CONWAY'S NAPKIN PROBLEM

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1. Introduction.

The problem studied in this article first appeared in the book *Mathematical Puzzles: A Connoisseur's Collection*, by Peter Winkler [5] and was inspired by a true story. Rather than recounting the problem and the story ourselves, we prefer to quote directly from *Mathematical Puzzles* [5, p. 22]:

> The Malicious Maitre D'
>
> At a mathematics conference banquet, 48 male mathematicians, none of them knowledgeable about table etiquette, find themselves assigned to a big circular table. On the table, between each pair of settings, is a coffee cup containing a cloth napkin. As each person is seated (by the maitre d'), he takes a napkin from his left or right; if both napkins are present, he chooses randomly (but the maitre d' doesn't get to see which one he chose).
>
> In what order should the seats be filled to maximize the expected number of mathematicians who don't get napkins?
>
> ...This problem can be traced to a particular event. Princeton mathematician John H. Conway came to Bell Labs on March 30, 2001 to give a "General Research Colloquium." At lunchtime, [Winkler] found himself sitting between Conway and computer scientist Rob Pike (now of Google), and the napkins and coffee cups were as described in the puzzle. Conway asked how many diners would be without napkins if they were seated in *random* order, and Pike said: "Here's an easier question—what's the *worst* order?"

The problem of the malicious maitre d' is not horribly difficult; if you're having trouble finding a solution, you can see Winkler's book for a nice explanation. In this paper, it is Conway's problem that we focus on. Again, from the book [5, p. 122]:

> Napkins in a Random Setting
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> Remember the conference banquet, where a bunch of mathematicians find themselves assigned to a big circular table? Again, on the table, between each pair of settings, is a coffee cup containing a cloth napkin. As each person sits down, he takes a napkin from his left or right; if both napkins are present, he chooses randomly.
>
> This time there is no maitre d'; the seats are occupied in random order. If the number of mathematicians is large, what fraction of them (asymptotically) will end up without a napkin?

Let p be the probability that a diner prefers the left napkin and $q = 1 - p$ the probability that a diner prefers the right napkin. For the case $p = q = 1/2$, Winkler's book gives two proofs of the answer: $(2-\sqrt{e})^2 \approx .12339675$. One is combinatorial, while the other, taken from a more general result due to Aidan Sudbury [3, Theorem 6], is analytical. In fact

Sudbury gives the expected proportion of diners without a napkin as (asymptotically)

$$\frac{(1 - pe^q)(1 - qe^p)}{pq}. \tag{1}$$

(As an aside, we took an informal survey of fifty-five mathematicians and found about 69 percent would prefer the napkin on the left. According to Sudbury's result, we would thus expect about 10.58 percent of the guests to get stuck without napkins.)

In this paper, we use combinatorial methods to produce the generating function for the probability that at a table for n people i of them have no napkin and j of them have a napkin, but not the napkin they prefer. This generating function allows for a thorough statistical analysis of the problem, including a new proof of (1).

However, the techniques we use are just as important as the results in and of themselves. We hope that those readers not familiar with generating functions in a combinatorial setting will find this problem an accessible case study. Combinatorialists often use the following two-step approach to such an enumerative or probabilistic question: (1) use a discrete combinatorial model to arrive at an expression for the relevant generating function, and if exact computation of the coefficients is infeasible, (2) apply techniques from analysis to obtain asymptotic estimates.

In section 2 we define the generating functions that we use throughout the paper. Section 3 explores the question of when everyone gets a napkin and makes connections with combinatorial objects called *ordered bipartitions*, due to Dominique Foata and Doron Zeilberger [1]. This connection makes subsequent proofs much simpler, as in section 4, where we derive powerful identities involving our generating functions that ultimately lead to exact formulas. In section 5 we answer the original question of how many guests are expected to be without a napkin, as well as provide (asymptotic estimates of) some other statistics of interest.

2. Preliminaries.

From our point of view, the number of people without a napkin is a statistic for signed permutations, just not one so well studied as inversions, descents, and such. We consider the order in which guests sit down at a place as a permutation of $[n] = \{1, 2, \ldots, n\}$, while their preference for the right or left napkin is given by a plus or minus sign, respectively. We label the places $1, 2, 3, \ldots, n$ counterclockwise, so that place i has place $i-1$ on its left, place $i+1$ on its right, and place n is to the left of place 1. If $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, then π_i records the order in which the guest sitting in place i arrived, along with his napkin preference. With this convention, the signed permutation $(2, -3, 4, -1)$ describes the following sequence of events at a table for four. The person sitting in place 4 sits down first and takes the napkin on his left. The person in place 1 sits next and takes the napkin on his right. The person at place 2 sits third and wants to take the napkin on his left, but since that napkin is already taken, he is forced to take the napkin on the right. Finally, the poor person in seat 3 sits last to find no napkin at his place.

In Figure 2 we illustrate the result of this sequence of events. The symbols "$\bullet$" and "人" represent a seat and a napkin, respectively. Moreover, a solid line symbolizes an arm taking a napkin and a broken line symbolizes an imaginary arm reaching for the preferred napkin. When looking at these diagrams, keep in mind that the table is circular: the guests at places 1 and 4 (on the far left and far right) are neighbors.

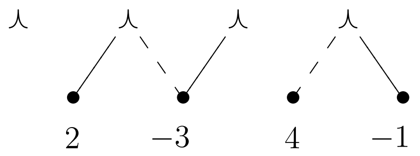

FIGURE 1. A table for four.

