

PARTITION AND COMPOSITION MATRICES

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ABSTRACT. This paper introduces two matrix analogues for set partitions. A composition matrix on a finite set X is an upper triangular matrix whose entries partition X , and for which there are no rows or columns containing only empty sets. A partition matrix is a composition matrix in which an order is placed on where entries may appear relative to one-another.

We show that partition matrices are in one-to-one correspondence with inversion tables. Non-decreasing inversion tables are shown to correspond to partition matrices with a row ordering relation. Partition matrices which are s -diagonal are classified in terms of inversion tables. Bidiagonal partition matrices are enumerated using the transfer-matrix method and are equinumerous with permutations which are sortable by two pop-stacks in parallel.

We show that composition matrices on X are in one-to-one correspondence with $(2 + 2)$ -free posets on X . Also, composition matrices whose rows satisfy a column-ordering relation are shown to be in one-to-one correspondence with parking functions. Finally, we show that pairs of ascent sequences and permutations are in one-to-one correspondence with $(2 + 2)$ -free posets whose elements are the cycles of a permutation, and use this relation to give an expression for the number of $(2 + 2)$ -free posets on $\{1, \dots, n\}$.

1. INTRODUCTION

We present two matrix analogues for set partitions that are intimately related to both permutations and $(2 + 2)$ -free posets.

Example 1. Here is an instance of what we shall call a partition matrix:

$$A = \begin{bmatrix} \{1, 2, 3\} & \emptyset & \{5, 7, 8\} & \{9\} \\ \emptyset & \{4\} & \{6\} & \{11\} \\ \emptyset & \emptyset & \emptyset & \{13\} \\ \emptyset & \emptyset & \emptyset & \{10, 12\} \end{bmatrix}.$$

Definition 2. Let X be a finite subset of $\{1, 2, \dots\}$. A *partition matrix* on X is an upper triangular matrix over the powerset of X satisfying the following properties:

- (i) each column and row contain at least one non-empty set;
- (ii) the non-empty sets partition X ;
- (iii) $\text{col}(i) < \text{col}(j) \implies i < j$,

where $\text{col}(i)$ denotes the column in which i is a member. Let Par_n be the collection of all partition matrices on $[1, n] = \{1, \dots, n\}$.

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For instance,

$$\text{Par}_1 = \{\{\{1\}\}\};$$

$$\text{Par}_2 = \left\{ [\{1,2\}], \begin{bmatrix} \{1\} & \emptyset \\ \emptyset & \{2\} \end{bmatrix} \right\};$$

$$\text{Par}_3 = \left\{ [\{1,2,3\}], \begin{bmatrix} \{1,2\} & \emptyset \\ \emptyset & \{3\} \end{bmatrix}, \begin{bmatrix} \{1\} & \{2\} \\ \emptyset & \{3\} \end{bmatrix}, \begin{bmatrix} \{1\} & \{3\} \\ \emptyset & \{2\} \end{bmatrix}, \begin{bmatrix} \{1\} & \emptyset \\ \emptyset & \{2,3\} \end{bmatrix}, \begin{bmatrix} \{1\} & \emptyset & \emptyset \\ \emptyset & \{2\} & \emptyset \\ \emptyset & \emptyset & \{3\} \end{bmatrix} \right\}.$$

In Section 2 we present a bijection between Par_n and the set of *inversion tables*

$$\mathcal{I}_n = [0, 0] \times [0, 1] \times \cdots \times [0, n-1], \text{ where } [a, b] = \{i \in \mathbb{Z} : a \leq i \leq b\}.$$

Non-decreasing inversion tables are shown to correspond to partition matrices with a row ordering relation. Partition matrices which are s -diagonal are classified in terms of inversion tables. Bidiagonal partition matrices are enumerated using the transfer-matrix method and are equinumerous with permutations which are sortable by two pop-stacks in parallel.

In Section 3 we show that composition matrices on X are in one-to-one correspondence with $(2+2)$ -free posets on X . We also show that composition matrices whose rows satisfy a column-ordering relation are in one-to-one correspondence with parking functions.

Finally, in Section 4 we show that pairs of ascent sequences and permutations are in one-to-one correspondence with $(2+2)$ -free posets whose elements are the cycles of a permutation, and use this relation to give an expression for the number of $(2+2)$ -free posets on $[1, n]$.

Taking the entry-wise cardinality of the matrices in Par_n one gets the matrices of Dukes and Parviainen [5]. In that sense, we generalize the paper of Dukes and Parviainen in a similar way as Claesson and Linusson [4] generalized the paper of Bousquet-Mélou et al. [2]. We note, however, that if we restrict our attention to those inversion tables that enjoy the property of being an *ascent sequence*, then we do *not* recover the bijection of Dukes and Parviainen.

