PERMUTATIONS SORTABLE BY $n-4$ PASSES THROUGH A STACK

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ABSTRACT. We characterise and enumerate permutations that are sortable by $n-4$ passes through a stack. We conjecture the number of permutations sortable by $n-5$ passes, and also the form of a formula for the general case $n-k$, which involves a polynomial expression.

1. BACKGROUND

We view permutations as words without repeated letters; if π is a permutation of an n element set and $\pi(i)=\pi_i$, then we write $\pi=\pi_1\dots\pi_n$. The stack sorting operator S can be defined recursively on permutations of finite subsets of $\{1,2,\dots\}$ as follows. If π is empty then $S(\pi)=\pi$. If π is nonempty write π as the concatenation $\pi=LnR$, where n is the greatest element of π and L and R are the subwords to the left and right of n respectively. Then

$$S(\pi)=S(L)S(R)n.$$

For example, $S(42513)=24135$. We say that a permutation π is *k-stack sortable* if $S^k(\pi)=\mathrm{id}$, where $S^k=S\circ S^{k-1}$, S^0 is the identity operator and id is the identity permutation $12\dots n$. Let the *(stack sorting) complexity* of π, denoted $\operatorname{ssc}\pi$, be the smallest k such that π is k-stack sortable. Let $\mathfrak{S}_n$ be the set of permutations of $\{1,\dots,n\}$. For the permutations in $\mathfrak{S}_3$ we have

π =	123	132	213	231	312	321
$\operatorname{ssc}\pi$ =	0	1	1	2	1	1

Let $W_{n,k}$ be the set of all k-stack sortable permutations in $\mathfrak{S}_n$; in other words, $W_{n,k}$ is the set of permutations in $\mathfrak{S}_n$ whose complexity is at most k. Let $E_{n,k}$ be the set of permutations in $\mathfrak{S}_n$ whose complexity is exactly k. Note that

$$E_{n,k}=W_{n,k}-W_{n,k-1};$$
$$W_{n,k}=\mathfrak{S}_n-(E_{n,k+1}\cup\dots\cup E_{n,n-1}).$$

It is easy to see that $W_{n,n-1}=\mathfrak{S}_n$. Knuth [1, 2.2.1.5] leaves as an exercise to the reader to show that $W_{n,1}=\mathfrak{S}_n(231)$. In his PhD thesis West [3] showed that

$$W_{n,2}=\mathfrak{S}_n(2341,3\bar{5}241),$$

where $\mathfrak{S}_n(2341,3\bar{5}241)$ is the set of permutations in $\mathfrak{S}_n$ that avoid the pattern 2341 and the "barred" pattern $3\bar{5}241$; for information on these, see [2], especially Section 7. West also showed that $W_{n,n-2}$ are precisely those permutations that do not have suffix $n1$. This last statement is easily shown by proving that the permutations in $E_{n,n-1}$ are those with suffix $n1$.

In addition, West characterized $E_{n,n-2}$. To state that result it is convenient to introduce some notation for special sets of words over the alphabet $\{1,2,\dots\}$. Let an asterisk $(*)$ stand for any word of zero or more characters, and let a question

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mark (?) stand for any single letter. These conventions are adopted from the so called glob patterns in computer science. For a word w over $\{*, ?\} \cup \{1, 2, \dots\}$, let $\langle\, w\,\rangle$ denote the set of words of the form w. For instance, $\langle *n1 \rangle$ consists of all words with suffix $n1$, and $\mathfrak{S}_5 \cap \langle *51? \rangle = \{23514, 24513, 32514, 34512, 42513, 43512\}$. Let

$$\langle\, w_1, \dots, w_k \,\rangle = \langle\, w_1 \,\rangle \cup \cdots \cup \langle\, w_k \,\rangle.$$

West's characterisation of $E_{n,n-1}$ and $E_{n,n-2}$ can then be stated as in the following two lemmas, which follow from Theorems 4.2.4 and 4.2.17 in [3] and their proofs.