We denote the set of signed permutations of $[n]$ by $\mathcal{C}_n$. For any π in $\mathcal{C}_n$ let $|\pi|_-$ and $|\pi|_+$ be the number of negative and positive entries in π, respectively. Also, let $|\pi| = |\pi|_- + |\pi|_+ = n$ be the length of π. Let $o(\pi)$ signify the number of guests without a napkin after every guest has been seated as described by π. Furthermore, let $m(\pi)$ be the number of people who get a napkin, but not their first choice. We say a guest is *napkinless* if he has no napkin, and a guest is *frustrated* if he gets a napkin, but not the napkin he originally wanted.[1] Otherwise, we say the guest is *happy*. Define the *weight* $w(\pi)$ of π by

$$w(\pi) = p^{|\pi|_-} q^{|\pi|_+} x^{o(\pi)} y^{m(\pi)},$$

where p and $q = 1 - p$ are the constants indicating the probability of a diner preferring the left or the right napkin, and x and y are indeterminates. For example, if $\pi = (2, -1, 3, 4)$, then $|\pi|_- = 1$, $|\pi|_+ = 3$, and $|\pi| = 4$. Also, the napkin between places 2 and 3 is unused (guest 4 is the unlucky one), and although person 1 gets a napkin, it was not the one he wanted. Thus, $o(\pi) = 1$, $m(\pi) = 1$, and two of the guests are happy: $w(\pi) = pq^3xy$.

The generating function we are interested in is

$$C(p; x, y, z) := \sum_{n\geq 1} \sum_{\pi\in\mathcal{C}_n} w(\pi) \frac{z^n}{n!} = \sum_{\substack{i,j\geq 0\\ n\geq 1}} \mathrm{pr}(i, j, n) x^i y^j z^n, \tag{2}$$

where $\mathrm{pr}(i, j, n)$ denotes the probability that at a table for n people i of them are napkinless and j of them are frustrated. Theorem 3 gives an exact formula for $C(p; x, y, z)$. Our approach is to first "straighten" the table.

Suppose that instead of a circular table with n places and n napkins, we look at a straight table with n places and $n + 1$ napkins, so that each place has a napkin on both its left and right. If we know all that can possibly happen in this situation, then in order to determine the circular case, we just consider that the last person to enter the room sits "between" the first and last person on the linear table. Let us make this connection more precise.

Let $\mathcal{N}_n$ be the set of all signed permutations of $[n]$ that result in taking neither the leftmost nor the rightmost napkin from the table. Similarly, $\mathcal{L}_n$ (respectively, $\mathcal{R}_n$) denotes the set of all signed permutations of $[n]$ that result in the left end napkin being taken but not the right (respectively, right but not left), and $\mathcal{B}_n$ denotes those signed permutations that result in both end napkins being taken. We note that $\mathcal{C}_n = \mathcal{N}_n \cup \mathcal{L}_n \cup \mathcal{R}_n \cup \mathcal{B}_n$.

[1]When discussing this problem, French mathematician Sylvie Corteel argued that if the mathematicians were French, they would never take the "incorrect" napkin. If the napkin they wanted was not there, they would simply cross their arms and refuse to eat. But of course, that is a different problem (How does the number of napkinless guests change as the proportion of French diners changes?).

For the straight table we define the weight $w_S(\pi)$ of π by

$$w_S(\pi) = p^{|\pi|_-} q^{|\pi|_+} x^{o_S(\pi)} y^{m_S(\pi)},$$

where $o_S(\pi)$ and $m_S(\pi)$ are the number of napkinless and frustrated people on the straight table. Notice that $w_S(\pi)$ and $w(\pi)$ might or might not agree. We introduce the following generating functions for the straight table:

$$S(p; x, y, z) := \sum_{n\geq 0} \sum_{\pi \in \mathcal{C}_n} w_S(\pi) \frac{z^n}{n!};$$

$$N(p; x, y, z) := \sum_{n\geq 0} \sum_{\pi \in \mathcal{N}_n} w_S(\pi) \frac{z^n}{n!}; \quad L(p; x, y, z) := \sum_{n\geq 1} \sum_{\pi \in \mathcal{L}_n} w_S(\pi) \frac{z^n}{n!};$$

$$R(p; x, y, z) := \sum_{n\geq 1} \sum_{\pi \in \mathcal{R}_n} w_S(\pi) \frac{z^n}{n!}; \quad B(p; x, y, z) := \sum_{n\geq 1} \sum_{\pi \in \mathcal{B}_n} w_S(\pi) \frac{z^n}{n!}.$$

By construction, we have

$$S(p; x, y, z) = N(p; x, y, z) + L(p; x, y, z) + R(p; x, y, z) + B(p; x, y, z). \tag{3}$$

Further, we can observe that by symmetry

$$\begin{aligned} C(p; x, y, z) &= C(q; x, y, z), \quad & S(p; x, y, z) &= S(q; x, y, z), \\ N(p; x, y, z) &= N(q; x, y, z), \quad & B(p; x, y, z) &= B(q; x, y, z), \end{aligned}$$

and

$$R(p; x, y, z) = L(q; x, y, z). \tag{4}$$

From now on we will usually suppress the p and write $C(x, y, z)$ instead of $C(p; x, y, z)$, $S(x, y, z)$ instead of $S(p; x, y, z)$, etc.

Now let us recast the generating function $C(x, y, z)$ in terms of the generating functions for the linear table. Everything that has happened before the last person sits down at a table for n people can be considered the result of a signed permutation of $[n-1]$ playing out on a linear table. If the last person sits at place n, this is obvious, but in general the labeling of the seats is unimportant (i.e., we can always cyclically permute the labels on the seats without changing the weight of the permutation). Therefore, in what follows we can assume without loss of generality that $\pi = (\pi_1, \pi_2, \ldots, \pi_{n-1}, \pm n)$. Let $\pi' = (\pi_1, \pi_2, \ldots, \pi_{n-1})$.