2. PARTITION MATRICES AND INVERSION TABLES

For w a sequence let $\text{Alph}(w)$ denote the set of distinct entries in w . In other words, if we think of w as a word, then $\text{Alph}(w)$ is the (smallest) alphabet on which w is written. Also, let us write $\{a_1, \dots, a_k\}_<$ for a set whose elements are listed in increasing order, $a_1 < \dots < a_k$. Given an inversion table $w = (x_1, \dots, x_n) \in \mathcal{I}_n$ with $\text{Alph}(w) = \{y_1, \dots, y_k\}_<$ define the $k \times k$ matrix $A = \Lambda(w) \in \text{Par}_n$ by

$$A_{ij} = \{ \ell : x_\ell = y_i \text{ and } y_j < \ell \leq y_{j+1} \},$$

where we let $y_{k+1} = n$. For example, with

$$w = (0, 0, 0, 3, 0, 3, 0, 0, 0, 8, 3, 8) \in \mathcal{I}_{12}$$

we have $\text{Alph}(w) = \{0, 3, 8\}$ and

$$\Lambda(w) = \begin{bmatrix} \{1, 2, 3\} & \{5, 7, 8\} & \{9\} \\ \emptyset & \{4, 6\} & \{11\} \\ \emptyset & \emptyset & \{10, 12\} \end{bmatrix} \in \text{Par}_{12}.$$

We now define a map $K : \text{Par}_n \rightarrow \mathcal{I}_n$. Given $A \in \text{Par}_n$, for $\ell \in [1, n]$ let $x_\ell = \min(A_{*i}) - 1$ where i is the row containing ℓ and $\min(A_{*i})$ is the smallest

entry in column i of A . Define

$$K(A) = (x_1, \dots, x_n).$$

Theorem 3. *The map $\Lambda : \mathcal{I}_n \rightarrow \text{Par}_n$ is a bijection and K is its inverse.*

Proof. It suffices to show the following four statements:

- (1) $\Lambda(\mathcal{I}_n) \subseteq \text{Par}_n$;
- (2) $K(\text{Par}_n) \subseteq \mathcal{I}_n$;
- (3) $K(\Lambda(w)) = w$ for all w in \mathcal{I}_n ;
- (4) $\Lambda(K(A)) = A$ for all A in Par_n .

Proof of (1): Assume that $w = (x_1, \dots, x_n) \in \mathcal{I}_n$ with $\text{Alph}(w) = \{y_1, \dots, y_k\}_<$, and let $A = \Lambda(w)$. We first need to see that A is upper triangular. Let $i > j$ and consider the entry A_{ij} . Assume that $x_\ell = y_i$. Since $w \in \mathcal{I}_n$ we have $\ell > x_\ell$ and thus $\ell > y_i$. Since $y_1 < \dots < y_k$ and $i \geq j + 1$ we have $\ell > y_i \geq y_{j+1}$. Thus $A_{ij} = \emptyset$ if $i > j$; that is, A is upper triangular.

Denote by A_{i*} and A_{*j} the union of the sets in the i th row and the j th column of A , respectively. By definition, we have $A_{i*} = \{\ell : x_\ell = y_i\}$ and $A_{*j} = [y_j + 1, y_{j+1}]$ and clearly both sets are non-empty. Thus A satisfies condition (i) of Definition 2. To show (ii), it suffices to note that the entries A_{i*} form a partition of $[1, n]$, and so do the entries A_{*j} . To show (iii), let $u, v \in [1, n]$ with $\text{col}(u) < \text{col}(v)$. Also, let $p = \text{col}(u)$ and $q = \text{col}(v)$. Then $u \leq y_{p+1}$ and $y_q < v$. Since $p + 1 \leq q$ and the numbers y_i are increasing, it follows that $u \leq y_{p+1} \leq y_q < v$.

Proof of (2): Given $A \in \text{Par}_n$ choose any $\ell \in [1, n]$. Suppose that ℓ is in row i of A and let $a = \min(A_{*i})$ be the smallest entry in column i of A . If $\text{col}(a) = \text{col}(\ell)$ then $a \leq \ell$, and so $x_\ell = a - 1 \leq \ell - 1$. Otherwise, $\text{col}(a) < \text{col}(\ell)$ and so, from condition (iii) of Definition 2, we have $a < \ell$. Thus $x_\ell < \ell - 1$.

Proof of (3): Let $w = (x_1, \dots, x_n) \in \mathcal{I}_n$, $\text{Alph}(w) = \{y_1, \dots, y_k\}_<$, $A = \Lambda(w)$ and $K(A) = (z_1, \dots, z_m)$. From the definitions of Λ and K it is clear that $n = m$. Suppose that $\ell \in [1, n]$ is in row i of A ; then $x_\ell = y_i$. Also, by the definition of Λ , the smallest entry in column i of A is $y_i + 1$. From the definition of K we have $z_\ell = (y_i + 1) - 1 = y_i = x_\ell$. So $x_\ell = z_\ell$ for all $\ell \in [1, n]$, and hence $w = z = K(A)$.

Proof of (4): Let $A \in \text{Par}_n$, $K(A) = w = (x_1, \dots, x_n)$, $\text{Alph}(w) = \{y_1, \dots, y_k\}_<$ and $P = \Lambda(w)$. Also, define $z_j = \min(A_{*j}) - 1$. Then, for $\ell \in [1, n]$, we have

$$\ell \in A_{ij} \iff x_\ell = z_i \text{ and } \ell \in [z_j + 1, z_{j+1}] \quad (1)$$

by the definitions of K and z_j . In particular, this means that each x_ℓ equals some z_i and, similarly, each z_i equals some x_ℓ . Hence $\text{Alph}(w) = \{z_1, \dots, z_{\dim(A)}\}_<$ and it follows that $\dim(A) = k$ and $y_j = z_j$ for all $j \in [1, k]$. So we can restate (1) as

$$\ell \in A_{ij} \iff x_\ell = y_i \text{ and } \ell \in [y_j + 1, y_{j+1}].$$

By the definition of Λ , the right hand side is equivalent to $\ell \in P_{ij}$. Thus $A = P$. \square

2.1. Statistics on partition matrices and inversion tables. Given $A \in \text{Par}_n$, let $\text{Min}(A) = \{\min(A_{*j}) : j \in [1, \dim(A)]\}$. For instance, the matrix A in Example 1 has $\text{Min}(A) = \{1, 4, 5, 9\}$. From the definition of Λ the following proposition is apparent.

