $v(K(i)) = v(K(j+1))$, whereas $\mathrm{sort}(x)(K(i)) = x(i) > x(j+1) = \mathrm{sort}(x)(K(j+1))$, and hence $\mathrm{sort}(x) \neq v$. Otherwise, consider the column $(x(t), t)$ immediately after the column $(x(j+1), j+1)$ in $(\mathrm{sort}(x), \pi)$. In other words, suppose that $K(t) = K(j+1)+1$. Since $x(i) = x(j+1)+1$, either $x(t) = x(j+1)$ or $x(t) = x(j+1)+1$. Suppose that $x(t) = x(j+1)$. We shall prove by contradiction that $v(K(t)) \neq v(K(j+1))$, and thus $\mathrm{sort}(x) \neq v$. If $v(K(t)) = v(K(j+1))$, then the site between $K(j+1)$ and $K(t)$ in π is not η-active. Therefore $j+1 > t$ is a descent. But then $x(t) = x(j+1)$ would precede $x(j+1)$ in x, which contradicts $x(i)x(j)x(j+1)$ being an occurrence of $\mathfrak{c}$ (since $x(t)$ would be placed in a forbidden region). Finally, suppose that $x(t) = x(j+1)+1$. We wish to show that $v(K(t)) = v(K(j+1))$. By contradiction, suppose that $v(K(j+1)) < v(K(t))$; that is, the site between $K(j+1)$ and $K(t)$ is η-active. Since $K(j) > K(j+1)$, we have that $j+1 < t$ is an ascent. But then $x(t) = x(j+1)+1 \leq x(i)$, which contradicts $x(i)x(j)x(j+1)$ being an occurrence of $\mathfrak{c}$ (again $x(t)$ would be placed in a forbidden region).

The pattern $\mathfrak{d}$ can be treated similarly, so we leave it to the reader.

Conversely, suppose that x avoids $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$. Let $\pi = \gamma(x)$. We wish to prove that $x = \eta(\pi)$ or, equivalently, $\mathrm{sort}(x) = \tilde{v}(\pi)$. Due to the great amount of technical details, we just sketch the proof. To prove the contrapositive statement, suppose that $\mathrm{sort}(x) \neq \tilde{v}(\pi)$. There are two possibilities for $\mathrm{sort}(x)$ to be different from the η-labeling of π. Either $\mathrm{sort}(x)$ labels two consecutive elements $\pi(i)$ with k and $\pi(i+1)$ with $k+1$, but i is not η-active. Or $\mathrm{sort}(x)$ labels $\pi(i)$ and $\pi(i+1)$ with the same integer k, but the site i is η-active. In the first case, $J(i) > i$ and $\pi(i) > \pi(i+1)$. If $J(i) = i+1$, then the labels k of $\pi(i)$ and $k+1$ of $\pi(i+1)$ necessarily result in an occurrence of $\mathfrak{d}$ in x. Similarly, if $J(i) > i+1$, then the labels of $\pi(i)$, $\pi(i+1)$ and $\pi(J(i))$ result in an occurrence of $\mathfrak{c}$ in x. Analogously, if $\pi(i)$ and $\pi(i+1)$ are labeled with the same integer k, but the site i is η-active, then it is possible to show that x contains an occurrence of $\mathfrak{a}$. This completes the proof. □

Theorems 7.3 and 7.5 characterize $\hat{A}$ and X as pattern avoiding Cayley permutations. As a result, we can interpret the transports of patterns described in Theorems 5.1 and 7.2 as Wilf-equivalences.