Picture π in $\mathcal{C}_n$ acting on a circular table for n guests. If the last person walks in and has both napkins available, then π' must have resulted in leaving both end napkins on a linear table, that is, π' belongs to $\mathcal{N}_{n-1}$. Whether the last guest prefers the left napkin or the right napkin, he will get his choice, so the weight of π is given by:

$$\text{Case } \mathcal{N}: \qquad w(\pi) = \begin{cases} p w_S(\pi') & \text{if person } n \text{ prefers left,} \\ q w_S(\pi') & \text{if person } n \text{ prefers right.} \end{cases}$$

If person n walks in to find only the left napkin available, then π' is in $\mathcal{L}_{n-1}$, and the last person will take that napkin regardless of preference, getting the one he wants with probability p, and getting frustrated with probability q:

$$\text{Case } \mathcal{L}: \qquad w(\pi) = \begin{cases} p w_S(\pi') & \text{if person } n \text{ prefers left,} \\ qy w_S(\pi') & \text{if person } n \text{ prefers right.} \end{cases}$$

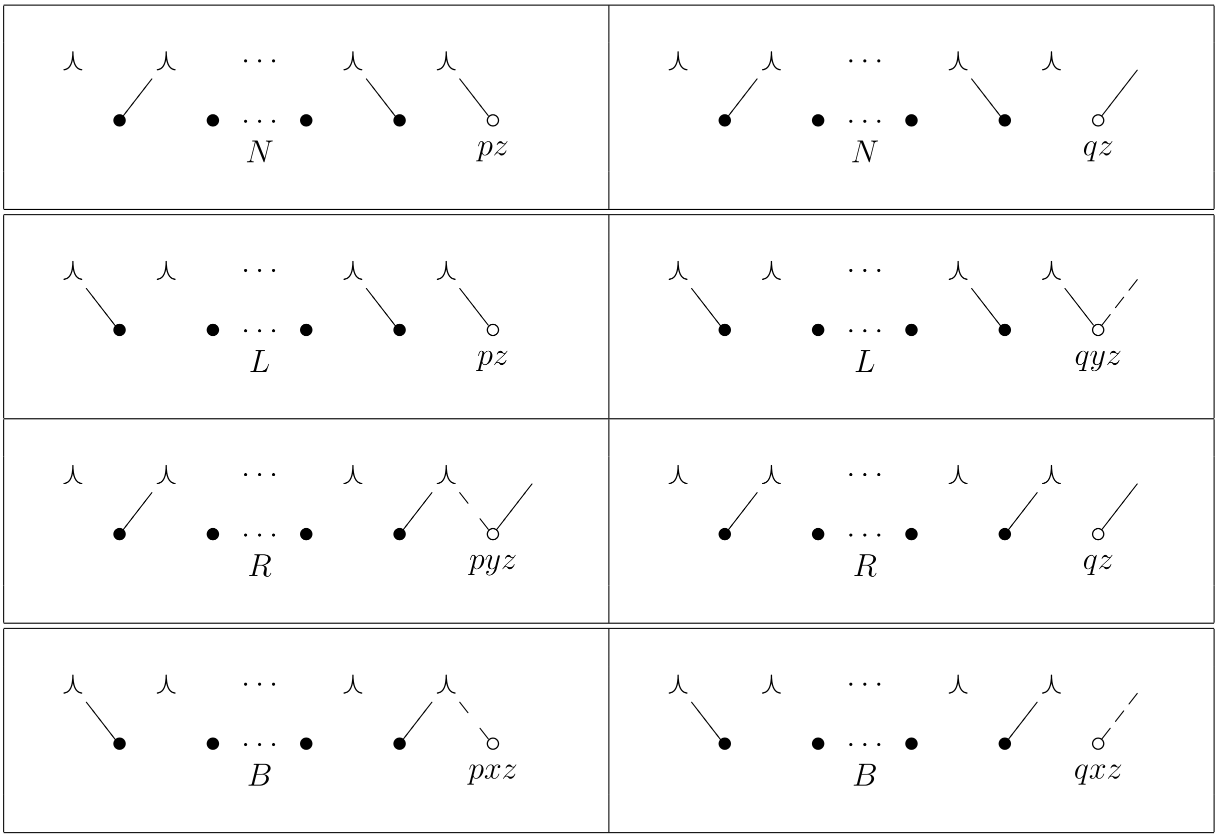

FIGURE 2. $C = z\big(N + (p+qy)L + (py+q)R + xB\big)$.

Similarly, if person n walks in to find only the right napkin available, then

$$\text{Case } \mathcal{R}: \qquad w(\pi) = \begin{cases} pyw_S(\pi') & \text{if person } n \text{ prefers left,} \\ qw_S(\pi') & \text{if person } n \text{ prefers right.} \end{cases}$$

Finally, guest n can walk in to find both napkins already taken. In this case, we know that π' belongs to $\mathcal{B}_{n-1}$ and that guest n will be one of the napkinless guests:

$$\text{Case } \mathcal{B}: \qquad w(\pi) = \begin{cases} pxw_S(\pi') & \text{if person } n \text{ prefers left,} \\ qxw_S(\pi') & \text{if person } n \text{ prefers right.} \end{cases}$$

Figure 2 illustrates these, altogether eight, possibilities, where the last person to sit is indicated with a "$\circ$." We can see that all other cases can be reduced to this case by cyclically shifting the picture. In terms of generating functions we have showed that

$$C(x,y,z) = z\big(N(x,y,z) + (p+qy)L(x,y,z) + (py+q)R(x,y,z) + xB(x,y,z)\big). \qquad (5)$$

3. A warm-up problem: when does everyone have a napkin?

A natural question to ask (perhaps easier than the general question) is: What is the probability that every guest has a napkin at a circular table for n people? The generating function for this probability is $C(0, y, z)$. By equation (5), we have

$$C(0, y, z) = z\big(N(0, y, z) + (p + qy)L(0, y, z) + (py + q)R(0, y, z)\big),$$

but since (on a straight table with at least one person) there is always at least one person without a napkin if neither end napkin is taken, $N(0, y, z) = 1$, and the equation reduces to

$$C(0, y, z) = z\big(1 + (p + qy)L(0, y, z) + (py + q)R(0, y, z)\big). \tag{6}$$

This makes intuitive sense because if everyone is to have a napkin, all diners need to take the left napkin, or all need to take the right napkin. Therefore we turn our attention to $L(0, y, z)$ (since $R(0, y, z)$ follows by swapping p and q).