Lemma 1.1. *For all $n \geq 2$,*

$$E_{n,n-1} = \mathfrak{S}_n \cap \langle *n1 \rangle.$$

Thus, the cardinality of $E_{n,n-2}$ is $(n-2)!$.

Lemma 1.2. *For all $n \geq 4$,*

$$E_{n,n-2} = \mathfrak{S}_n \cap \langle\ *n2,\ *(n-1)1n,\ *n1?,\ *n?1,\ *n*(n-2)*(n-1)1\ \rangle.$$

Thus, the cardinality of $E_{n,n-3}$ is $(n-3)!(7n-12)/2$.

By subtracting the cardinalities in Lemmas 1.1 and 1.2 from $n!$, we get the following result.

Proposition 1.3. *For all $n \geq 4$ the cardinality of $W_{n,n-3}$ is*

$$\frac{(n-3)!}{2}\left(2n^3 - 6n^2 - 5n + 16\right).$$

2. Permutations requiring exactly $(n-3)$-stack sorts

Theorem 2.1. *For all $n \geq 6$, the set of permutations $E_{n,n-3}$ are those given in column labeled "Type" of Table 1. The number of such permutations is*

$$\frac{(n-4)!}{3}\left(47\binom{n-6}{2} + 194\binom{n-6}{1} + 297\right). \tag{2.1}$$

To prove this we require some terminology and results from West [3, §4.2]. A *forbidden pattern of order k* in a permutation π is a triple (B, c, a), where B is a subsequence of length k in π and (c, a) is a pair of entries in π such that for every $b \in B$ the subsequence bca is an occurrence of the pattern 231, that is, $a < b < c$. In such a situation we say that the pair (c, a) *witnesses* the forbidden pattern B. We call a forbidden pattern (B, c, a) *uninterrupted* if there is no subsequence bxb' in π where $b, b' \in B$ and $x > c$.

Lemma 2.2. [3, Theorems 4.2.10 and 4.2.14] *Let π be a permutation.*

(i) $\mathrm{ssc}(\pi) \leq k$ *if π does not contain any forbidden pattern of order k;*
(ii) $\mathrm{ssc}(\pi) > k$ *if π contains an uninterrupted forbidden pattern of order k.*

Proof of Theorem 2.1. Let $\pi \in \mathfrak{S}_n$. Using the contrapositive of (i) and (ii) in Lemma 2.2, if $\mathrm{ssc}(\pi) = n-3$, then π contains a forbidden pattern of order $n-4$ and does not contain an uninterrupted forbidden pattern of order $n-3$. The forbidden pattern of order $n-4$ must be witnessed by entries that appear in some two of the positions $n-3$, $n-2$, $n-1$ or n of the permutation. We will first condition on the position of the largest entry in the permutations, then condition on those permutations that contain a forbidden pattern of the required order, and finally single out those permutations that are in $E_{n,n-3}$.

Suppose $\pi_n = n$. A permutation $\pi = \pi' n$ is a member of $E_{n,n-3}$ precisely when π' is a member of $E_{n-1,n-3}$. Thus entries 1(a)–1(e) of Table 1 immediately follow from Lemma 1.2.