Proposition 4. *If $w \in \mathcal{I}_n$, $\text{Alph}(w) = \{y_1, \dots, y_k\}_<$ and $A = \Lambda(w)$, then*

$$\text{Min}(A) = \{y_1 + 1, \dots, y_k + 1\} \text{ and } \dim(A) = |\text{Alph}(w)|.$$

Corollary 5. *The statistic dim on Par_n is Eulerian.*

Proof. The statistic “number of distinct entries in the inversion table” is Eulerian; that is, it has the same distribution on \mathfrak{S}_n (the symmetric group) as the number of descents. For a proof due to Deutsch see [4, Corollary 19]. \square

Let us say that i is a *special descent* of $w = (x_1, \dots, x_n) \in \mathcal{I}_n$ if $x_i > x_{i+1}$ and i does not occur in w . Let $\text{sdes}(w)$ denote the number of special descents of w , so

$$\text{sdes}(w) = |\{i : x_i > x_{i+1} \text{ and } x_\ell \neq i \text{ for all } \ell \in [1, n]\}|.$$

Claesson and Linusson [4] conjectured that sdes has the same distribution on \mathcal{I}_n as the so-called bivincular pattern $p = (231, \{1\}, \{1\})$ has on \mathfrak{S}_n . An occurrence of p in a permutation $\pi = a_1 \dots a_n$ is a subword $a_i a_{i+1} a_j$ such that $a_{i+1} > a_i = a_j + 1$. We shall define a statistic on partition matrices that is equidistributed with sdes . Given $A \in \text{Par}_n$ let us say that i is a *column descent* if $i + 1$ is in the same column as, and above, i in A . Let $\text{cdes}(A)$ denote the number of column descents in A , so

$$\text{cdes}(A) = |\{i : \text{row}(i) > \text{row}(i + 1) \text{ and } \text{col}(i) = \text{col}(i + 1)\}|.$$

Proposition 6. *The special descents of $w \in \mathcal{I}_n$ equal the column descents of $\Lambda(w)$.*

Proof. Given $t \in [1, n - 1]$, let $u = \text{row}(t + 1)$ and $v = \text{row}(t)$. As before, let $w = (x_1, \dots, x_n)$ and $\text{Alph}(w) = \{y_1, \dots, y_k\}_<$. By the definition of Λ we have $x_{t+1} = y_u$ and $x_t = y_v$. So, since the numbers y_i are increasing, we have

$$\text{row}(t + 1) < \text{row}(t) \iff u < v \iff y_u < y_v \iff x_{t+1} < x_t.$$

Now, let $u = \text{col}(t + 1)$ and $v = \text{col}(t)$. Then

$$\begin{aligned} \text{col}(t + 1) = \text{col}(t) &\iff t + 1 \in [y_u + 1, y_{u+1}] \text{ and } t \in [y_v + 1, y_{v+1}] \\ &\iff y_u + 1 \leq t \leq y_{v+1} - 1 \\ &\iff x_j \neq t \text{ for all } j \in [1, n], \end{aligned}$$

which concludes the proof. \square

Corollary 7. *The statistic sdes on \mathcal{I}_n has the same distribution as cdes on Par_n .*

2.2. Non-decreasing inversion tables and partition matrices. Let us write Mono_n for the collection of matrices in Par_n which satisfy

$$(iv) \text{ row}(i) < \text{row}(j) \implies i < j,$$

where $\text{row}(i)$ denotes the row in which i is a member. We say that an inversion table (x_1, \dots, x_n) is *non-decreasing* if $x_i \leq x_{i+1}$ for all $1 \leq i < n$.

Theorem 8. *Under the map $\Lambda : \text{Par}_n \rightarrow \mathcal{I}_n$, matrices in Mono_n correspond to non-decreasing inversion tables.*

Proof. Let $w = (x_1, \dots, x_n) \in \mathcal{I}_n$ and $\text{Alph}(w) = \{y_1, \dots, y_k\}_<$. Looking at the definition of Λ , we see that ℓ is in row i of $\Lambda(w)$ if and only if $x_\ell = y_i$. Since the numbers y_i are increasing it follows that condition (iv) holds if and only if w is non-decreasing, as claimed. \square

Proposition 9. $|\text{Mono}_n| = \binom{2n}{n} / (n + 1)$, the n th Catalan number.