For n a positive integer let us consider a straight table for n guests, and let π be a member of $\mathcal{L}_n$. Further, let $\pi = (\sigma, \pm n, \tau)$, where σ and τ are the signed permutations consisting of the letters to the left and to the right of n, respectively.

If the last person to enter sits down at the rightmost seat (i.e., τ is empty) then, since π is in $\mathcal{L}_n$, he necessarily prefers the left napkin. In this event,

$$w_S(\pi) = w_S(\sigma)p.$$

Assume that the last person to enter sits down with at least one guest to his right (i.e., τ is not empty). Then he will be happy with probability p and frustrated with probability q:

$$w_S(\pi) = \begin{cases} w_S(\sigma)pw_S(\tau) & \text{if person } n \text{ prefers left,} \\ w_S(\sigma)qyw_S(\tau) & \text{if person } n \text{ prefers right.} \end{cases}$$

Thus, by comparing coefficients, we see that the function $L(0, y, z)$ satisfies the following differential equation:

$$\frac{d}{dz}L(0, y, z) = \big(L(0, y, z) + 1\big)\big(p + (p + qy)L(0, y, z)\big). \tag{7}$$

We can solve this differential equation to get a formula for $L(0, y, z)$, but let us investigate its coefficients first and try to understand better what's going on. Let L_n denote the coefficient of $z^n/(2^n n!)$ in $L(1/2; 0, 1, z)$, in other words, the number of signed permutations that result in everyone taking the left napkin. Using either (7) or brute force, we can compute the L_n for some small values: $L_1 = 1$, $L_2 = 3$, $L_3 = 13$, $L_4 = 75$, $L_5 = 541$, $\ldots$, and plug them into Sloane's *Encyclopedia* [2]. Luckily, we get a hit with sequence A000670! These numbers happen to be fairly well known as the "ordered Bell numbers," or the number of ordered set partitions of $[n]$. This observation gives us a hint about the combinatorial object we will use in section 4 to get our main theorem. First, we will establish a bijection between the ordered set partitions of $[n]$ and permutations for which everyone takes the left napkin. This bijection will also provide the general solution to (7) (without having to solve the differential equation!).

Ordered set partitions. We now give a bijection between ordered set partitions of $[n]$ and signed permutations that correspond to everyone at a linear table taking the napkin on the left. An *ordered set partition* α of $[n]$ is a word $\alpha = B_1B_2\cdots B_k$, where the "blocks" $B_1, B_2, \ldots, B_k$ are subsets of $[n]$ such that $\{B_1, B_2, \ldots, B_k\}$ is a set partition of $[n]$. By convention we require that the elements of each block be written in decreasing order.

We describe the bijection with an example. Let

$$\alpha = \{5, 2\}\{6\}\{7, 4, 1\}\{3\}.$$

First, we give a minus sign to the least element of each block, then we remove the braces, to obtain

$$\pi = (5, -2, -6, 7, 4, -1, -3),$$

which is a signed permutation corresponding to a situation where everyone takes the napkin on the left. Because we required that the elements of a block be in decreasing order, anytime someone has a preference for the napkin on the right, he finds that it has already been taken by the person sitting to his right.

Clearly this process is reversible. Given a permutation where everyone takes the napkin on the left, it must have at least one minus sign. In particular, it must have a minus sign on 1, since the first person to enter must take the napkin on his left. Anybody with a plus sign immediately to the left of a person with a minus sign must enter after that person. And if two or more people with plus signs are sitting (consecutively) to the left of a person with a minus sign, they must arrive in order of closeness to the minus sign; the closest to the minus sign first, followed by the second closest to the minus sign, and so on. This gives us the block structure of the partition on $[n]$. Reading π from left to right, we separate the blocks by putting walls immediately to the right of any number appearing with a minus sign. For example, if

$$\pi = (7, -5, -6, 4, -1, 3, -2),$$

we convert this permutation into

$$\alpha = \{7, 5\}\{6\}\{4, 1\}\{3, 2\}.$$

Now we can build the generating function for $L(0, y, z)$ by purely combinatorial means. Taking an approach from [1] (more on that paper in a bit), let

$$H(p; y, z) := \frac{pz}{1!} + \frac{pqyz^2}{2!} + \cdots + \frac{p(qy)^{n-1}z^n}{n!} + \cdots = \frac{p(e^{qyz} - 1)}{qy}.$$

Then $H(y, z) := H(p; y, z)$ is the generating function for single blocks $\{n, \ldots, 2, 1\}$, since for every such block only the person corresponding to the least number gets the napkin he wants, leaving the other $n-1$ people frustrated. For example, with $n = 5$ we see the following pattern:

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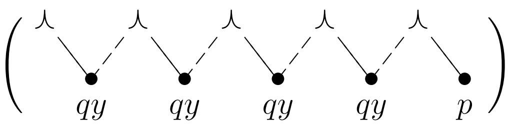

