DECOMPOSITIONS AND STATISTICS FOR $\beta(1,0)$-TREES AND NONSEPARABLE PERMUTATIONS

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ABSTRACT. The subject of pattern avoiding permutations has its roots in computer science, namely in the problem of sorting a permutation through a stack. A formula for the number of permutations of length n that can be sorted by passing it twice through a stack (where the letters on the stack have to be in increasing order) was conjectured by West, and later proved by Zeilberger. Goulden and West found a bijection from such permutations to nonseparable planar maps, and later, Jacquard and Schaeffer presented a bijection from these planar maps to certain labeled plane trees, called $\beta(1,0)$-trees. Using generating trees, Dulucq, Gire and West showed that nonseparable planar maps are equinumerous with permutations avoiding the (classical) pattern 2413 and the barred pattern $41\bar{3}52$; they called these permutations *nonseparable*.

We give a new bijection between $\beta(1,0)$-trees and permutations avoiding the dashed patterns 3-1-4-2 and 2-41-3. These permutations can be seen to be exactly the reverse of nonseparable permutations. Our bijection is built using decompositions of the permutations and the trees, and it translates seven statistics on the trees into statistics on the permutations. Among the statistics involved are ascents, left-to-right minima and right-to-left maxima for the permutations, and leaves and the rightmost and leftmost paths for the trees.

In connection with this we give a nontrivial involution on the $\beta(1,0)$-trees, which specializes to an involution on unlabeled rooted plane trees, where it yields interesting results.

Lastly, we conjecture the existence of a bijection between nonseparable permutations and two-stack sortable permutations preserving at least four permutation statistics.

1. INTRODUCTION

In Exercise 2.2.1.5 Knuth [13] asks the reader to "Show that it is possible to obtain a permutation $p_1p_2\dots p_n$ from $12\dots n$ using a stack if and only if there are no indices $i<j<k$ such that $p_j<p_k<p_i$." The equivalent inverse problem is to sort a permutation (into increasing order) through a stack, and the characterization in Knuth's exercise states that a permutation can be sorted through a stack if and only if it *avoids the pattern* 2-3-1. (A permutation avoids the pattern 2-3-1 if there are no indices $i<j<k$ such that $p_k<p_i<p_j$.)

In his Ph.D. thesis West [15] considered the problem of sorting a permutation by passing it twice through a stack, where the letters on the stack have to be in increasing order when read from top to bottom. He conjectured that the number of permutations on $[n]=\{1,\dots,n\}$ so sortable is $a_n = 2(3n)!/\big((2n+1)!(n+1)!\big)$. This was first proved by Zeilberger [16], who found the functional equation

$$x^2F^3 + x(2+3x)F^2 + (1-14x+3x^2)F + x^2 + 11x - 1 = 0$$

for the generating function $F=\sum_n a_nx^n$ and then used Lagrange's inversion formula to solve it.

Key words and phrases. stack sorting, trees, pattern avoidance, nonseparable, planar maps, involution, bijection.

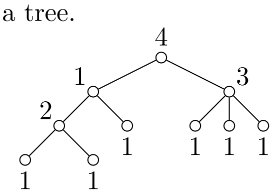
The research presented here was supported by grant no. 060005012 from the Icelandic Research Fund.

West also had noted that a_n is the number of rooted nonseparable planar maps on $n + 1$ edges as enumerated by Brown and Tutte [4, 5, 14]. Based on this fact two bijective proofs were later found. One by Dulucq et al. [9, 10] who establish the correspondence using generating trees and eight different families of permutations with forbidden subsequences as intermediate sets. The other by Goulden and West [11]: Using Raney paths they unearthed a recursive structure on the permutations that parallels the recursive structure that Brown and Tutte had found on the planar maps.

Jacquard and Schaeffer [12] presented a bijection from rooted nonseparable planar maps to certain labeled plane trees. These trees are the combinatorial structure that most transparently embodies the recursive structure that Brown and Tutte had found on the planar maps, and here is their definition: A $\beta(1,0)$-*tree* is a rooted plane tree labeled with positive integers such that

(1) Leaves have label 1.
(2) The root has label equal to the sum of its children's labels.
(3) Any other node has label no greater than the sum of its children's labels.

Below is an example of such a tree.



The main result (Theorem 9) of this paper is a one-to-one correspondence between $\beta(1,0)$-trees on n edges and permutations of length n that avoid the two dashed (or generalized) patterns 3-1-4-2 and 2-41-3 (we call these permutations *avoiders*). That correspondence "translates" seven different statistics from $\beta(1,0)$-trees to avoiders. To be more precise, we say that two vectors $(s_1, s_2, \dots, s_k)$ and $(t_1, t_2, \dots, t_k)$ of statistics on sets A and B, respectively, *have the same distribution* if

$$\sum_{a\in A} x_1^{s_1(a)} x_2^{s_2(a)} \cdots x_k^{s_k(a)} = \sum_{b\in B} x_1^{t_1(b)} x_2^{t_2(b)} \cdots x_k^{t_k(b)}.$$

The bijection f in Theorem 9 shows that the first vector below has the same distribution on $\beta(1,0)$-trees as the second has on avoiders (see Section 2 for definitions):

$$\begin{array}{lllllllll} (& \text{sub}, & \text{leaves}, & \text{root}, & \text{lpath}, & \text{rpath}, & \text{lsub}, & \text{beta} &) \\ (& \text{comp}, & 1+\text{asc}, & \text{lmax}, & \text{lmin}, & \text{rmax}, & \text{ldr}, & \text{lir} &) \end{array}$$

Bijections between two sets (of combinatorial objects) that take one vector of statistics to another typically reveal that the two sets are structurally similar, in a sense illuminated by the statistics in question. By doing exhaustive computations of large sets of statistics on our avoiders, two-stack sortable permutations and $\beta(1,0)$-trees, respectively, we have found that there are many more, and larger, vectors of statistics that are equidistributed between avoiders and $\beta(1,0)$-trees than between two-stack sortable permutations and the trees. Moreover, there are few and small equidistributions between avoiders and two-stack sortables. This suggests that the avoiders are structurally more similar to the $\beta(1,0)$-trees than two-stack sortable permutations are.

Thus, it seems that the avoiders are a better set of permutations than two-stack sortables to capture interesting properties of the trees and, by extension, the nonseparable planar maps. In fact, we were led to some of the statistics on trees we study here—statistics that belong to the vector of seven statistics mentioned above—by well-known statistics on permutations. In particular, the left subtrees presented themselves when we studied what the bijection f translates the permutation statistic ldr to on $\beta(1,0)$-trees. The left subtrees are obtained by "cutting" the tree at each 1 on the left path (except at the root). The statistic ldr is the place of the first ascent in a permutation (if there is at least one ascent, and equal to the length of the permutation otherwise).

In connection with Theorem 9 mentioned above it should be acknowledged that Dulucq et al. [9, 10] gave a bijection between rooted nonseparable planar maps and what they call *nonseparable permutations* (permutations avoiding the pattern 2-4-1-3 and the barred pattern 4-1-$\bar{3}$-5-2). It is not hard to see that those permutations are the reverse of the permutations avoiding 3-1-4-2 and 2-41-3. Dulucq et al. construct their bijection via a generating tree of nonseparable permutations and a generating tree of rooted nonseparable planar maps, and they show that their bijection sends the pair (degree of root face, number of nodes) to the pair (rmax, des).