Corollary 7.6. *Let σ be a permutation.*

1. *The two sets $\{11, \mathfrak{f}, \sigma\}$ and $\{\mathfrak{a}, \mathfrak{b}\} \cup B(\sigma)$ are Wilf-equivalent. That is,*

$$|\mathrm{Cay}_n(11, \mathfrak{f}, \sigma)| \;=\; |\mathrm{Cay}_n(\mathfrak{a}, \mathfrak{b}, B(\sigma))|.$$

2. *The two sets $\{11, \sigma\}$ and $\{\mathfrak{a}, \mathfrak{c}, \mathfrak{d}\} \cup B(\sigma)$ are Wilf-equivalent. That is,*

$$|\mathrm{Cay}_n(11, \sigma)| \;=\; |\mathrm{Cay}_n(\mathfrak{a}, \mathfrak{c}, \mathfrak{d}, B(\sigma))|.$$

The transport between RGF and $S(23\text{-}1)$, that is Theorem 6.4, can be interpreted as a Wilf-equivalence on Cayley permutations as well.

Lemma 7.7. *We have* $\mathrm{RGF} = \mathrm{Cay}(\mathfrak{g})$*, where $\mathfrak{g}$ is the Cayley-mesh pattern depicted in Figure 6.*

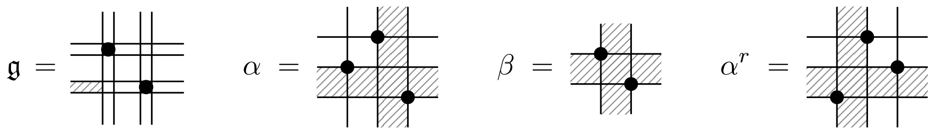


Figure 6: Mesh pattern such that RGF = Cay($\mathfrak{g}$) and bivincular patterns such that $\eta(S^0) = S(\alpha, \beta)$

Proof. A well known equivalent description of RGF is the following. A Cayley permutation x is a restricted growth function if and only if every occurrence of each integer $k \geq 2$ in x is preceded by at least one occurrence of each of the integers $1, 2, \ldots, k-1$. This property can in turn be expressed by the avoidance of the Cayley-mesh pattern $\mathfrak{g}$. ☐

Corollary 7.8. *Let σ be a permutation. Then the two sets $\{11, 23\text{-}1, \sigma\}$ and $\{\mathfrak{g}\} \cup [\sigma^{-1}]$ are Wilf-equivalent. That is,*

$$|\mathrm{Cay}_n(11, 23\text{-}1, \sigma)| = |\mathrm{Cay}_n(\mathfrak{g}, [\sigma^{-1}])|.$$

7.1 Permutations with no η-inactive sites

Let S^0 denote the set of permutations with no η-inactive sites. Note that if $\pi \in S^0$, then $\tilde{v}(\pi) = \mathrm{id}$, and so $\eta(\pi)$ contains no repeated letters. Indeed, $\eta(\pi) = \pi^{-1}$. Thus $\eta(S^0) = (S^0)^{-1} = X \cap S$. When restricting to permutations we can considerably simplify the mesh patterns $\mathfrak{a}$, $\mathfrak{c}$ and $\mathfrak{d}$ that characterize X: since the underlying pattern of $\mathfrak{a}$ is not a permutation we can remove it; the pattern $\mathfrak{c}$ is equivalent to the bivincular pattern $\alpha = (231, \{2\}, \{1\})$; and the pattern $\mathfrak{d}$ is equivalent to the bivincular pattern $\beta = (21, \{1\}, \{1\})$. Thus

$$\eta(S^0) = S(\alpha, \beta).$$

The patterns α and β are depicted in Figure 6.

We wish to construct a bijection between $\eta(S^0)$ and the set of ascent sequences with no flat steps (consecutive equal entries). An ascent sequence with no flat steps is said to be *primitive*. Primitive ascent sequences were enumerated by Dukes et al. [15]. Dukes and Parviainen [16] proved that primitive ascent sequences are in bijection with binary upper triangular matrices with non-negative entries such that all rows and columns contain at least one nonzero entry. The pattern α is closely related to the Fishburn pattern $\mathfrak{f}$. Let $\alpha^r = (132, \{1\}, \{1\})$, the reverse of α (see Figure 6). Recall from Section 2.3 the step-wise procedure that associates each Fishburn permutation π with an ascent sequence through the construction of π from 1 by inserting a new maximum, at each step, and recording its position. Parviainen [23] observed that an alternative description of A can be obtained by performing the same construction on $S(\alpha^r)$ instead of F. The avoidance of α^r gives rise to an analogous notion of α^r-active site and the resulting bijection $\psi' : \hat{A} \to S(\alpha^r)$ can be computed using the Burge transpose by replacing $\mathfrak{f}$-active sites with α^r-active sites.