$= p(qy)^4.$

Therefore

$$(1 - H(y, z))^{-1} = 1 + H(y, z) + (H(y, z))^2 + \cdots$$

is the generating function for ordered sequences of such blocks (work out the first three terms to convince yourself), whence

$$L(0, y, z) = (1 - H(y, z))^{-1} - 1 = \frac{p(e^{qyz} - 1)}{qy - p(e^{qyz} - 1)}.$$

The reader can easily check that this expression indeed satisfies the differential equation (7). Due to equations (4) and (6) we are now in a position to give the generating function for the probability that everyone gets a napkin on a circular table:

$$\begin{aligned} C(0,y,z) &= z\big(1+(p+qy)L(p;0,y,z)+(py+q)L(q;0,y,z)\big) \\ &= z\left(1+\frac{(p+qy)p(e^{qyz}-1)}{qy-p(e^{qyz}-1)}+\frac{(py+q)q(e^{pyz}-1)}{py-q(e^{pyz}-1)}\right). \end{aligned}$$

Ordered bipartitions. Now we will generalize the correspondence just described. The paper of Foata and Zeilberger [1] introduces objects called *ordered bipartitions*, which are easiest to think of as ordered set partitions with some of the subsets underlined. A *compatible bipartition* is an ordered bipartition in which all the underlined subsets are on the right, e.g.,

$$\alpha = \{5,2\}\{6\}\underline{\{1,4,7\}}\,\underline{\{3\}},$$

where we adopt the convention that underlined subsets have their elements written in ascending order. The bijection works as follows. For every nonunderlined group in α we perform the same operation as earlier, while for every underlined group we perform the "opposite" operation. Specifically, we put minus signs on all *but* the least element before removing the braces. This produces

$$\pi = (5,-2,-6,1,-4,-7,3),$$

a permutation where everyone on a linear table receives a napkin. All we really need to observe is that the underlined groups correspond to the part of the table at which people all take napkins on their right, whereas the nonunderlined groups all take napkins on the left. Thus,

$$B(0,y,z) = L(0,y,z)R(0,y,z).$$

We can take this more general correspondence and see now that the permutations for which everyone takes the napkin on the left correspond to the ordered bipartitions with no underlined subsets, the ordered bipartitions with all subsets underlined correspond to permutations where everyone takes the right napkin, and the compatible bipartitions with at least one underlined subset and one nonunderlined subset correspond to the permutations where everyone gets a napkin and both end napkins are taken. These observations give

$$S(0,y,z) = (L(0,y,z)+1)(R(0,y,z)+1).$$

4. Ordered bipartitions and generating functions.

Using ordered bipartitions (not simply the compatible ones), we can encode more than just those permutations where everyone gets a napkin. Using the algorithm that we are about to describe, we can encode the set of *all* signed permutations. Let φ_S be this map, where the subscript S reminds us that we are dealing with the straight table. The image of the injection φ_S will lead us to the main theorem of this section, which gives some wonderful relationships between the generating functions for the linear table. Given a signed permutation π of $[n]$, we form its image $\varphi_S(\pi)$ as follows:

(1) Find the least element $\pi(i)$ (ignoring signs) that is not already included in some subset.
(2a) If $\pi(i)$ is positive, then underline the set including $\pi(i)$, and set $j = i+1$.
While $|\pi(j)| > |\pi(j-1)|$ and $\pi(j)$ negative,
add $\pi(j)$ to the set containing $\pi(i)$, and set $j = j+1$.
(2b) If $\pi(i)$ is negative, then set $j = i-1$.
While $|\pi(j)| > |\pi(j+1)|$ and $\pi(j)$ positive,
add $\pi(j)$ to the set containing $\pi(i)$, and set $j = j-1$.

(3) If every element is contained in a set, then delete all minus signs and quit. Otherwise, go to (1).

Clearly, no two permutations can be mapped to the same bipartition. As an example,

$$\pi = (9, 1, -3, 2, 5, 6, -4, -7, 8)$$

results in the ordered bipartition

$$\underline{\{\,9\,\}}\;\underline{\{\,1,3\,\}}\;\underline{\{\,2\,\}}\;\underline{\{\,5\,\}}\;\{\,6,4\,\}\;\{\,7\,\}\;\underline{\{\,8\,\}}.$$

We now explain how the weight $w(\pi)$ of the signed permutation π can be read from the corresponding ordered bipartition $\alpha = \varphi_S(\pi)$. To start with, the number $o_S(\pi)$ of napkinless diners is exactly the number of occurrences of an underlined block immediately to the left of a nonunderlined block. The frustrated diners are those who are not the least element in a block, less the people without any napkin:

$$m(\pi) = |\alpha| - \ell(\alpha) - o_S(\pi),$$

where $|\alpha|$ is the number of elements of the underlying set of α (same as $|\pi|$ here) and $\ell(\alpha)$ is the number of blocks in α. Let $\underline{\alpha}$ (respectively, $\overline{\alpha}$) be the the bipartition formed from the blocks of α that are underlined (respectively, nonunderlined). For underlined blocks of α we know that the least element is positive in π and the other elements are negative in π. Similarly, for nonunderlined blocks of α, the least element is negative in π and the other elements are positive in π. Accordingly,

$$|\pi|_- = |\underline{\alpha}| - \ell(\underline{\alpha}) + \ell(\overline{\alpha}), \qquad |\pi|_+ = |\overline{\alpha}| - \ell(\overline{\alpha}) + \ell(\underline{\alpha}).$$

Notice that while φ_S is an injection, it is not a bijection. Let $\varphi_S(\mathcal{C}_n)$ be the image set of bipartitions corresponding to all signed permutations of $[n]$. By examining the algorithm describing φ_S, we see that the set $\varphi_S(\mathcal{C}_n)$ consists of all ordered bipartitions that never contain the patterns

$$\underline{\cdots a\}}\,\{b\}\cdots$$

or

$$\cdots\underline{\{b\}}\,\{a\cdots,$$

where $a < b$.