In a recent paper, Bonichon et al. [2] give a beautifully simple and direct bijection from Baxter permutations to plane bipolar orientations. The restriction of this bijection to nonseparable permutations (a subset of the Baxter permutations) is the same bijection to rooted nonseparable planar maps as the one given by Dulucq et al. [9]. Bonichon et al. also show that their bijection translates five natural statistics on permutations to statistics on the maps, thus strengthening the corresponding result of Dulucq et al. A question then presents itself: If we take the bijection given in the present paper and compose it with the bijection of Jacquard and Schaeffer [12], is the resulting bijection equal to or different from the bijection of Dulucq et al? The answer is that it is different, and it is different also "up to symmetry."

Let us make this a bit more precise. It can be seen that of the seven non-identity symmetries on permutations formed by composing reverse (r), complement (c) and inverse (i), the set of avoiders is closed under three: $r \circ c$, $r \circ i$ and $c \circ i$. We also consider a symmetry on $\beta(1,0)$-trees, namely mirror, which recursively reverses the order of subtrees. On planar maps we consider two symmetries: mirror and the planar dual (see [2]). We claim that our bijection composed with any of the mentioned symmetries gives a bijection that is different from the bijection of Dulucq et al. [9]. This can be seen by applying all these bijections to the avoider 245316.

Bousquet-Mélou [3] studied the generating function for 2-stack sortable permutations π with respect to 5 parameters: length of π, $\mathrm{des}(\pi)$, $\mathrm{lmax}(\pi)$, $\mathrm{rmax}(\pi)$, and the largest i such that $(n, n-1, \ldots, n+1-i)$ is a subsequence of π (where n is the length of π). She showed that this five-variable generating function is algebraic of degree 20. As mentioned above, we treat all these statistics, and more, on avoiders and relate them to statistics on $\beta(1,0)$-trees.

In Section 8 we introduce an involution h on $\beta(1,0)$-trees. It gives us three further results about equidistributions (see Theorem 10, Corollary 11, Corollary 12). Moreover, h gives an involution on *unlabeled* rooted plane trees. Using that, we obtain some interesting results on one-stack sortable permutations and also a genuinely new bijection between (1-2-3)-avoiding and (1-3-2)-avoiding permutations, yielding new equidistributions of statistics on these two classes of permutations. These results will be presented in a forthcoming paper [8].

2. Preliminaries

Let $V = \{v_1, v_2, \ldots, v_n\}$ with $v_1 < v_2 < \cdots < v_n$ be any finite subset of $\mathbb{N}$. The *standardization* of a permutation π on V is the permutation $\mathrm{std}(\pi)$ on $[n]$ obtained from π by replacing the letter v_i with the letter i. As an example, $\mathrm{std}(19452) = 15342$. If the set V is fixed, the inverse of the standardization map is well defined, and we denote it by $\mathrm{std}_V^{-1}(\sigma)$; for instance, with $V = \{1, 2, 4, 5, 9\}$, we have $\mathrm{std}_V^{-1}(15342) = 19452$. An *occurrence* of the pattern 3-1-4-2 in a permutation $\pi = a_1 a_2 \dots a_n$ is a subsequence $o = a_i a_j a_k a_\ell$ (where $i < j < k < \ell$) of π such that $\mathrm{std}(o) = 3142$; an occurrence of 2-41-3 is a subsequence $o = a_i a_j a_{j+1} a_k$ such that $\mathrm{std}(o) = 2413$. A permutation is said to *avoid* a pattern if it has no occurrences of it. (For more on dashed permutation patterns see [1, 6].)

From now on we will refer to a permutation avoiding the two patterns 3-1-4-2 and 2-41-3 simply as an ***avoider****.*

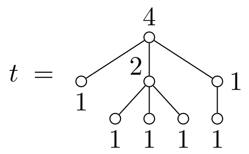
Here follows some more terminology that we shall use. An *interval* in a permutation is a factor (contiguous subsequence) that contains a set of contiguous values. For example, 423 is an interval in 1642375. In particular, every permutation is an interval of itself, and every letter is an interval too.

For words α and β over the alphabet $\mathbb{N}$ we define that $\alpha \prec \beta$ if for all letters a in α and all letters b in β we have $a < b$. For instance, $412 \prec 569$ and $2 \prec 348$. The reflexive closure of $\prec$ is a partial order on the set of nonempty words over $\mathbb{N}$. Indeed, if both α and β are nonempty then $\alpha \prec \beta$ if and only if $\max \alpha < \min \beta$.

We now define the statistics on permutations we shall be concerned with. Let $\pi = a_1 a_2 \dots a_n$ be any permutation. An *ascent* is a letter followed by a larger letter; a *descent* is a letter followed by a smaller letter. The number of ascents and descents are denoted $\mathrm{asc}(\pi)$ and $\mathrm{des}(\pi)$, respectively. A *left-to-right minimum* of π is a letter with no smaller letter to the left of it; the number of left-to-right minima is denoted $\mathrm{lmin}(\pi)$. The statistics *right-to-left minima* (rmin), *left-to-right maxima* (lmax), and *right-to-left maxima* (rmax) are defined similarly. The statistic $\mathrm{ldr}(\pi)$ is defined as the largest integer i such that $a_1 > a_2 > \cdots > a_i$ (the leftmost decreasing run). Similarly, $\mathrm{lir}(\pi)$ is defined as the largest integer i such that $a_1 < a_2 < \cdots < a_i$ (the leftmost increasing run). A *component* of π is a nonempty factor τ of π such that $\pi = \sigma\tau\rho$ with $\sigma \prec \tau \prec \rho$, and such that if $\tau = \alpha\beta$ and $\alpha \prec \beta$ then α or β is empty. By $\mathrm{comp}(\pi)$ we denote the number of components of π. For instance, $\mathrm{comp}(213645) = 3$, the components being 21, 3, and 645.

We also define some statistics on $\beta(1,0)$-trees. By $\mathrm{leaves}(t)$ we denote the number of leaves in t; by $\mathrm{int}(t)$ we denote the number of internal nodes (or nonleaves) in t. Note that the root is an internal node. By $\mathrm{root}(t)$ we denote the label of the root. The number of subtrees (or, equivalently, the number of children of the root) is denoted $\mathrm{sub}(t)$. The *left-path* is the path from the root to the leftmost leaf, and the *right-path* is the path from the root to the rightmost leaf. The lengths of (number of edges on) the left and right paths are denoted $\mathrm{lpath}(t)$ and $\mathrm{rpath}(t)$, respectively. By $\mathrm{stem}(t)$ we denote the number of internal nodes that are common to the left and the right-path. Also, $\mathrm{lsub}(t)$ and $\mathrm{rsub}(t)$ denote the number of 1's below the root on the left and right paths, respectively.