Lemma 7.9. *Let x be a modified ascent sequence and let $\pi = \psi'(x)$. Then π contains an occurrence of β^r if and only if x contains a flat step.*

Proof. Suppose that $\pi(i)\pi(i+1)$ is an occurrence of β^r in π, or, equivalently, that $\pi(i+1) = \pi(i)+1$. Note that the site between $\pi(i)$ and $\pi(i+1)$ is not α^r-active, since inserting a new maximum $n+1$ in this position would create an occurrence $\pi(i), n+1, \pi(i+1)$ of α^r. Therefore the labels of $\pi(i)$ and $\pi(i+1)$ are equal. Since $\pi(i+1) = \pi(i)+1$, this results in a flat step $x(\pi(i))\,x(\pi(i+1))$ in x. Conversely, suppose that $x(i)x(i+1)$ is a flat step in x. Then, by definition of Burge transpose, the elements $i+1$ and i are in consecutive positions in π, and $i+1$ precedes i. Thus $i+1, i$ is an occurrence of β, as desired. □

As a consequence of the proof of Lemma 7.9, ψ' is a bijection between the set of modified ascent sequences with no flat steps and $S(\alpha^r, \beta^r)$. Moreover, $\pi \mapsto \pi^r$ is a bijection between $S(\alpha^r, \beta^r)$ and $S(\alpha, \beta) = \eta(S^0)$. Finally, since flat steps are preserved when mapping a modified ascent sequence to its corresponding ascent sequence, we obtain by composition the desired bijection between $\eta(S^0)$ and the set of primitive ascent sequences. We close this section by stating this as a theorem.

Theorem 7.10. *There is a one-to-one correspondence between permutations with no η-inactive sites and the set of primitive ascent sequences.*

8 Future directions

In this paper we have laid the theoretical foundations for the development of a theory of transport of patterns from Fishburn permutations to ascent sequences, and more generally between S and [Cay], leaving most applications for future work. Given a set of pattern avoiding Fishburn permutations, we have provided a construction for the basis of the corresponding set of modified ascent sequences. Using the bijection $\hat{A} \to A$, this result can be interpreted in terms of (plain) ascent sequences. Nevertheless, a more direct construction for a basis would be of interest.

Open Problem 8.1. *Given a permutation σ, determine a set of Cayley permutations $C(\sigma)$ such that*

$$\varphi(A(C(\sigma))) = F(\sigma).$$

To find analogous sets in the other direction also remains an open problem.

Open Problem 8.2. *Given a Cayley permutation x, determine a set $B'(x)$ such that*

$$\psi(\hat{A}(x)) = F(B'(x)),$$

and a set $C'(x)$ such that

$$\varphi(A(x)) = F(C'(x)).$$

Understanding how the avoidance of a pattern on ascent sequences affects the corresponding set of modified ascent sequences, and vice versa, seems to be necessary if we want to answer these questions. In other words, we would like to describe the set of sequences obtained by modifying $A(x)$ in terms of avoidance of patterns, as well as the set obtained by applying the inverse construction to $\hat{A}(x)$.

Our work suggests that a natural setting for the transport of patterns is the set of Cayley permutations. Indeed, we showed how a transport theorem often can be regarded as an example of Wilf-equivalence over Cayley permutations. On the other hand, not all the ascent sequences are Cayley permutations. This raises at least two more questions. First, is there an analogue of the Burge transpose that allows us to incorporate (plain) ascent sequences in the same framework? Secondly, what natural superset Y do ascent sequences belong to? Ideally, since we would like to transport patterns between S and Y, the set Y should be equinumerous with the set of permutations. A reasonable guess could then be the set of inversion sequences, which properly contains A, but this remains to be investigated.

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