To simplify notation in what follows, we write C for $C(x, y, z)$, S for $S(x, y, z)$, etc. Further, we write H for $H(p; y, z)$ and $\underline{H}$ for $H(q; y, z)$. The following theorem takes advantage of our combinatorial model. Its power is reflected in the subsequent results it implies.

Theorem 1. *The following formulas hold:*

$$L + B = HS; \tag{8}$$

$$B = HS\underline{H}. \tag{9}$$

Proof. If we add a nonunderlined block of size r to the left of any bipartition in $\varphi_S(\mathcal{C}_n)$, then clearly we get a bipartition in $\varphi_S(\mathcal{C}_{n+r})$ that corresponds to a permutation in $\mathcal{L}_{n+r} \cup \mathcal{B}_{n+r}$. Furthermore, this new block does not change the number of people without a napkin (occurrences of underlined blocks immediately to the left of nonunderlined blocks), and the number of new people who get a napkin they don't want is exactly $r - 1$. Therefore,

$$L + B = HS.$$

Similarly, if we add a non-underlined block of size r to the left of a a bipartition in $\varphi_S(\mathcal{C}_n)$, and an underlined block of size s to the right, then we get a bipartition in $\varphi_S(\mathcal{C}_{n+r+s})$ that corresponds to a permutation in $\mathcal{B}_{n+r+s}$. Thus,

$$B = HS\underline{H},$$

which concludes the proof. $\square$

Corollary 1. *The following formulas hold:*

$$\begin{aligned}
N &= S\big(1-H\big)\big(1-\underline{H}\big);\\
L &= SH\big(1-\underline{H}\big);\\
R &= S\big(1-H\big)\underline{H};\\
B &= SH\underline{H};\\
C &= zS\big(1+q(y-1)H+p(y-1)\underline{H}+(x-y)H\underline{H}\big).
\end{aligned}$$

Proof. From Theorem 1 we immediately get the formulas for B and L. If we plug these formulas into (4) and (3), the formulas for R and N follow. Finally, (5) yields the formula for C. $\square$

What Corollary 1 tells us is that if we can find an explicit formula for S, then we will have explicit formulas for all the other generating functions. We will derive such an explicit formula shortly. First we need to introduce the notion of a *cyclic* bipartition. A cyclic bipartition is a bipartition for which one element is distinguished, and only the cyclic ordering of the blocks matters. As a convention, we put the block containing the distinguished element at the far right if that block is not underlined and at the far left if it is underlined. In our notation we enclose the distinguished element in parentheses. For example,

$$\underline{\{1,6\}}\,\{8,2\}\,\underline{\{3,5\}}\,\{9,(7),4\}$$

is a cyclic bipartition, which we also write

$$7,4\}\,\underline{\{1,6\}}\,\{8,2\}\,\underline{\{3,5\}}\,\{9,$$

so that the distinguished element is equivalently the first element in a bipartition where a block can "wrap around." Similarly to how we encoded signed permutations playing out on a straight table with ordered bipartitions, we can encode the case of the circular table with cyclic bipartitions. Here the distinguished element corresponds to a "distinguished guest," who sits in place number 1 at the circular banquet table.

Let φ be the map encoding the circular table case. For any signed permutation π of $[n]$ we form its image $\varphi(\pi)$ with the same method as before, only now our searches are cyclic. Rather than a full description of the algorithm, we give an example. The permutation

$$\pi = (-7, 1, -3, 4, -2, 5, -6)$$

corresponds to

$$\underline{7\}}\,\underline{\{1,3\}}\,\{4,2\}\,\underline{\{5,6,}$$

or

$$\underline{\{5,6,(7)\}}\,\underline{\{1,3\}}\,\{4,2\}.$$

Let $\varphi(\mathcal{C}_n)$ be the set of all cyclic bipartitions corresponding to signed permutations on a circular table. We use this set of cyclic bipartitions along with $\varphi_S(\mathcal{C}_n)$ to obtain the following theorem regarding the derivative of $S(x, y, z)$:

Theorem 2. *The following formula relates the generating function for the straight table and the generating function for the circular table:*

$$z\frac{d}{dz}S(x, y, z) = C(x, y, z)S(x, y, z). \tag{10}$$

Proof. To derive equation (10), it suffices to equate coefficients and prove that

$$nS_n(x, y) = \sum_{i=1}^{n} \binom{n}{i} C_k(x, y)S_{n-i}(x, y), \tag{11}$$

in which the polynomials

$$S_n(x, y) := \sum_{\pi \in \mathcal{C}_n} w_S(\pi), \qquad C_n(x, y) := \sum_{\pi \in \mathcal{C}_n} w(\pi)$$

are the coefficients of $z^n/n!$ in $S(x, y, z)$ and $C(x, y, z)$, respectively. We prove (11) with a bijection.