Case	Type	Number
$\pi_n = n$		
1(a)	$*(n-1)2n$	$(n-3)!$
1(b)	$*(n-1)?1n$	$(n-3)!$
1(c)	$*(n-1)1?n$	$(n-3)!$
1(d)	$*(n-1)*(n-3)*(n-2)1n$	$(n-3)!/2$
1(e)	$*(n-2)1(n-1)n$	$(n-4)!$
$\pi_{n-1} = n$		
2(a)	$*n3$	$(n-2)!$
2(b)	$*(n-1)1n?$	$(n-5)(n-4)!$
2(c)	$*(n-2)1n(n-1)$	$(n-4)!$
$\pi_{n-2} = n$		
3(a)	$*n2?$	$(n-3)(n-3)!$
3(b)	$*n?2$	$(n-3)(n-3)!$
$\pi_{n-3} = n$		
4(a)	$*nxy1$, but not $n(n-2)(n-1)1$	$(n-2)! - (n-4)!$
4(b)	$*nx1y$	$(n-2)!$
4(c)	$*n1xy$	$(n-2)!$
4(d)	$*n(n-2)(n-1)2$	$(n-4)!$
$\pi_{n-i} = n$ and $i > 3$		
5(a)	$*nA(n-2)B(n-1)2$, where $A \cup B \neq \emptyset$	$(n-2)!/2 - (n-4)!$
5(b)	$*n*(n-3)*(n-1)(n-2)1$	$(n-3)!/2$
5(c)	$*n*(n-1)*(n-3)*(n-2)1$	$(n-2)!/6$
5(d)	$*(n-1)*n*\left\{\begin{matrix}(n-3)*(n-4)\\(n-4)*(n-3)\end{matrix}\right\}*(n-2)1$	$(n-2)!/12$
5(e)	$*n*(n-2)*(n-1)\left\{\begin{matrix}1?\\?1\end{matrix}\right\}$	$(n-4)(n-3)!$
5(f)	$*n*(n-3)*(n-1)1(n-2)$	$(n-3)!/2$
5(g)	$*n*(n-3)*(n-2)1(n-1)$	$(n-3)!/2$
5(h)	$*(n-2)*n*\left\{\begin{matrix}(n-3)*(n-4)\\(n-4)*(n-3)\end{matrix}\right\}*(n-1)1$	$(n-2)!/12$

TABLE 1. Permutations in $E_{n,n-3}$

Suppose $\pi_{n-1} = n$. We must have $\pi_n \geq 3$, for otherwise, by Lemmas 1.1 and 1.2, we would have $\pi \in E_{n,n-1}$ or $\pi \in E_{n,n-2}$. If $\pi_n = 3$ then the permutation is in $E_{n,n-3}$, and hence we get 2(a). If $k = \pi_n > 3$ then it is impossible for (n,k) to witness a forbidden pattern F of order $n-4$ since there could be at most $n-k-1 < n-4$ elements in F. Thus the forbidden pattern F must be witnessed by (π_{n-3}, π_{n-2}). As there are now $n-2$ values from which to form the forbidden pattern F, we are forced to choose the extreme values from this set as the values for (π_{n-3}, π_{n-2}). Consequently, if $k = n-1$ then we must have $(\pi_{n-3}, \pi_{n-2}) = (n-2, 1)$, giving case 2(c). Otherwise $3 < k < n-1$ and $(\pi_{n-3}, \pi_{n-2}) = (n-1, 1)$ which gives case 2(b). One easily verifies that all such permutations are in $E_{n,n-3}$.

Suppose $\pi_{n-2} = n$. By Lemma 1.2 the value 1 cannot be to the right of n. Let k be the smaller of the two values π_{n-1} and π_n. If $k > 2$ then the pair (n,k) witnesses a forbidden pattern of order at most $n-k-1-1 < n-4$. For the same reason, (π_{n-1}, π_n) cannot witness a forbidden pattern of order $n-4$. So exactly one of the entries to the right of n must be 2, giving 3(a) and (b). One easily verifies that all such permutations are in $E_{n,n-3}$.

Suppose $\pi_{n-3} = n$. If 1 is to the right of n in π then $(n,1)$ witnesses a forbidden pattern of order $n-4$. However, it is possible that there is an uninterrupted forbidden pattern of order $n-3$ in π. This happens when $\pi \in \langle\, {*}n(n-2)(n-1)1 \,\rangle$ as in Lemma 1.2. In all other cases the permutation is in $E_{n,n-3}$. This gives 4(a), (b) and (c).