Proof. Consider the set of lattice paths in the plane from $(0,0)$ to (n,n) which take steps in the set $\{(1,0), (0,1)\}$ and never go above the diagonal line $y = x$. Such paths are commonly known as *Dyck paths*, and they can be encoded as a sequence (x_1, \dots, x_n) where x_i is the y -coordinate of the i th horizontal step $(1,0)$. The restriction on such a sequence, for it to be Dyck path, is precisely that it is a non-decreasing inversion table. The number of Dyck paths from $(0,0)$ to (n,n) is given by the n th Catalan number. \square

We want to remark that a matrix $A \in \text{Mono}_n$ is completely determined by the cardinalities of its entries. Thus, we can identify A with an upper triangular matrix that contains non-negative entries which sum to n and such that there is at least one non-zero entry in each row and column. Also, the number of such matrices with k rows is equal to the Narayana number

$$\frac{1}{k} \binom{n}{k} \binom{n}{k-1}.$$

2.3. s -diagonal partition matrices. A $k \times k$ matrix A is called *s -diagonal* if A is upper triangular and $A_{ij} = 0$ for $j - i \geq s$. For $s = 1$ we get the collection of diagonal matrices.

Theorem 10. *Let $w = (x_1, \dots, x_n) \in \mathcal{I}_n$, $A = \Lambda(w)$ and $\text{Alph}(w) = \{y_1, \dots, y_k\} <$. Define $y_{k+1} = n$. The matrix A is s -diagonal if and only if for every $i \in [1, n]$ there exists an $a(i) \in [1, k]$ such that*

$$y_{a(i)} < i \leq y_{a(i)+1} \text{ and } x_i \in \{y_{a(i)}, y_{a(i)-1}, \dots, y_{\max(1, a(i)-s+1)}\}.$$

Proof. From the definition of Λ we have that i is in row $a - j$ of A precisely when $x_i = y_{a-j}$, and that i is in column a of A precisely when $y_a < i \leq y_{a+1}$. The matrix A is s -diagonal if for every entry i , there exists $a(i)$ such that i is in column $a(i)$ and row $a(i) - j$ for some $0 \leq j < s$. \square

Setting $s = 1$ in the above theorem gives us the class of diagonal matrices. These admit a more explicit description which we will now present.

In computer science, *run-length encoding* is a simple form of data compression in which consecutive data elements (runs) are stored as a single data element and its multiplicity. We shall apply this to inversion tables, but for convenience rather than compression purposes. Let $\text{RLE}(w)$ denote the run-length encoding of the inversion table w . For example,

$$\text{RLE}(0, 0, 0, 0, 1, 1, 0, 2, 3, 3) = (0, 4)(1, 2)(0, 1)(2, 1)(3, 2).$$

A sequence of positive integers (u_1, \dots, u_k) which sum to n is called an *integer composition* of n and we write this as $(u_1, \dots, u_k) \models n$.

Corollary 11. *The set of diagonal matrices in Par_n is the image under Λ of*

$$\{w \in \mathcal{I}_n : (u_1, \dots, u_k) \models n \text{ and } \text{RLE}(w) = (p_0, u_1) \dots (p_{k-1}, u_k)\},$$

where $p_0 = 0$, $p_1 = u_1$, $p_2 = u_1 + u_2$, $p_3 = u_1 + u_2 + u_3$, etc.

Since diagonal matrices are in bijection with integer compositions, the number of diagonal matrices in Par_n is 2^{n-1} . Although the bidiagonal matrices do not admit as compact a description in terms of the corresponding inversion tables, we can

We would like to enumerate paths that start with ϵ and end in a configuration with no empty rows or columns. Letting M denote the above transfer-matrix, this amounts to calculating the first coordinate in

$$(1 - xM)^{-1}[11101101001101]^T.$$

Proposition 12. *We have*

$$\sum_{n \geq 0} \sum_{A \in \text{BiPar}_n} q^{\dim(A)} x^n = \frac{2x^3 - (q+5)x^2 + (q+4)x - 1}{2(q^2 + q + 1)x^3 - (q^2 + 4q + 5)x^2 + 2(q+2)x - 1},$$

where BiPar_n is the collection of bidiagonal matrices in Par_n .

We find it interesting that the number of bidiagonal matrices in Par_n is given by the sequence [8, A164870], which corresponds to permutations of $[1, n]$ which are sortable by two pop-stacks in parallel. In terms of pattern avoidance those are the permutations in the class

$$\mathfrak{S}_n(3214, 2143, 24135, 41352, 14352, 13542, 13524).$$

See Atkinson and Sack [1]. Moreover, there are exactly 2^{n-1} permutations of $[1, n]$ which are sortable by one pop-stack; hence equinumerous with the diagonal partition matrices. One might then wonder about permutations which are sortable by three pop-stacks in parallel. Are they equinumerous with tridiagonal partition matrices? Computations show that this is not the case: For $n = 6$ there are 646 tridiagonal partition matrices, but only 644 permutations which are sortable by three pop-stacks in parallel. For more on the enumeration of permutations sortable by three pop stacks in parallel see Smith and Vatter [9].