The left-hand side of equation (11) can be thought of as counting the weights of permutations corresponding to bipartitions in $\varphi_S(\mathcal{C}_n)$ with a distinguished element, or a straight table with a distinguished guest. We simply put parentheses around one of the n elements of the bipartition, as in

$$\underline{\{\,2, 8\,\}}\ \{\,1\,\}\ \{\,7, (4), 3\,\}\ \underline{\{\,9\,\}}\ \underline{\{\,5, 6\,\}}.$$

Given any bipartition with a distinguished element, we can form a pair (c, s), where c is a cyclic bipartition of a subset A of $[n]$ (corresponding to one in $\varphi(\mathcal{C}_n)$), and s is an ordered bipartition of $[n] \setminus A$ (corresponding to one in $\varphi_S(\mathcal{C}_n)$). We simply split the table into two pieces just before or after the block containing the distinguished element. If the block containing the distinguished element is not underlined, we make the split just after that block. The foregoing example yields the pair

$$\big(\,\underline{\{\,2, 8\,\}}\ \{\,1\,\}\ \{\,7, (4), 3\,\},\ \underline{\{\,9\,\}}\ \underline{\{\,5, 6\,\}}\,\big).$$

If instead the block with the distinguished element is underlined, as in

$$\{\,2, 8\,\}\ \{\,1\,\}\ \underline{\{\,3, (4), 7\,\}}\ \underline{\{\,9\,\}}\ \underline{\{\,5, 6\,\}},$$

we make the split before the underlined block. Now the right half becomes the circular table, and the left half is the straight table:

$$\big(\,\underline{\{\,3, (4), 7\,\}}\ \underline{\{\,9\,\}}\ \underline{\{\,5, 6\,\}},\ \{\,2, 8\,\}\ \{\,1\,\}\,\big).$$

By splitting the bipartition as we do, all of the guests retain their status as happy, frustrated, or napkinless. In other words, the product of the weights of the pair of tables equals the weight of original table.

Now, given any pair (c, s) with c in $\varphi(\mathcal{C}_i)$ and s in $\varphi_S(\mathcal{C}_{n-i})$, we can form a bipartition in $\varphi_S(\mathcal{C}_n)$ with a distinguished element. First, in any of $\binom{n}{i}$ ways, we choose a subset $A = \{a_1 < a_2 < \cdots < a_i\}$ of $[n]$ and replace k with a_k in c. For s we replace k with b_k, where $[n] \setminus A = \{b_1 < b_2 < \cdots < b_{n-i}\}$. Now we concatenate the bipartitions. If the distinguished element of c is in an underlined block, we put the straight table on the left: sc. If the distinguished element is not in an underlined block, we put the straight table on the right: cs. $\square$

Now, thanks to Corollary 1 and Theorem 2, we have:

$$S' = \big((x-y)H\underline{H} + q(y-1)H + p(y-1)\underline{H} + 1\big)S^2.$$

We can solve this differential equation to get the following exact formula:

$$S(x,y,z) = \frac{pqy^3}{D}, \tag{12}$$

where the denominator D is

$$\begin{aligned} pq(y-x)e^{yz} + (qx - pqy - q^2y^2)e^{pyz} + (px - pqy - p^2y^2)e^{qyz} \\ + pq\big(y(y-1)^2 + x(1-yz)\big) + y^2 - x. \end{aligned} \tag{13}$$

Now we can obtain exact formulas for any of the other generating functions discussed here by plugging equation (12) into the formulas of Corollary 1. In particular, we have our main result.

Theorem 3. *At a table for n people, the probability that i people are napkinless and j people are frustrated is given by the coefficient of $x^i y^j z^k$ in the following function:*

$$C(x,y,z) = \frac{pqyz\big((x-y)e^{yz} + (qy^2 + py - x)e^{pyz} + (py^2 + qy - x)e^{qyz} + x\big)}{D}, \tag{14}$$

where D is given in (13).

5. The expected number of napkinless guests and other statistics.

With the generating function from (14), we can in principle extract any sort of statistics related to the proportion of napkinless guests or frustrated guests. We will highlight a few statistical results that are of interest to us. In particular, we find the expected number of napkinless and frustrated guests, the variance for each of these distributions, and the covariance for their joint distribution.[2]

The expected number of napkinless guests. If we want to obtain the expected number of people without a napkin, we want to compute the weighted average

$$E_n\big(o(\pi)\big) := \sum_{\pi \in \mathcal{C}_n} p^{|\pi|_-} q^{|\pi|_+} o(\pi).$$

Recall the definition of the polynomial $C_n(x,y) = \sum_{\pi \in \mathcal{C}_n} w(\pi)$ from the proof of Theorem 2. Suppose we know exactly what $C_n(x,y)$ is for some n. Then we simply differentiate $C_n(x,1)$ with respect to x and set $x = 1$ (we set $y = 1$ since we're not interested in frustrated diners at the moment):

$$C_n'(x,1) = \sum_{\pi \in \mathcal{C}_n} p^{|\pi|_-} q^{|\pi|_+} o(\pi) x^{o(\pi)-1},$$

$$C_n'(1,1) = \sum_{\pi \in \mathcal{C}_n} p^{|\pi|_-} q^{|\pi|_+} o(\pi).$$

[2]This task is daunting by hand, but luckily we have technology to help us. Computer software such as Maple, for example, is very helpful, both with solving the differential equation leading to (12) and in obtaining residues.

Therefore, we want to find the generating function for the numbers $E_n\big(o(\pi)\big) = C'_n(1,1)$, or

$$\begin{aligned} E(z) &:= \sum_{n\geq 0} E_n\big(o(\pi)\big) z^n \\ &= \frac{d}{dx}\Big[C(x,1,z)\Big]_{x=1} \\ &= \frac{z\big(pq(2-z)e^z + (p^2+pqz-1)e^{pz} + (q^2+pqz-1)e^{qz} + 1\big)}{pq(1-z)^2}. \end{aligned} \tag{15}$$

Theorem 4. *The expected number of napkinless guests on a circular table for n people is*

$$E_n\big(o(\pi)\big) = \frac{n}{pq}\big(1 - p\exp_n(q) - q\exp_n(p) + pq\exp_n(1)\big), \tag{16}$$

where $\exp_n(x) = \sum_{k=0}^{n} x^k/k!$ *is the truncated exponential function.*