Alternatively, if 1 is to the left of n, let k be the value of the smallest entry to the right of n. Since there are at most $n-5$ entries to the right of n in π that take values between k and n, the pair (n,k) cannot witness a forbidden pattern of order $n-4$. In order for π to contain a forbidden pattern F of order $n-4$ it must be witnessed by the pair (π_{n-1}, π_n) and have as the block of $n-4$ values the value π_{n-2} along with the $n-5$ elements to the right of n that are not 1. Thus $\pi_n = 2$, $\pi_{n-1} = n-1$ and

$$\pi \in \langle\, {*}nk(n-1)2 \,\rangle$$

for some $k \neq 1$. One easily checks that the only value of k for which $\pi \in E_{n,n-3}$ is $k = n-2$. This gives 4(d).

Suppose $\pi_{n-i} = n$ for some $i > 3$. Then (n,k) can never witness a forbidden pattern of order $n-4$. In the remainder of the proof, we condition on the relative positions of $(n-2)$, $(n-1)$ and n. If $\pi = AnB(n-1)C(n-2)D$ where $|B \cup C \cup D| \geq 2$, then

$$S(\pi) = S(A)S(B)S(C)S(D)(n-2)(n-1)n.$$

This gives

$$\begin{aligned} \pi \in E_{n,n-3} &\iff S(A)S(B)S(C)S(D) \in E_{n-3,n-4} \\ &\iff S(A)S(B)S(C)S(D) \in \langle\, {*}(n-3)1 \,\rangle. \end{aligned}$$

Since $|B \cup C \cup D| \geq 2$, the ways in which this can happen are restricted to

(i) $D = 1$ and $n-3 \in C$, giving case 5(c),
(ii) $D = 1$, $C = \emptyset$ and $n-3 \in B$, giving case 5(b),
(iii) $D = \emptyset$, $C = 1$ and $n-3 \in B$, giving case 5(f).

If $\pi = AnB(n-2)C(n-1)D$ where $|B \cup C \cup D| \geq 2$, then

$$S(\pi) = S(A)S(B)S(C)(n-2)S(D)(n-1)n.$$

This implies that π belongs to $E_{n,n-3}$ if and only if

$$S(A)S(B)S(C)(n-2)S(D) \in E_{n-2,n-4}.$$

If $D = \emptyset$ then π belongs to $E_{n,n-3}$ if and only if $S(A)S(B)S(C)$ belongs to $E_{n-3,n-4}$, which happens if and only if $C = 1$ and $(n-3) \in B$, from which we get 5(g). Otherwise $S(A)S(B)S(C)(n-2)S(D)$ belongs to the types listed in Lemma 1.2. There are 3 cases to consider:

(i) $S(D) = 2$, so $\pi = AnB(n-2)C(n-1)2$ where $B \cup C \neq \emptyset$, giving case 5(a);
(ii) $S(D) = 1k$, so $D = 1k$ or $D = k1$, giving case 5(e); and
(iii) it is not possible that $S(A)S(B)S(C)(n-2)S(D)$ matches the last type in Lemma 1.2 since this would mean that $S(D)$ ends in 1, its smallest entry.

If $\pi = A(n-1)BnC(n-2)D$ where $|C \cup D| \geq 3$, then a forbidden pattern of order $n-4$ can only be witnessed by the pair $(n-2, 1)$. Furthermore, all elements 2 through $n-3$ must be to the left of $n-2$, so $\pi_{n-1} = n-2$ and $\pi_n = 1$, that is $\pi = A(n-1)BnC(n-2)1$ where $|C| \geq 2$. From this we have $S^2(\pi) = S(S(A)S(B))S(S(C)1)(n-2)(n-1)n$ and