3. COMPOSITION MATRICES AND $(2+2)$ -FREE POSETS

Consider Definition 2. Define a *composition matrix* to be a matrix that satisfies conditions (i) and (ii), but not necessarily (iii). Let $\text{Comp}_n \supseteq \text{Par}_n$ denote the set of all composition matrices on $[1, n]$. The smallest example of a composition matrix that is not a partition matrix is

$$\begin{bmatrix} \{2\} & \emptyset \\ \emptyset & \{1\} \end{bmatrix}.$$

In this section we shall give a bijection from Comp_n to the set of $(2+2)$ -free posets on $[1, n]$. This bijection will factor through a certain union of Cartesian products that we now define. Given a set X , let us write $\binom{X}{x_1, \dots, x_\ell}$ for the collection of all sequences (X_1, \dots, X_ℓ) that are ordered set partitions of X and $|X_i| = x_i$ for all $i \in [1, \ell]$. For a sequence (a_1, \dots, a_i) of numbers let

$$\text{asc}(a_1, \dots, a_i) = |\{j \in [1, i-1] : a_j < a_{j+1}\}|.$$

Following Bousquet-Mélou et al. [2] we define a sequence of non-negative integers $\alpha = (a_1, \dots, a_n)$ to be an *ascent sequence* if $a_1 = 0$ and $a_{i+1} \in [0, 1 + \text{asc}(a_1, \dots, a_i)]$ for $0 < i < n$. Let \mathcal{A}_n be the collection of ascent sequences of length n . Define the *run-length record* of α to be the sequence that records the multiplicities of adjacent values in α . We denote it by $\text{RLR}(\alpha)$. In other words, $\text{RLR}(\alpha)$ is the sequence of second coordinates in $\text{RLE}(\alpha)$, the run-length encoding of α . For instance,

$$\text{RLR}(0, 0, 0, 0, 1, 1, 0, 2, 3, 3) = (4, 2, 1, 1, 2).$$

Finally we are in a position to define the set through which our bijection from Comp_n to $(2+2)$ -free posets on $[1, n]$ will factor. Let

$$\mathfrak{A}_n = \bigcup_{\alpha \in \mathcal{A}_n} \{\alpha\} \times \binom{[1, n]}{\text{RLR}(\alpha)}.$$

Let \mathcal{M}_n be the collection of upper triangular matrices that contain non-negative integers whose entries sum to n and such that there is no column or row of all zeros. Dukes and Parviainen [5] presented a bijection

$$\Gamma : \mathcal{M}_n \rightarrow \mathcal{A}_n.$$

Given $A \in \mathcal{M}_n$, let $\text{nz}(A)$ be the number of non-zero entries in A . Since it follows from [5, Thm. 4] that A may be uniquely constructed, in a step-wise fashion, from the ascent sequence $\Gamma(A)$, we may associate to each non-zero entry A_{ij} its time of creation $T_A(i, j) \in [1, \text{nz}(A)]$. By defining $T_A(i, j) = 0$ if $A_{ij} = 0$ we may view T_A as a $\dim(A) \times \dim(A)$ matrix. Define $\text{Seq}(A) = (y_1, \dots, y_{\text{nz}(A)})$ where $y_t = A_{ij}$ and $T_A(i, j) = t$.

Example 13. With

$$A = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

we have

$$T_A = \begin{bmatrix} 1 & 0 & 5 & 8 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

and

$$\text{Seq}(A) = (3, 1, 2, 1, 3, 1, 1, 1).$$

Lemma 14. *Given $A \in \mathcal{M}_n$, we have that $\text{Seq}(A) = \text{RLR}(\Gamma(A))$.*

Proof. This is a straightforward consequence of the construction rules given by Dukes and Parviainen [5]. \square

For a matrix $A \in \text{Comp}_n$ define $\text{Card}(A)$ as the matrix obtained from A by taking the cardinality of each of its entries. Note that $A \mapsto \text{Card}(A)$ is a surjection from Par_n onto \mathcal{M}_n . Define $E(A)$ as the ordered set partition $(X_1, \dots, X_{\text{nz}(A)})$, where $X_t = A_{ij}$ for $t = T_{\text{Card}(A)}(i, j)$. Finally, define $f : \text{Comp}_n \rightarrow \mathfrak{A}_n$ by

$$f(A) = (\Gamma(\text{Card}(A)), E(A)).$$

Example 15. Let us calculate $f(A)$ for

$$A = \begin{bmatrix} \{3, 8\} & \{6\} & \emptyset \\ \emptyset & \{2, 5, 7\} & \emptyset \\ \emptyset & \emptyset & \{1, 4\} \end{bmatrix}.$$

We have

$$\text{Card}(A) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \quad T_{\text{Card}(A)} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

It is straightforward to check that ϕ and ψ are each others inverses. Consequently, we have the following theorem.

Theorem 18. *The map $\phi : \mathfrak{A}_n \rightarrow \mathfrak{P}_n$ is a bijection and ψ is its inverse.*

Let \mathcal{C}_n be the collection of composition matrices M on $[1, n]$ with the following property: in every row of M , the entries are increasing from left to right. An example of a matrix in \mathcal{C}_8 is

$$M' = \begin{bmatrix} \{4, 6\} & \{8\} & \emptyset \\ \emptyset & \{1\} & \{7\} \\ \emptyset & \emptyset & \{2, 3, 5\} \end{bmatrix}.$$

Every matrix $M \in \mathcal{C}_n$ may be written as a unique pair $(w(M), \hat{\chi}(M))$ where:

- $w(M)$ is the non-decreasing inversion table that via Theorem 8 corresponds to $\text{mono}(M)$, the matrix in Mono_n that is formed from M the following way: replace the entries in M from left to right, beginning with the first row, with the values $1, \dots, n$, in that order. For the example above we have

$$\text{mono}(M') = \begin{bmatrix} \{1, 2\} & \{3\} & \emptyset \\ \emptyset & \{4\} & \{5\} \\ \emptyset & \emptyset & \{6, 7, 8\} \end{bmatrix} \in \text{Mono}_8$$

and $w(M') = (0, 0, 0, 2, 2, 4, 4, 4)$.