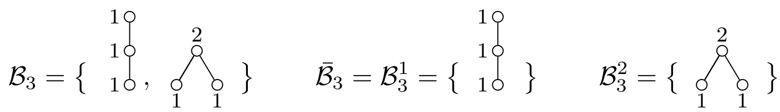
We define $\mathrm{beta}(t)$ as follows: Order the leaves of a tree t from left to right and call them $\ell_1, \ell_2, \dots, \ell_m$ (where ℓ_1 is leftmost and so on). Look at the path from ℓ_1 to the root. If no node on that path, except for the leaf ℓ_1, has label 1, reduce the labels on all nodes on that path by 1 and delete ℓ_1. Note that the resulting tree is a $\beta(1,0)$-tree and that its leaves are $\ell_2, \dots, \ell_m$. Now look at ℓ_2 and repeat the process, until we come to a leaf ℓ_i whose path to the root contains a node (other than ℓ_i) that now has label 1. Then $\mathrm{beta}(t) = i$. We end these preliminaries with an example:



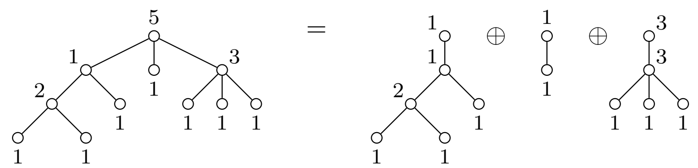
leaves$(t) = 5$; root$(t) = 4$;
int$(t) =$ sub$(t) =$ beta$(t) = 3$;
rpath$(t) =$ rsub$(t) = 2$;
lsub$(t) =$ lpath$(t) =$ stem$(t) = 1$.

3. The structure of $\beta(1,0)$-trees

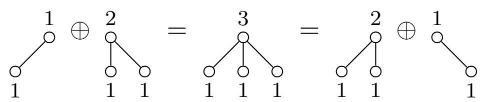
We say a $\beta(1,0)$-tree on two or more nodes is *indecomposable* if its root has exactly one child and *decomposable* if it has more than one child. The $\beta(1,0)$-tree on one node is neither indecomposable nor decomposable. Let $\mathcal{B}_n$ be the set of all $\beta(1,0)$-trees on n nodes, and let $\bar{\mathcal{B}}_n$ be the subset of $\mathcal{B}_n$ consisting of the indecomposable trees. Let $\mathcal{B}_n^k$ be the subset of $\mathcal{B}_n$ consisting of the trees with root label k. For instance,



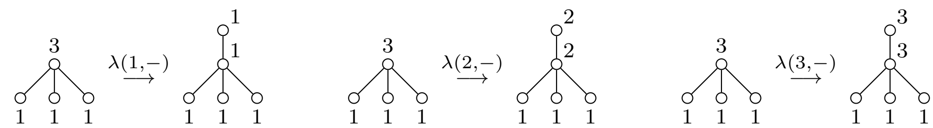
Decomposable trees can be regarded as sums of indecomposable ones:



In fact we do not need to require u and v to be indecomposable for the sum $u \oplus v$ to make sense. In general, we define that the root label of $u \oplus v$ is the sum of the root label of u and the root label of v, and that the subtrees of $u \oplus v$ are those of u followed by those of v. So,



Further, there is a simple one-to-one correspondence λ between the Cartesian product $[k] \times \mathcal{B}_{n-1}^k$ and the disjoint union $\cup_{i=1}^k \bar{\mathcal{B}}_n^i$, where $\bar{\mathcal{B}}_n^k$ is the subset of $\bar{\mathcal{B}}_n$ consisting of the trees with root label k:

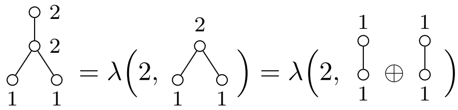


In general, if t is a tree with root label k and i is an integer such that $1 \le i \le k$, then $\lambda(i,t)$ is obtained from t by joining a new root via an edge to the old root; and both the new root and the old root are assigned the label i.

Thus each $\beta(1,0)$-tree, t, is of exactly one the following three forms:

$t = \circ$,	(the one node tree)
$t = u \oplus v$,	(decomposable)
$t = \lambda(i,u)$, where $1 \le i \le \operatorname{root} u$,	(indecomposable)

in which u and v are $\beta(1,0)$-trees. Note that any tree that is decomposable with respect to $\oplus$ (second case above) is indecomposable with respect to λ; that is, it is not of the form $\lambda(i,u)$. Also, any tree that is indecomposable with respect to $\oplus$ (third case above) is decomposable with respect to λ. Thus each tree with at least one edge is decomposable with respect to exactly one of $\oplus$ or λ, and if we keep decomposing until only single node trees remain we get an unambiguous encoding of $\beta(1,0)$-trees, an example of which is given by



$$= \lambda\Big(2,\ \lambda(1,\circ) \oplus \lambda(1,\circ)\Big).$$

4. The structure of avoiders

In this paper we construct a bijection between avoiders and $\beta(1,0)$-trees by defining a sum on permutations analogous to the sum on trees, and a function ϕ analogous to λ; the sum on permutations works like this:

$$21 \oplus 132 = 21354.$$

In general, $\sigma \oplus \tau = \sigma\tau'$ where τ' is obtained from τ by adding $|\sigma|$ to each of its letters. We call a nonempty permutation π *decomposable* if it can be written $\pi = \sigma \oplus \tau$ with σ and τ both nonempty; otherwise we call it *indecomposable*.

Describing how the map ϕ works is quite a bit harder. In the case of $\beta(1,0)$-trees, $\lambda(i,t)$ is indecomposable and has root label i. In the case of avoiders, $\phi(i,\pi)$ is indecomposable and has i left-to-right maxima. For instance,

$$\pi = \underline{2}1\underline{58}6473$$

has 3 left-to-right maxima, namely the underlined elements, 2, 5, and 8. The 3 images of π under the function ϕ are $\phi(1,\pi)$, $\phi(2,\pi)$, and $\phi(3,\pi)$; they should have 1, 2, and 3 left-to-right maxima, respectively. To achieve this we start by inserting 9 immediately before the ith left-to-right maximum:

$$\pi_1 = \underline{9}21586473; \qquad \pi_2 = \underline{2}1\underline{9}586473; \qquad \pi_3 = \underline{2}1\underline{59}86473.$$

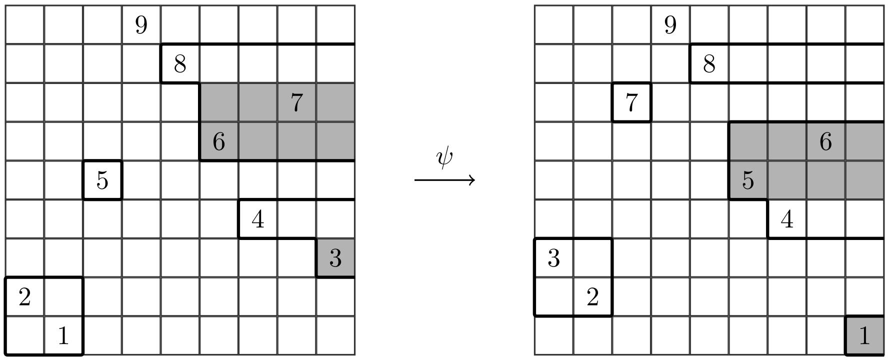
Clearly π_i has i left-to-right maxima. Note also that it is necessary to insert 9 *immediately* before the ith left-to-right maximum, or else a 2-41-3 pattern would be formed. To be concrete, there is only one position in π between the first left-to-right maximum, 2, and the second left-to-right maximum, 5, where we can insert 9. Placing 9 before 1 would yield 291586473 and, as witnessed by the subsequence 2915, that permutation contains the pattern 2-41-3. In general, inserting n between two consecutive left-to-right maxima, say b and c with $b < c$, but not immediately before c, will lead to an occurrence of 2-41-3 formed by b, n, the element immediately to the right of n, and c.