$$\pi \in E_{n,n-3} \iff \pi' = S(S(A)S(B))S(S(C)1) \in E_{n-3,n-5}.$$

The conditions on π' (and therefore π) are easily derived by comparing π' to the types in Lemma 1.2. Since $|C| \geq 2$ one cannot have $(n-3)2$ as a suffix of $S(S(C)1)$. Similarly, $(n-3)1k$ and $(n-3)k1$ cannot be suffixes of $S(S(C)1)$. Also, it is only possible that

$$\pi' \in \langle\, *(n-3)*(n-5)*(n-4)1 \,\rangle$$

if $C = \emptyset$, which is not allowed. The only possibility for π' is that it has $(n-4)1(n-3)$ as a suffix, and so $n-4, n-3 \in C$ since $|C| \geq 2$. Under these conditions, $S^2(\pi)$ has suffix $(n-4)1(n-3)(n-2)(n-1)n$ and $\pi \in E_{n,n-3}$. Thus $D = 1$ and $n-4, n-3 \in C$, from which we get case 5(d).

If $\pi = A(n-2)BnC(n-1)D$ where $|C \cup D| \geq 3$ then

$$S(\pi) = S(A)S(B)(n-2)S(C)S(D)(n-1)n.$$

Now, π belongs to $E_{n,n-3}$ if and only if $\pi' = S(A)S(B)(n-2)S(C)S(D)$ belongs to $E_{n-2,n-4}$. Since $|C \cup D|$ is at least 3, the entry $n-2$ has at least 3 entries to its right in π'. By comparing this to the possible types that it may take in Lemma 1.2, we find that the only possibility is

$$\pi' \in \langle\, *(n-2)*(n-4)*(n-3)1 \,\rangle.$$

This happens if and only if $D = 1$ and $n-4, n-3 \in C$, which gives case 5(h).

For the two final cases in which $n-2$ and $n-1$ are to the left of n, it is not possible for π to contain a forbidden pattern of order $n-4$, since at least one of $n-2$, $n-1$ and n is needed in order to witness such a pattern, and none of them are in the rightmost four positions of the permutation.

The number of permutations of each type is shown in column three of Table 1. Adding these gives $(47n^2 - 223n + 240)(n-4)!/6$ which may be be rewritten as in formula (2.1). □

Corollary 2.3. *For $n \geq 6$, the collection of $(n-4)$-stack sortable permutations in $\mathfrak{S}_n$ are those permutations that are not of the types listed in Lemma 1.1, Lemma 1.2 or Table 1. The number of these is $(n-4)!(3n^4 - 18n^3 - 4n^2 + 158n - 192)/3$.*

It is now straightforward to write down (a lengthy expression for) the descent polynomial of the $(n-4)$-stack sortable permutations, that is, the polynomial whose k-th coefficient is the number of $(n-4)$-stack sortable permutations with exactly k descents (a descent in $\pi = a_1a_2\ldots a_n$ is an i such that $a_i > a_{i+1}$).

The following conjecture is based on computer generated data for $n \leq 13$.

Conjecture 2.4. *For all $n \geq 8$, the number of permutations in $E_{n,n-4}$ is*

$$\frac{(n-5)!}{10}\left(854\binom{n-8}{3} + 5099\binom{n-8}{2} + 12545\binom{n-8}{1} + 16130\right),$$

or equivalently, the number of $(n-5)$*-stack sortable permutations is*

$$\frac{(n-5)!}{60}\left(60n^5 - 600n^4 + 506n^3 + 11241n^2 - 38369n + 34236\right).$$

We end with a conjecture about the form of an expression for the number of permutations needing exactly $n-k$ stack sorts. This has been verified for all $n \leq 13$ and all relevant k.

Conjecture 2.5. *For all* $n \geq 2k$*, the number of permutations in* $E_{n,n-k}$ *may be written as*

$$\frac{(k-1)!(n-k-1)!}{\big(2(k-1)\big)!}\sum_{i=0}^{k-1} a_i \binom{n-2k}{i}$$

where $a_i \in \mathbb{N}$.

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