- $\chi(M) = (X_1, \dots, X_k)$ where X_i is the union of the entries in row i of M , and $\hat{\chi}(M)$ is the permutation of $[n]$ achieved by removing the parentheses from $\chi(M)$; see paragraph after Theorem 16. For the above example, $\chi(M') = (\{4, 6, 8\}, \{1, 7\}, \{2, 3, 5\})$ and $\hat{\chi}(M') = (4, 6, 8, 1, 7, 2, 3, 5)$.

Given $M \in \mathcal{C}_n$ with $w(M) = (w_1, \dots, w_n)$ and $\hat{\chi}(M) = (\hat{\chi}_1, \dots, \hat{\chi}_n)$, let us define the sequence $\eta(M) = (a_1, \dots, a_n)$ by $a_i = 1 + w_j$ where $i = \hat{\chi}_j$. (For the small example above, we have $\eta(M') = (3, 5, 5, 1, 5, 1, 3, 1)$.) Recall [11, p. 94] that a sequence $\eta = (a_1, \dots, a_n) \in [n]^n$ is a *parking function* if and only if the increasing rearrangement $b_1 \leq b_2 \leq \dots \leq b_n$ of a_1, \dots, a_n satisfies $b_i \leq i$.

Theorem 19. *Matrices in \mathcal{C}_n are in one-to-one correspondence with parking functions of order n . The parking function that corresponds to the matrix $M \in \mathcal{C}_n$ is (a_1, \dots, a_n) where $a_{\hat{\chi}_j} = 1 + w_j$ and $M \leftrightarrow (w(M), \hat{\chi}(M))$.*

3.1. Diagonal and bidiagonal composition matrices. Consider the set of diagonal matrices DiComp_n in Comp_n . Under the bijection f these matrices map to pairs $(\alpha, \chi) \in \mathfrak{A}_n$ where α is a non-decreasing ascent sequence. Applying ϕ to $f(\text{DiComp}_n)$ we get the collection of $(2+2)$ -free posets P which have the property that every element at level j covers every element at level $j-1$, for all levels j but the first. Reading the levels from bottom to top we get an ordered set partition of $[1, n]$. It is not hard to see that this ordered set partition is, in fact, χ . For instance,

$$\begin{bmatrix} \{4\} & \emptyset & \emptyset \\ \emptyset & \{1, 3\} & \emptyset \\ \emptyset & \emptyset & \{2, 5\} \end{bmatrix} \leftrightarrow ((0, 1, 1, 2, 2), \{4\}\{1, 3\}\{2, 5\}) \leftrightarrow \begin{array}{c} \begin{array}{cc} 2 & 5 \\ \bullet & \bullet \end{array} \\ \begin{array}{ccc} \bullet & & \bullet \\ | & \diagdown & / \\ \bullet & & \bullet \\ | & & | \\ 1 & & 3 \\ \bullet & & \bullet \\ \diagup & & \diagdown \\ & 4 & \end{array} \end{array}.$$

It follows that there are exactly $k!S(n, k)$ diagonal composition matrices of size n and dimension k , where $S(n, k)$ is the number of partitions of an n element set into k parts (a *Stirling number of the second kind*). For bidiagonal composition matrices the situation is a bit more complicated:

Proposition 20. *We have*

$$\sum_{n \geq 0} \sum_{A \in \text{BiComp}_n} q^{\dim(A)} \frac{x^n}{n!} = \frac{qe^{2x} - qe^x - 1}{(1-q)qe^{2x} + 2q^2e^x - q^2 - q - 1},$$

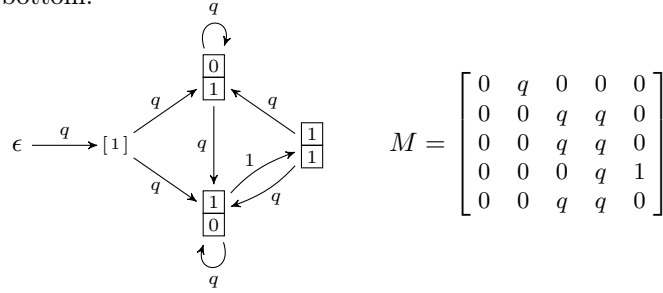
where BiComp_n is the collection of bidiagonal matrices in Comp_n .

Proof. Let \mathcal{B}_n be the collection of binary bidiagonal matrices in \mathcal{M}_n . A matrix $A \in \text{BiComp}_n$ can in a natural and simple way be identified with a pair (B, χ) where $B \in \mathcal{B}_k$, $k = \text{nz}(\text{Card}(A))$, and χ is an ordered set partition of $[1, n]$:

$$\begin{bmatrix} \{3, 8\} & \{6\} & \emptyset \\ \emptyset & \{2, 5, 7\} & \emptyset \\ \emptyset & \emptyset & \{1, 4\} \end{bmatrix} \leftrightarrow \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \{3, 8\}\{6\}\{2, 5, 7\}\{1, 4\} \right)$$

Thus, if $F(q, x)$ is the ordinary generating function for matrices in \mathcal{B}_n counted by dimension and size, then $F(q, e^x - 1)$ is the exponential generating function we require. This is because $F(q, x)$ is also the generating function for pairs (B, π) where $B \in \mathcal{B}_n$ and $\pi \in \mathfrak{S}_n$, and $e^x - 1$ is the exponential generating function for non-empty sets.