A permutation obtained by inserting a new largest element in an avoider as described above will have the correct number of left-to-right maxima; it will also avoid 3-1-4-2 and 2-41-3; it will however *not* be indecomposable, in general. Above, π_1 is indecomposable but π_2 and π_3 are not. There is in fact a simple criterion for indecomposability of avoiders: If π is a permutation in which the largest letter precedes the smallest letter, then clearly π is indecomposable. For (3-1-4-2)-avoiding permutations the converse is also true.

Lemma 1. *In any indecomposable* (3-1-4-2)*-avoiding permutation the largest letter precedes the smallest letter, when read from left to right.*

Proof. We shall demonstrate the contrapositive statement: if $\pi \in \mathcal{S}_n$ and 1 precedes n in π, then either π contains an occurrence of 3-1-4-2 or π is decomposable. To this end, let a permutation $\pi = \sigma n \tau$ in $\mathcal{S}_n$ with $1 \in \sigma$ be given. If τ is empty, then n is a component and thus π is decomposable. If $\sigma \prec \tau$ then $n\tau$ is a component; otherwise the subword $\rho = (\, x \in \sigma : x > \min\tau \,)$ of σ is nonempty, and there are two possibilities: either ρ is a factor in σ and $\pi = \sigma'\rho n\tau$ with $1 \in \sigma'$; or there is an x in ρ to the left of a y in σ such that $y < \min \tau$. In the former case, $\rho n\tau$ is a component; in the latter case, $(x, y, n, \min \tau)$ is an occurrence of the pattern 3-1-4-2. □

We shall now describe a function ψ that can turn a decomposable avoider into an indecomposable one, while preserving many of its other properties. For now, let us concentrate on $\pi_3 = 215986473$ (from above), and let us describe how to get $\psi(\pi_3)$. In doing so it will be convenient to refer to the following picture.



To the left of 9, the largest element in π_3, we have $\sigma = 215$; to the right we have $\tau = 86473$. The patterns (standardizations) of the left and right parts are $\bar\sigma = 213$ and $\bar\tau = 53241$. We shall make a new permutation $\pi_3' = \psi(\pi_3)$ that will be similar to π_3 in the sense that the letters to the left and right of 9 in π_3' also will form the patterns $\bar\sigma$ and $\bar\tau$. Thus, to specify π_3' it is sufficient to give the 3 element set, call it L, from which the left part of π_3' is built. Further, due to Lemma 1, π_3' will be indecomposable if 1 is not a member of L. The underlying set of σ is

$$\{1, 2, 5\}.$$

Considering $\sigma = 215$ as a permutation of this set, we can divide it into two intervals 21 and 5. To each of the letters in the first interval we will add some positive number $m_1 + 1$. Similarly, to the letters in the second interval we will add some positive number m_2+1. (In the picture above, m_1+1 and m_2+1 are the number of elements in the gray areas.) The resulting underlying set of the left part of π_3' is

$$L = \{\, 1 + (m_1 + 1),\ 2 + (m_1 + 1),\ 5 + (m_2 + 1)\,\}.$$

That $m_1 + 1$ is positive implies that 1 is not in L, which in turn implies that π_3' is indecomposable.

We now describe how m_1 and m_2 are determined. Let w_1 be the subsequence of τ whose elements are between 21 and 5 in value. So, $w_1 = 43$. Similarly, let w_2 be the subsequence of τ whose elements are larger than 5 in value. So, $w_2 = 867$.

Denote by $\hat{0}_i$ the smallest element of w_i. Then m_i is the number of elements to the right of $\hat{0}_i$ in w_i that are smaller than all elements to the left of $\hat{0}_i$ in w_i.

We find that $m_1 = 0$ and $m_2 = 1$. Consequently, $L = \{2, 3, 7\}$. The final step is to fill in the left part using the elements of L and to fill in the right part using the remaining elements $[8] \setminus L = \{1, 4, 5, 6, 8\}$, while preserving the patterns $\bar\sigma$ and $\bar\tau$. The resulting permutation is 327985461.

We shall now go through the definition of ψ again, this time dealing with the general case. Let π be an avoider whose first letter is not n. In addition, assume that n precedes $n-1$ in π. Let σ and τ be defined by $\pi = \sigma n\tau$. Note that, by assumption, both σ and τ are then nonempty. Let $\sigma_1, \ldots, \sigma_k$ be the intervals of σ, and, by convention, let $\sigma_0 = \epsilon$ and $\sigma_{k+1} = \epsilon$. For each $i = 0, \ldots, k$ define a subsequence w_i of τ by

$$w_i = (\, x \in \tau \,:\, \sigma_i \prec x \prec \sigma_{i+1} \,).$$

Denote by u_i and v_i the parts of w_i that are to the left and to the right of the smallest element of w_i. In short, $w_i = u_i \hat{0} v_i$ where $\hat{0}_i = \min w_i$. We shall now specify the set L of elements to the left of n in $\psi(\pi)$; this is the key object in the definition of ψ: let $m_i = \operatorname{card}\{\, x \in v_i : x \prec u_i \,\}$ and $L_i = \{\, x + m_i + 1 : x \in \sigma_i \,\}$; then $L = \cup_{i=1}^{k} L_i$.

Finally we define $\psi(\pi)$ as the result of filling in the elements of L to the left of n while respecting the pattern $\bar{\sigma} = \operatorname{std}(\sigma)$, and filling in the elements of $R = [n-1] \setminus L$ to the right of n while respecting the pattern $\bar{\tau} = \operatorname{std}(\tau)$:

$$\psi(\pi) = \sigma' n \tau', \text{ where } \sigma' = \operatorname{std}_L^{-1}(\bar{\sigma}) \text{ and } \tau' = \operatorname{std}_R^{-1}(\bar{\tau}). \tag{1}$$

For the definition of std, see Section 2.

Lemma 2. *Let $n \geq 2$, and let $\tilde{\mathcal{A}}_n$ be the set of avoiders whose first letter is not n. Then the function ψ, as defined in the preceding paragraph, is a bijection from*

$$\{\, \pi \in \tilde{\mathcal{A}}_n \mid n \text{ precedes } n-1 \text{ in } \pi \,\} \quad \text{onto} \quad \{\, \pi \in \tilde{\mathcal{A}}_n \mid \pi \text{ is indecomposable} \,\}.$$

Proof. We shall use the same notation as above, so $\pi = \sigma n \tau$ is an avoider with σ nonempty and the letter $n-1$ belongs to τ. Further, we shall split this proof into 4 parts, showing that:

a. the permutation $\psi(\pi)$ is indecomposable;
b. the letter n is not the first letter in $\psi(\pi)$;
c. the permutation $\psi(\pi)$ avoids 3-1-4-2 and 2-41-3;
d. the function ψ is injective;
e. the function ψ is surjective.