Proof. Let $f(n) = n\big(1 - p\exp_n(q) - q\exp_n(p) + pq\exp_n(1)\big)/(pq)$ be the right-hand side of equation (16), and let $F(z)$ be the ordinary generating function for the numbers $f(n)$. Note that

$$nf(n+1) = (n+1)f(n) + \frac{1-p^n-q^n}{(n-1)!},$$

which implies that

$$z\frac{d}{dz}\left[\frac{1}{z}F(z)\right] = \frac{d}{dz}\big[zF(z)\big] + z(e^z - pe^{pz} - qe^{qz})$$

or, equivalently,

$$(1-z)F'(z) = \left(1+\frac{1}{z}\right)F(z) + z(e^z - pe^{pz} - qe^{qz}).$$

It is easy to check that the function $E(z)$ given by formula (15) satisfies this differential equation. □

Formula (16) implies Sudbury's result (equation (1)):

Corollary 2. *The expected value of* $o(\pi)$ *satisfies*

$$E_n\big(o(\pi)\big) = \frac{n(1-pe^q)(1-qe^p)}{pq} + O\Big(\frac{1}{n!}\Big).$$

In particular, when $p = q = 1/2$

$$E_n\big(o(\pi)\big) = n(2-\sqrt{e})^2 + O\Big(\frac{1}{n!}\Big).$$

In other words, the answer to Winkler's problem of napkins in a random setting is $(2-\sqrt{e})^2 \approx 0.12339675$. It required quite a bit of work to obtain this answer, but of course our work pays off in being able to find the following statistics as well.

Other statistics. In the rest of this section we present asymptotic estimates for further statistics regarding the napkin problem. Since the formulas are less messy, we primarily restrict our attention to the case $p = q = 1/2$, but our approach is general.

It is straightforward to obtain asymptotic estimates for functions with a finite number of poles such as $E(z)$. See, for example, chapter 5 of Herbert Wilf's book [4]. We use the same technique to obtain all the estimates given here. We briefly outline the approach.

Suppose that f is a meromorphic function on the complex plane whose only singularity is a pole of order m at the point $z = 1$ and that $f(z) = \sum a_n z^n$ is the power series expansion of f about the origin (valid when $|z| < 1$). The Laurent expansion of f around 1 has the form:

$$f(z) = \frac{b_{-m}}{(z-1)^m} + \cdots + \frac{b_{-1}}{(z-1)} + b_0 + b_1(z-1) + b_2(z-1)^2 + \cdots .$$

If we let

$$g(z) = \frac{b_{-m}}{(z-1)^m} + \cdots + \frac{b_{-1}}{(z-1)} = \sum c_n z^n$$

(called the *principal part* of f), then the function $h(z) = f(z) - g(z)$ is entire, and its coefficients vanish very quickly. Thus for large n the coefficient of z^n in $g(z)$ closely approximates the coefficient of z^n in $f(z)$ (i.e.,

$$a_n \sim c_n = (-1)^m \binom{m+n-1}{m-1} b_{-m} + \cdots + \binom{n+1}{1} b_{-2} - b_{-1} \text{).}$$

As an illustration, we apply this technique to $f(z) = E(z)$ to obtain a second derivation of Sudbury's result.

We first expand $E(z)$ as a series in $u = z - 1$ to get

$$\begin{aligned} E(u) &= \sum_{n \geq -2} b_n u^n \\ &= \frac{(1+u)\big(pq(1-u)e^{1+u} + (p^2 + pq(1+u) - 1)e^{p(1+u)} + (q^2 + pq(1+u) - 1)e^{q(1+u)} + 1\big)}{pqu^2}. \end{aligned}$$

We consider $\overline{E}(u) := u^2 E(u)$ and let $u \to 0$ to obtain

$$\overline{E}(0) = b_{-2} = \frac{pqe + (p^2 + pq - 1)e^p + (q^2 + pq - 1)e^q + 1}{pq}.$$

Next, we differentiate $\overline{E}$ and again let $u \to 0$, which gives

$$\overline{E}'(0) = b_{-1} = \frac{pq(e + e^p + e^q) + (1+p)(p^2 + pq - 1)e^p + (1+q)(q^2 + pq - 1)e^q - 1}{pq}.$$

Now we can simply read off the desired estimate:

$$a_n \sim (n+1)b_{-2} - b_{-1} = \frac{n(1 - pe^q)(1 - qe^p)}{pq}.$$

Using the same method, we find that the variance is asymptotically

$$\mathrm{Var}_n(o(\pi)) \sim \frac{n(1 - pe^q)(1 - qe^p)\big(1 - (p^2 - pq)e^q - (q^2 - pq)e^p - pq(e+1)\big)}{p^2 q^2},$$

or $n(3 - e)(2 - \sqrt{e})^2 \approx n(.0347631)$ for the $p = q = 1/2$ case.

X:	$o(\pi)$ (Napkinless)	$m(\pi)$ (Frustrated)
$E(X)$	$n(2-\sqrt{e})^2 \approx n(.12339675)$	$n(6\sqrt{e}-e-7) \approx n(.174046)$
$\mathrm{Var}(X)$	$n(3-e)(2-\sqrt{e})^2 \approx n(.0347631)$	$n(6\sqrt{e^3}-e^2-e-38\sqrt{e}+46)$ $\approx n(.13138819)$
$\mathrm{Cov}(X,Y)$	$n\big(-(2-\sqrt{e})(\sqrt{e^3}-3e-5\sqrt{e}+12)\big) \approx n(-.029239461)$	

Table 1. Statistics for napkinless and frustrated guests with $p = q = 1/2$.

The statistics for both the frustrated and napkinless guests are summarized in Table 1. Notice in particular that if the expected number of napkinless guests is $n(2-\sqrt{e})^2$ and the expected number of frustrated guests is $n(6\sqrt{e}-e-7)$, then the expected number of happy guests is $n(4-2\sqrt{e}) \approx n(.702557)$. Seventy percent of the guests are happy!

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