We now derive $F(q, x)$ using the transfer-matrix method. We grow the matrices in \mathcal{B}_n from left to right by adding new columns, and within a column we add ones from top to bottom:



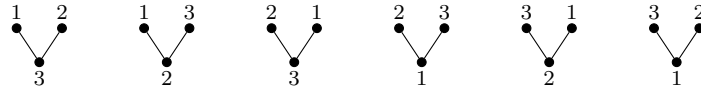
Here ϵ is the empty matrix; $[1]$ is the 1×1 identity matrix; $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ denotes any matrix in \mathcal{B}_n of dimension 2 or more whose bottom most entries in the last column are 0 and 1; etc. Calculating the first entry in $(1 - xM)^{-1}[11101]^T$ we find that

$$F(q, x) = \sum_{n \geq 0} \sum_{A \in \mathcal{B}_n} q^{\dim(A)} x^n = \frac{qx^2 + qx - 1}{(1-q)qx^2 + 2qx - 1},$$

and on simplifying $F(q, e^x - 1)$ we arrive at the claimed generating function. \square

4. THE NUMBER OF $(2 + 2)$ -FREE POSETS ON $[1, n]$

Let us consider *plane $(2+2)$ -free posets* on $[1, n]$. That is, $(2 + 2)$ -free posets on $[1, n]$ with a canonical embedding in the plane. For instance, these are six *different* plane $(2 + 2)$ -free posets on $[1, 3] = \{1, 2, 3\}$:



By definition, if u_n is the number of unlabeled $(2 + 2)$ -free posets on n nodes, then $u_n n!$ is the number of plane $(2 + 2)$ -free posets on $[1, n]$. In other words, we may identify the set of plane $(2 + 2)$ -free posets on $[1, n]$ with the Cartesian product

$\mathcal{P}_n \times \mathfrak{S}_n$, where \mathcal{P}_n denotes the set of unlabeled $(2+2)$ -free posets on n nodes and \mathfrak{S}_n denotes the set of permutations on $[1, n]$. We shall demonstrate the isomorphism

$$\bigcup_{\pi \in \mathfrak{S}_n} \mathfrak{P}(\text{Cyc}(\pi)) \simeq \mathcal{P}_n \times \mathfrak{S}_n, \quad (2)$$

where $\text{Cyc}(\pi)$ is the set of (disjoint) cycles of π and $\mathfrak{P}(\text{Cyc}(\pi))$ is the set of $(2+2)$ -free posets on those cycles. As an illustration we consider the case $n = 3$. On the right-hand side we have $|\mathcal{P}_3 \times \mathfrak{S}_3| = |\mathcal{P}_3| |\mathfrak{S}_3| = 5 \cdot 6 = 30$ plane $(2+2)$ -free posets. Taking the cardinality of the left-hand side we get

$$\begin{aligned} & |\mathfrak{P}\{(1),(2),(3)\}| + |\mathfrak{P}\{(1),(23)\}| + |\mathfrak{P}\{(12),(3)\}| + |\mathfrak{P}\{(2),(13)\}| + |\mathfrak{P}\{(123)\}| + |\mathfrak{P}\{(132)\}| \\ &= |\mathfrak{P}_3| + 3|\mathfrak{P}_2| + 2|\mathfrak{P}_1| = 19 + 3 \cdot 3 + 2 \cdot 1 = 30. \end{aligned}$$

Bousquet-Mélou et al. [2] gave a bijection Ψ from \mathcal{P}_n to \mathcal{A}_n , the set of ascent sequences of length n . Recall also that in Theorem 18 we gave a bijection ϕ from \mathfrak{P}_n to \mathfrak{A}_n . Of course, there is nothing special about the ground set being $[1, n]$ in Theorem 18; so, for any finite set X , the map ϕ can be seen as a bijection from $(2+2)$ -free posets on X to the set

$$\mathfrak{A}(X) = \bigcup_{\alpha \in \mathcal{A}_{|X|}} \{\alpha\} \times \binom{X}{\text{RLR}(\alpha)}.$$

In addition, the fundamental transformation [3] is a bijection between permutations with exactly k cycles and permutations with exactly k left-to-right minima. Putting these observations together it is clear that to show (2) it suffices to show

$$\bigcup_{\pi \in \mathfrak{S}_n} \mathfrak{A}(\text{LMin}(\pi)) \simeq \mathcal{A}_n \times \mathfrak{S}_n, \quad (3)$$

where $\text{LMin}(\pi)$ is the set of segments obtained by breaking π apart at each left-to-right minima. For instance, the left-to-right minima of $\pi = 5731462$ are 5, 3 and 1; so $\text{LMin}(\pi) = \{57, 3, 1462\}$.