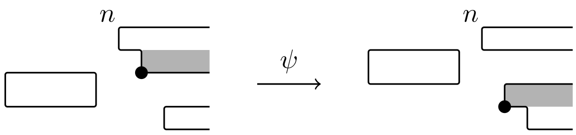
Part a. Looking at the definition of ψ we see that all the m_is are nonnegative; hence all members of L (i.e., all elements to the left of n in $\psi(\pi)$) are bigger than 1 and it follows by Lemma 1 that $\psi(\pi)$ is indecomposable.

Part b. By assumption σ is nonempty; by definition ψ preserves the position of n. Thus n is not the first letter in $\psi(\pi)$.

Part c. We shall show that $\psi(\pi) = \sigma' n \tau'$ avoids 3-1-4-2 and 2-41-3, given that $\pi = \sigma n \tau$ does. Consider the contrapositive statement, and assume that there is an occurrence o of 3-1-4-2 or 2-41-3 in $\psi(\pi)$. Since the L_is are intervals and the underlying permutations of the patterns 3-1-4-2 and 2-41-3 do not contain any nontrivial intervals we know that either o is entirely contained in σ' or only its first letter belongs to σ'. For the first case it suffices to recall that ψ preserves the patterns of σ and τ. The second case is more intricate: Let $o = c'a'd'b'$ be any occurrence of 3-1-4-2 in $\psi(\pi)$ in which c' is a letter of σ' and $a'd'b'$ is a subword of τ'. Let a, b, c, and d be the preimages of a', b', c', and d' under ψ. Since ψ preserves the pattern of τ we know that $a < b < d$. Also, τ' is the disjoint union of the sequences $\psi(w_i)$; let us call these the *blocks* of τ'. The assumption that o is an occurrence of 3-1-4-2 implies that d' belongs to a different block than a' and b'. This, in turn, implies that there is a letter x in τ such that $b < x < d$ and x is to the left of a. Thus $xadb$ is an occurrence of 3-1-4-2 in π.

Let $o = b'd'a'c'$ be any occurrence of 2-41-3 in $\psi(\pi)$ in which b' is a letter of σ' and $d'a'c'$ is a subword of τ'. As before, let a, b, c, and d be the preimages of a', b', c', and d' under ψ. Since ψ preserves the pattern of τ we have $a < c < d$. The assumption that o is an occurrence of 2-41-3 implies that a' belongs to a different block than c' and d'. If $a' = a$ then $bdac$ is an occurrence of 2-41-3 in π. If $a' \neq a$ then there is a letter x in τ such that $a < x < c$ and x is to the left of d, and so $xdac$ is an occurrence of 2-41-3 in π.

Part d. Note that the smallest letter in w_1 is to the left of any letter in w_0; otherwise, an occurrence $bdac$ of 2-41-3 is materialized by letting $a = \psi(\pi)(i)$ be the first letter of w_0; b be any letter in σ_1; c be the smallest letter in w_1; and $d = \psi(\pi)(i-1)$ be the left neighbor of a. The following picture illustrates this argument in the special case when σ is an interval.



An almost identical argument entails the more general conclusion: the smallest letter in w_i is to the left of any letter in w_{i-1}. Thus we can recover m_i as the number of elements of w_i that are bigger than the first letter of w_i. Consequently, the map ψ is injective.

Part e. Let π be an indecomposable avoider whose first letter is not n. To reverse ψ we do as described in Part d. Let w_i be defined as above. By Lemma 1, w_0 contains the letter 1 and, in particular, it is nonempty. Thus, in the preimage of π under ψ, the topmost w_i will be nonempty, and hence $n-1$ will be to the right of n. By similar reasoning as in Part d it also follows that the preimage avoids 3-1-4-2 and 2-41-3. $\square$

Given an avoider π on $[n-1]$ and a positive integer i that is no greater than the number of left-to-right maxima in π, we define

$$\begin{aligned} \phi(1,\pi) &= \hat{\pi}, \\ \phi(i,\pi) &= \psi(\hat{\pi}) \ \text{ if } i > 1, \end{aligned} \tag{2}$$

where $\hat{\pi}$ is obtained from π by inserting n immediately to the left of the ith left-to-right maximum in π.

Let $\mathcal{A}_n$ be the set of avoiders on $[n]$, and let $\mathcal{A}_n^k$ be the subset of $\mathcal{A}_n$ consisting of those avoiders that have k left-to-right maxima. Similarly, let $\bar{\mathcal{A}}_n$ be the set of indecomposable avoiders on $[n]$, and let $\bar{\mathcal{A}}_n^k$ be the subset of $\bar{\mathcal{A}}_n$ consisting of those indecomposable avoiders that have k left-to-right maxima.

Lemma 3. *Let $n \geq 1$. The function ϕ, as defined by* (2)*, is a bijection between the Cartesian product $[k] \times \mathcal{A}_{n-1}^k$ and the disjoint union $\cup_{i=1}^k \bar{\mathcal{A}}_n^i$.*

Proof. We start by showing that $\phi(i,\pi)$ has exactly i left-to-right maxima. By definition, $\phi(i,\pi) = \psi(\hat{\pi})$ where $\hat{\pi}$ is obtained from π by inserting n immediately to the left of the ith left-to-right maximum in π. For the case $i = 1$ we have $\phi(1,\pi) = \hat{\pi} = n\pi$ and that permutation clearly has one left-to-right maximum. For the case $i > 1$ we note that by construction $\operatorname{lmax}\hat{\pi} = i$, and according to Lemma 6 the number of left-to-right maxima is preserved under ψ.

We now show that ϕ has the claimed codomain. The case $i = 1$ is simple: $\phi(1,\pi) = n\pi$ has 1 left-to-right maximum and is indecomposable by Lemma 1. Also, $n\pi$ is an avoider precisely when π is. Thus $\phi(1,\pi)$ is a member of $\bar{\mathcal{A}}_n^1$. The case $i > 1$ follows from Lemma 2.

Finally, to show that ϕ is bijective we give its inverse: if $\pi = n\tau$ then $\phi^{-1}(\pi) = (1,\tau)$; otherwise, $\phi^{-1}(\pi) = (\operatorname{lmax}\pi, \tau)$ where τ is obtained from $\psi^{-1}(\pi)$ by removing the largest element. That this really is the inverse of ϕ is an immediate consequence of the definition of ϕ and Lemma 2. $\square$

As a corollary to the preceding lemma we get that each avoider π is of exactly one the following three forms:

$\pi = \epsilon,$ (the empty permutation)

$$\pi = \sigma \oplus \tau, \qquad \text{(decomposable)}$$
$$\pi = \phi(i, \sigma), \text{ where } 1 \le i \le \operatorname{lmax}(\sigma), \qquad \text{(indecomposable)}$$

Note the striking similarity with the decomposition of $\beta(1,0)$-trees as given at the end of Section 3. With avoiders as with $\beta(1,0)$-trees we can keep decomposing until only "atoms" remain. For trees "atom" means the one node tree. For avoiders "atom" means the empty permutation. We thus get an unambiguous encoding of avoiders; for example, the encoding of 523147896 is

$$\phi\big(1, \phi(2, \phi(1,\epsilon) \oplus \phi(1,\epsilon)) \oplus \phi(1,\epsilon)\big) \oplus \phi\big(3, \phi(1,\epsilon) \oplus \phi(1,\epsilon) \oplus \phi(1,\epsilon)\big).$$

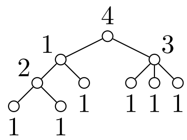
See also the example in the next section.

5. The bijection between trees and avoiders

Using recursion it is now easy to define a bijection—let us call it f—between $\beta(1,0)$-trees and avoiders:

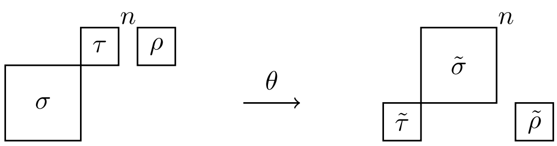
$$f(\circ) = \epsilon, \quad f(\lambda_i t) = \phi_i f(t), \quad \text{and} \quad f(u \oplus v) = f(u) \oplus f(v), \tag{3}$$

where we have written $\lambda_i(t)$ instead of $\lambda(i,t)$, and $\phi_i(\pi)$ instead of $\phi(i,\pi)$. Hence, to find the image of a $\beta(1,0)$-tree t under the bijection f, write out t as a string using λ and $\oplus$, replace $\circ$ with ϵ (or $\substack{\circ\\\circ}$ with 1), replace λ with ϕ, and replace $\oplus$ (on trees) with $\oplus$ on avoiders. Finally, translate the derived expression to a permutation. Clearly this is an invertible process, and so describes a bijection. For instance,



$$\begin{aligned}
&= \lambda_1\Big(\lambda_2\big(\substack{\circ\\\circ} \oplus \substack{\circ\\\circ}\big) \oplus \substack{\circ\\\circ}\Big) \oplus \lambda_3\big(\substack{\circ\\\circ} \oplus \substack{\circ\\\circ} \oplus \substack{\circ\\\circ}\big) \\
&\xrightarrow{f} \phi_1\big(\phi_2(1 \oplus 1) \oplus 1\big) \oplus \phi_3\big(1 \oplus 1 \oplus 1\big) \\
&= \phi_1(\phi_2(12) \oplus 1) \oplus \phi_3(123) \\
&= \phi_1(231 \oplus 1) \oplus 2341 \\
&= \phi_1(2314) \oplus 2341 \\
&= 52314 \oplus 2341 = 523147896.
\end{aligned}$$

Remark. In the course of discovering the map ψ, defined in (1), we first discovered a different map that we call θ. The map θ can be used instead of ψ in the above proofs with the exception that rpath on $\beta(1,0)$-trees would not be mapped to lmin on avoiders; thus our main result would be weakened. On the other hand, θ is easier than ψ to define: Let π be an avoider on $[n]$ such that n is not the first letter of π and n precedes $n-1$ (as in Lemma 2). Also, let us write $\pi = \sigma\tau n\rho$ in which $\tau n\rho$ is the rightmost component of π. Then θ is defined as in this picture:



Here $\tilde{\tau}\tilde{\rho}$ is the standardization of $\tau\rho$ and $\tilde{\sigma}$ is obtained from σ by adding $|\tau\rho|$ to all of the letters of σ. Note that by Lemma 1, since $\tau n\rho$ is irreducible, the smallest letter of $\tau n\rho$ is found in ρ. Thus the letter 1 in $\theta(\pi)$ belongs to $\tilde{\rho}$ and again using Lemma 1 we conclude that $\theta(\pi)$ is irreducible.

6. Statistics on $\beta(1,0)$-trees

The proofs in this section will use induction on the number of edges in a tree. For that reason we note that, by definition, we have

$$\begin{aligned}&\mathrm{leaves}(\circ) = 1 \ \text{ and}\\ &\mathrm{sub}(\circ) = \mathrm{root}(\circ) = \mathrm{lpath}(\circ) = \mathrm{rpath}(\circ) = \mathrm{lsub}(\circ) = \mathrm{stem}(\circ) = 0,\end{aligned} \tag{4}$$

where $\circ$ is the unique $\beta(1,0)$-tree with a single node. This will serve as the basis for induction.

Lemma 4. *The map $\lambda : [k] \times \mathcal{B}^k_{n-1} \to \cup_{i=1}^k \bar{\mathcal{B}}^i_n$ has the following properties:*

$$\begin{aligned}
\mathrm{leaves}\,\lambda(i,t) &= \mathrm{leaves}\,t; &&\\
\mathrm{root}\,\lambda(i,t) &= i; &&\\
\mathrm{lpath}\,\lambda(i,t) &= \mathrm{lpath}\,t + 1; &&\\
\mathrm{rpath}\,\lambda(i,t) &= \mathrm{rpath}\,t + 1; &&\\
\mathrm{lsub}\,\lambda(i,t) &= \mathrm{lsub}\,t + 1 && \text{if } i = 1;\\
\mathrm{lsub}\,\lambda(i,t) &= \mathrm{lsub}\,t && \text{if } i > 1;\\
\mathrm{beta}\,\lambda(i,t) &= i && \text{if } i \le \mathrm{beta}\,t;\\
\mathrm{beta}\,\lambda(i,t) &= \mathrm{beta}\,t && \text{if } i > \mathrm{beta}\,t.
\end{aligned}$$

Proof. Straightforward and omitted. □

Lemma 5. *If $t = u \oplus v$, with $u \neq \circ$ and $v \neq \circ$, then*

$$\begin{aligned}
\mathrm{leaves}\,t &= \mathrm{leaves}\,u + \mathrm{leaves}\,v;\\
\mathrm{root}\,t &= \mathrm{root}\,u + \mathrm{root}\,v;\\
\mathrm{lpath}\,t &= \mathrm{lpath}\,u;\\
\mathrm{rpath}\,t &= \mathrm{rpath}\,v;\\
\mathrm{lsub}\,t &= \mathrm{lsub}\,u;
\end{aligned}$$

and, if k is the largest integer such that $t = \left(\oplus^k \, \substack{\circ \\ | \\ \circ}\right) \oplus v$ for some $v \neq \circ$, then

$$\mathrm{beta}\,t = k + \mathrm{beta}\,v.$$

Proof. Straightforward and omitted. □

7. Statistics on avoiders

The proofs in this section will use induction on the number of letters in a permutation, and we therefore note that, by definition, we have

$$\mathrm{asc}(\epsilon) = \mathrm{comp}(\epsilon) = \mathrm{lmax}(\epsilon) = \mathrm{lmin}(\epsilon) = \mathrm{rmax}(\epsilon) = \mathrm{ldr}(\epsilon) = \mathrm{lir}(\epsilon) = 0, \tag{5}$$

where ϵ is the empty permutation.

Lemma 6. *The function ψ, as defined by* (1)*, preserves left-to-right maxima, right-to-left maxima, ascents, leftmost decreasing run, and leftmost increasing run; in addition, it increases the number of left-to-right minima by one.*

Proof. Let $\pi = \sigma n \tau$, with σ nonempty and $n-1$ in τ, be an avoider. Since

$$\mathrm{lmax}\,\pi = 1 + \mathrm{lmax}\,\sigma, \quad \mathrm{rmax}\,\pi = 1 + \mathrm{rmax}\,\tau, \quad \mathrm{asc}\,\pi = 1 + \mathrm{asc}\,\sigma + \mathrm{asc}\,\tau,$$
$$\mathrm{ldr}\,\pi = \mathrm{ldr}\,\sigma, \quad \mathrm{lir}\,\pi = \mathrm{lir}(\sigma n),$$

and ψ preserves the patterns of σ and τ, it immediately follows that ψ preserves lmax, rmax, asc, ldr, and lir.