Let us now prove (3) by giving a bijection h from the left-hand side to the right-hand side. To this end, fix a permutation $\pi \in \mathfrak{S}_n$ and let $k = |\text{LMin}(\pi)|$ be the number of left-to-right minima in π . Assume that $\alpha = (a_1, \dots, a_k)$ is an ascent sequence in \mathcal{A}_k and that $\chi = (X_1, \dots, X_r)$ is an ordered set partition in $\binom{\text{LMin}(\pi)}{\text{RLR}(\alpha)}$. To specify the bijection h let

$$h(\alpha, \chi) = (\beta, \tau)$$

where $\beta \in \mathcal{A}_n$ and $\tau \in \mathfrak{S}_n$ are defined in the next paragraph.

For each $i \in [1, r]$, first order the blocks of X_i decreasingly with respect to first (and thus minimal) element, then concatenate the blocks to form a word \hat{X}_i . Define the permutation τ as the concatenation of the words \hat{X}_i :

$$\tau = \hat{X}_1 \dots \hat{X}_k.$$

Let $i_1 = 1$, $i_2 = i_1 + |X_1|$, $i_3 = i_2 + |X_2|$, etc. By definition, these are the indices where the ascent sequence α changes in value. Define β by

$$\text{RLE}(\beta) = (a_{i_1}, x_1) \dots (a_{i_k}, x_k), \text{ where } x_i = |\hat{X}_i|.$$

Consider the permutation $\pi = \text{A9B68D4F32C175E} \in \mathfrak{S}_{15}$ (in hexadecimal notation). Then $\text{LMin}(\pi) = \{\text{A}, \text{9B}, \text{68D}, \text{4F}, \text{3}, \text{2C}, \text{175E}\}$. Assume that

$$\begin{aligned}\alpha &= (0, 0, 1, 2, 2, 2, 0); \\ \chi &= \{\text{2C}, \text{68D}\}\{\text{9B}\}\{\text{3}, \text{175E}, \text{4F}\}\{\text{A}\}.\end{aligned}$$

Then we have $\hat{X}_1 = \text{68D2C}$, $\hat{X}_2 = \text{9B}$, $\hat{X}_3 = \text{4F3175E}$ and $\hat{X}_4 = \text{A}$. Also, $i_1 = 1$, $i_2 = 1 + 2 = 3$, $i_3 = 3 + 1 = 4$ and $i_4 = 4 + 3 = 7$. Consequently,

$$\begin{aligned}\beta &= (0, 0, 0, 0, 0, 1, 1, 2, 2, 2, 2, 2, 2, 2, 0); \\ \tau &= \text{6 8 D 2 C 9 B 4 F 3 1 7 5 E A}.\end{aligned}$$

It is clear how to reverse this procedure: Split τ into segments according to where β changes in value when reading from left to right. With τ as above we get

$$(\text{68D2C}, \text{9B}, \text{4F3175E}, \text{A}) = (\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4)$$

We have thus recovered \hat{X}_1, \hat{X}_2 , etc. Now $X_i = \text{LMin}(\hat{X}_i)$, and we thus know χ . It only remains to recover α . Assume that $\text{RLE}(\beta) = (b_1, x_1) \dots (b_k, x_k)$, then $\text{RLE}(\alpha) = (b_1, |X_1|) \dots (b_k, |X_k|)$. This concludes the proof of (3). Let us record this result.

Theorem 21. *The map $h : \cup_{\pi \in \mathfrak{S}_n} \mathfrak{A}(\text{LMin}(\pi)) \rightarrow \mathcal{A}_n \times \mathfrak{S}_n$ is a bijection.*

As previously explained, (2) also follows from this proposition. Let us now use (2) to derive an exponential generating function $L(t)$ for the number of $(2+2)$ -free posets on $[1, n]$. Bousquet-Mélou et al. [2] gave the following *ordinary* generating function for *unlabeled* $(2+2)$ -free posets on n nodes:

$$\begin{aligned}P(t) &= \sum_{n \geq 0} \prod_{i=1}^n (1 - (1-t)^i) \\ &= 1 + t + 2t^2 + 5t^3 + 15t^4 + 53t^5 + 217t^6 + 1014t^7 + 5335t^8 + O(t^9).\end{aligned}$$

This is, of course, also the exponential generating function for plane $(2+2)$ -free posets on $[1, n]$. Moreover, the exponential generating function for cyclic permutations is $\log(1/(1-t))$. On taking the union over $n \geq 0$ of both sides of (2) it follows that $L(\log(1/(1-t))) = P(t)$; so $L(t) = P(1 - e^{-t})$.

Corollary 22. *The exponential generating function for $(2+2)$ -free posets is*

$$\begin{aligned}L(t) &= \sum_{n \geq 0} \prod_{i=1}^n (1 - e^{-ti}) \\ &= 1 + t + 3\frac{t^2}{2!} + 19\frac{t^3}{3!} + 207\frac{t^4}{4!} + 3451\frac{t^5}{5!} + 81663\frac{t^6}{6!} + 2602699\frac{t^7}{7!} + O(t^8).\end{aligned}$$

This last result also follows from a result of Zagier [12, Eq. 24] and a bijection, due to Bousquet-Mélou et al [2], between unlabeled $(2+2)$ -free posets and certain matchings. See also Exercises 14 and 15 in Chapter 3 of the second edition of Enumerative Combinatorics volume 1 (available on R. Stanley's homepage).

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