The reason why ψ increases lmin by one is a bit more involved: Note first that, with the same notation as in the definition of ψ, we have

$$\operatorname{lmin}\pi = \operatorname{lmin}\sigma_1 + \operatorname{lmin} w_0.$$

Moreover, any left-to-right minimum in w_0 is to the right of the smallest element of w_1. This is because otherwise we have a 2-41-3 pattern $bdac$ in which $c = \min w_1$; a is a left-to-right minimum to the left of c; d is the element immediately to the left of a in π; and b is any element in σ_1. (Here, the assumption that $n-1$ is in τ entails that w_1 is nonempty and the assumption that τ is nonempty entails that σ_1 is nonempty.) The subsequence w_0 of τ is fixed under ψ. Also, ψ preserves the pattern of σ_1 but adds $m_i + 1$ to each of its elements. Exactly one new left-to-right minimum is thus obtained, namely $1 + \max w_0$, the image of $\min w_1$ under ψ. □

Lemma 7. *The function ϕ, as defined by (2), has the following properties.*

$$\operatorname{asc}\phi(i,\pi) = \operatorname{asc}\pi; \tag{6}$$
$$\operatorname{lmax}\phi(i,\pi) = i; \tag{7}$$
$$\operatorname{lmin}\phi(i,\pi) = \operatorname{lmin}\pi + 1; \tag{8}$$
$$\operatorname{rmax}\phi(i,\pi) = \operatorname{rmax}\pi + 1; \tag{9}$$
$$\operatorname{ldr}\phi(i,\pi) = \operatorname{ldr}\pi + 1 \qquad \textit{if } i = 1; \tag{10}$$
$$\operatorname{ldr}\phi(i,\pi) = \operatorname{ldr}\pi \qquad \textit{if } i > 1; \tag{11}$$
$$\operatorname{lir}\phi(i,\pi) = i \qquad \textit{if } i \leq \operatorname{lir}\pi; \tag{12}$$
$$\operatorname{lir}\phi(i,\pi) = \operatorname{lir}\pi \qquad \textit{if } i > \operatorname{lir}\pi. \tag{13}$$

Proof. For the case $i = 1$ we have $\phi(1,\pi) = \hat{\pi} = n\pi$, and all of the statements (6) through (13) follow immediately. The interesting case is $i > 1$, which we now consider. By definition, $\phi(i,\pi) = \psi(\hat{\pi})$ where $\hat{\pi}$ is obtained from π by inserting n immediately to the left of the ith left-to-right maximum in π. By construction $\operatorname{lmax}\hat{\pi} = i$, and according to Lemma 6 the number of left-to-right maxima is preserved under ψ; thus $\operatorname{lmax}\phi(i,\pi) = i$, proving (7). Further,

$$\operatorname{asc}\hat{\pi} = \operatorname{asc}\pi; \quad \operatorname{lmin}\hat{\pi} = \operatorname{lmin}\pi; \quad \operatorname{rmax}\hat{\pi} = \operatorname{rmax}\pi + 1; \quad \operatorname{ldr}\hat{\pi} = \operatorname{ldr}\pi.$$

Hence statements (6), (8), (9), and (11) follow from the corresponding statements in Lemma 6. For (12) and (13) note that an element in the leftmost increasing run is a left-to-right maximum, and that inserting n in front of the ith of those elements (in order to build $\hat{\pi}$) results in $\operatorname{ldr}\hat{\pi} = i$. On the other hand, inserting n in front of a left-to-right maximum that is not in the leftmost increasing run results in $\operatorname{ldr}\hat{\pi} = \operatorname{lir}\pi$. Thus, also, (12) and (13) follow from the corresponding statements in Lemma 6. □

Lemma 8. *With $\pi = \sigma \oplus \tau$ we have*

$$\begin{aligned}
\operatorname{comp}\pi &= \operatorname{comp}\sigma + \operatorname{comp}\tau;\\
\operatorname{asc}\pi &= 1 + \operatorname{asc}\sigma + \operatorname{asc}\tau;\\
\operatorname{lmax}\pi &= \operatorname{lmax}\sigma + \operatorname{lmax}\tau;\\
\operatorname{lmin}\pi &= \operatorname{lmin}\sigma;\\
\operatorname{rmax}\pi &= \operatorname{rmax}\tau;\\
\operatorname{ldr}\pi &= \operatorname{ldr}\sigma;
\end{aligned}$$

and, if k is the largest integer such that $\pi = (\oplus^k 1) \oplus \tau$ for some nonempty τ, then

$$\operatorname{lir}\pi = k + \operatorname{lir}\tau$$

Proof. Straightforward and omitted. □

Theorem 9. *Let f be the bijection from $\beta(1,0)$-trees on $n+1$ nodes onto length n avoiders, as defined by* (3)*. It sends the first 7-tuple of statistics, below, to the second 7-tuple.*

$$\begin{array}{lllllllll} (& \text{sub}, & \text{leaves}, & \text{root}, & \text{lpath}, & \text{rpath}, & \text{lsub}, & \text{beta} &) \\ (& \text{comp}, & 1+\text{asc}, & \text{lmax}, & \text{lmin}, & \text{rmax}, & \text{ldr}, & \text{lir} &) \end{array}$$

Proof. The proof proceeds by induction. The base case follows from Lemma 5 and Lemma 4. Let t be any $\beta(1,0)$-tree with $n+1$ nodes. We split into two cases: (1) $t=\lambda_i u$ is indecomposable; (2) $t=u\oplus v$ is decomposable.

Case 1: That $\operatorname{comp} f(\lambda_i s) = \operatorname{comp} \phi_i f(s) = 1 = \operatorname{sub} \lambda_i s$ is clear since both $\phi_i \pi$ and $\lambda_i t$ are indecomposable for any permutation π and any $\beta(1,0)$-tree t. Also,

$$\begin{aligned} 1+\operatorname{asc} f(\lambda_i s) &= 1+\operatorname{asc} \phi_i f(s) && \text{by definition of } f \\ &= 1+\operatorname{asc} f(s) && \text{by Lemma 7} \\ &= \operatorname{leaves} s && \text{by induction} \\ &= \operatorname{leaves} \lambda_i s && \text{by Lemma 4} \end{aligned}$$

Thus we have proved that, for indecomposable trees, f sends sub to comp and leaves to $1+\text{asc}$. The proofs of the remaining statements follow the same pattern: recall the definition of f, apply Lemma 7, use the induction hypothesis, and finish by applying Lemma 4.

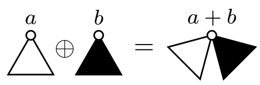
Case 2: We have

$$\begin{aligned} \operatorname{comp} f(u\oplus v) &= \operatorname{comp}(f(u)\oplus f(v)) && \text{by definition of } f \\ &= \operatorname{comp} f(u) + \operatorname{comp} f(v) && \text{by Lemma 8} \\ &= \operatorname{sub} u + \operatorname{sub} v && \text{by induction} \\ &= \operatorname{sub}(u\oplus v) && \text{by Lemma 5} \end{aligned}$$

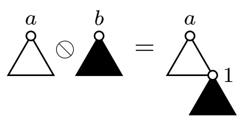
Again, the remaining statements follow similarly. □

8. An involution on $\beta(1,0)$-trees

In this section we define an involution on $\beta(1,0)$-trees. To that end we now describe a new way of decomposing $\beta(1,0)$-trees. Schematically the sum $\oplus$ on $\beta(1,0)$-trees is described by



An alternative sum is



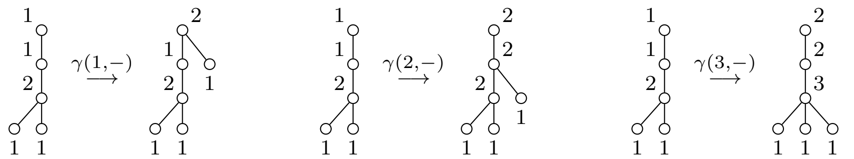
That is, to get $u \oslash v$ we join u and v by identifying the rightmost leaf in u with the root of v, and that node is assigned the label 1. Note that

$$\operatorname{root}(u\oplus v) = \operatorname{root}(u) + \operatorname{root}(v) \tag{14}$$
$$\operatorname{rpath}(u\oplus v) = \operatorname{rpath}(v) \tag{15}$$

while

$$\operatorname{root}(u\oslash v) = \operatorname{root}(u) \tag{16}$$
$$\operatorname{rpath}(u\oslash v) = \operatorname{rpath}(u) + \operatorname{rpath}(v). \tag{17}$$

for $u \neq \circ$ and $v \neq \circ$. Thus, with respect to $\oslash$, rpath plays the role of root, and vice versa. There is also a map γ that plays a role analogous to that of λ:

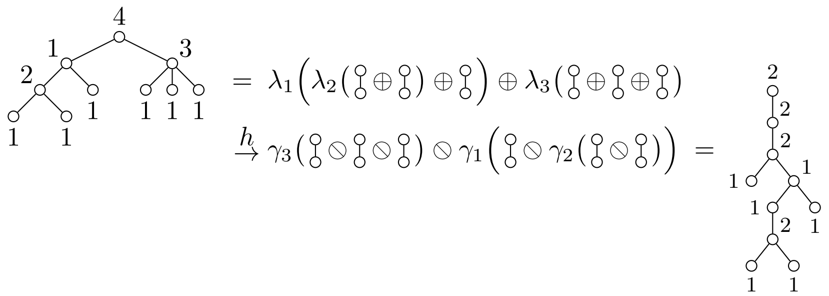


Here is how $\gamma(i, t)$ is defined in general: Assume that the length of the right path of t is k and that i is an integer such that $1 \leq i \leq k$. Let us by x refer to the ith node on the right path of t. Then $\gamma(i, t)$ is obtained from t by joining a new leaf via an edge to x, making the new leaf the rightmost leaf in $\gamma(i, t)$; and, lastly, adding 1 to the label of each node on the new right path, except for the new leaf (in which the new right path ends). Note, in particular, that $\operatorname{rpath} \gamma(i, t) = i$.

We now connect the two ways we have to decompose $\beta(1, 0)$-trees by defining an endofunction $h : \mathcal{B} \to \mathcal{B}$:

$$h(\circ) = \circ, \quad h(\lambda_i t) = \gamma_i h(t), \quad \text{and} \quad h(u \oplus v) = h(v) \oslash h(u).$$

For instance,



It should be clear that h is defined to translate an encoding based on λ and $\oplus$ into an encoding based on γ and $\oslash$. As it turns out, h is an involution! For the proof of the following theorem we refer the reader to a forthcoming paper by the present authors [8], but first we need to define the statistic gamma(t). This is simply the statistic beta on the mirror image of t, where the mirror image m is the involution on $\beta(1, 0)$-trees that recursively reverses the order of subtrees (see the end of Section 2 for the definition of beta). To be precise, we have $m(\circ) = \circ$, $m(\lambda_i t) = \lambda_i m(t)$, and $m(u \oplus v) = m(v) \oplus m(u)$. Then we have $\operatorname{gamma}(t) = \operatorname{beta}(m(t))$.

Another way to define gamma(t) is as follows: Using γ and $\oslash$ we can write $t = \gamma_{i_1}(\gamma_{i_2}(\ldots \gamma_{i_k}(u)))$ in which u is either a single node or decomposable with respect to $\oslash$, and we then let $\operatorname{gamma}(t) = k$.

Theorem 10. *On $\beta(1, 0)$-trees with at least one edge, the function h is an involution, and it sends the first tuple below to the second.*

$$\begin{array}{lllllllll} (& \text{leaves}, & \text{int}, & \text{root}, & \text{rpath}, & \text{sub}, & \text{rsub}, & \text{stem}, & \text{gamma} \;) \\ (& \text{int}, & \text{leaves}, & \text{rpath}, & \text{root}, & \text{rsub}, & \text{sub}, & \text{gamma}, & \text{stem} \;) \end{array}$$

Corollary 11. *On length n avoiders, the involution $f^{-1} \circ h \circ f$ sends*

$$(\text{asc}, \text{lmax}, \text{rmax}) \;\; \textit{to} \;\; (\text{des}, \text{rmax}, \text{lmax}).$$

Proof. Follows from combining Theorem 10 with Theorem 9. □

Corollary 12. *On length n avoiders, the involution $f^{-1} \circ m \circ h \circ m \circ f$ sends*

$$(\text{asc}, \text{lmax}, \text{lmin}, \text{comp}, \text{ldr}) \;\; \textit{to} \;\; (\text{des}, \text{lmin}, \text{lmax}, \text{ldr}, \text{comp}).$$

Proof. Follows from Theorems 10 and 9 together with the definition of m. □

We end this section with the observation that h restricted to $\beta(1,0)$-trees with all nodes labeled 1 (except the root) induces an involution on unlabeled rooted plane trees. This involution appears to be new and its consequences will be explored in a forthcoming paper [8]. In particular, this yields results akin to Corollaries 11 and 12 for one-stack sortable permutations. Moreover, this also gives rise to a genuinely new bijection between (1-2-3)-avoiding and (1-3-2)-avoiding permutations and yields new equidistributions of statistics on these two classes of permutations. (See [7] for a classification of the known bijections between (1-2-3)-avoiding and (1-3-2)-avoiding permutations.)

9. A conjecture about two-stack sortable permutations

Dulucq et al [9] proved that the pair $(\mathrm{asc}, \mathrm{lmax})$ on avoiders is equidistributed with the pair $(\mathrm{des}, \mathrm{rmax})$ on 2-stack sortable permutations. We make the following conjecture:

Conjecture 13. *The quadruple* $(\,\mathrm{comp},\ \mathrm{asc},\ \mathrm{ldr},\ \mathrm{rmax}\,)$ *has the same distribution on length n avoiders as it has on 2-stack sortable permutations of length n.*

10. Acknowledgments

The first author did some of the work on this paper while visiting the Mittag-Leffler Institute, and he is grateful for their hospitality.

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